RESIDUATION THEORY AND MATRIX MULTIPLICATION ON ORTHOMODULAR LATTICES

by J. H. BEVIS *and* C. K. MARTIN

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In this paper we consider mappings induced by matrix multiplication which are defined on lattices of matrices whose coordinates come from a fixed orthomodular lattice L (i.e. a lattice with an orthocomplementation denoted by $'$ in which $a \leqq b \Rightarrow a \vee (a' \wedge b) = b$). $\mathscr{A}_{mn}$ will denote the set of all $m \times n$ matrices over L with partial order and lattice operations defined coordinatewise. For conformal matrices A and B the (i, j)th coordinate of the matrix product AB is defined to be $(AB)_{ij} = \bigvee_k (A_{ik} \wedge B_{kj})$. We assume familiarity with the notation and results of [**1**]. $\mathscr{A}_{mn}$ is an orthomodular lattice and the (lattice) *centre* of $\mathscr{A}_{mn}$ is defined as $\mathscr{C}(\mathscr{A}_{mn}) = \{A \in \mathscr{A}_{mn} \mid A\mathscr{C}B \text{ for all } B \in \mathscr{A}_{mn}\}$, where we say that A *commutes with* B and write $A\mathscr{C}B$ if $(A \vee B') \wedge B = A \wedge B$. In § 1 it is shown that mappings from $\mathscr{A}_{mn}$ into $\mathscr{A}_{mr}$ characterized by right multiplication $X \rightarrow XP$ $(P \in \mathscr{A}_{nr})$ are residuated if and only if $P \in \mathscr{C}(\mathscr{A}_{nr})$. (Similarly for left multiplication.) This result is used to show the existence of residuated pairs. Hence, in § 2 we are able to extend a result of Blyth [**3**] which relates invertible and cancellable matrices (see Theorem 3 and its corollaries). Finally, for right (left) multiplication mappings, characterizations are given in § 3 for closure operators, quantifiers, range closed mappings, and Sasaki projections.

1. After Croisot [**4**] a monotone mapping $\phi: \mathscr{A} \rightarrow \mathscr{B}$ from a lattice $\mathscr{A}$ into a lattice $\mathscr{B}$ is *residuated* if there is a monotone mapping $\phi^+: \mathscr{B} \rightarrow \mathscr{A}$ called the *residual* mapping corresponding to ϕ such that $a \leqq a\phi\phi^+$ for all a in $\mathscr{A}$ and $b\phi^+\phi \leqq b$ for all b in $\mathscr{B}$. One may show that ϕ and ϕ^+ determine each other uniquely.

THEOREM 1. *Given $P \in \mathscr{A}_{nr}$, the mapping $\phi: \mathscr{A}_{mn} \rightarrow \mathscr{A}_{mr}$ defined by $A\phi = AP$ is residuated if and only if $P \in \mathscr{C}(\mathscr{A}_{nr})$. If ϕ is residuated, $B\phi^+ = (B'P^t)'$, where P^t is the transpose of P.*

Proof. According to [**4**], a residuated mapping preserves joins. Hence, by Lemma 2 of [**1**], if $A \rightarrow AP$ is residuated, then $P \in \mathscr{C}(\mathscr{A}_{nr})$. If $P \in \mathscr{C}(\mathscr{A}_{nr})$, then

$$[(AP)'P^t]_{ij} = \bigvee_k [P_{jk} \wedge \bigwedge_h (A'_{ih} \vee P'_{hk})] = \bigvee_k [P_{jk} \wedge A'_{ij} \wedge \bigwedge_{h \neq j} (A'_{ih} \vee P'_{hk})] \leqq A'_{ij}.$$

Hence $A \leqq [(AP)'P^t]'$. Similarly $(B'P^t)'P \leqq B$.

For left multiplication we have the result:

THEOREM 1*. *Given $P \in \mathscr{A}_{nr}$, the mapping $\phi: \mathscr{A}_{rm} \rightarrow \mathscr{A}_{nm}$ defined by $A\phi = PA$ is residuated if and only if $P \in \mathscr{C}(\mathscr{A}_{nr})$. If ϕ is residuated, $B\phi^+ = (P^tB')'$.*

Extending the definition of Birkhoff [**2**, XIII], for P in $\mathscr{A}_{nr}$ and B in $\mathscr{A}_{mr}$ (B in $\mathscr{A}_{nm}$), we define the *right-residual* $B:P$ (*left-residual* $B::P$) of B by P as the largest X in $\mathscr{A}_{mn}$ ($\mathscr{A}_{rm}$), if it exists, satisfying $XP \leqq B$ ($PX \leqq B$). Such a pair P, B is said to be *residuated on the right* (*left*) if $B:P$ ($B::P$) exists.

The first two lemmas are due to Croisot [**4**], and are used in the proof of Theorem 2.

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LEMMA 1. *Let $\phi:\mathscr{A}\to\mathscr{B}$ be a residuated mapping, and let ϕ^+ be the corresponding residual mapping. For b in $\mathscr{B}$, $b\phi^+$ is the greatest element in the non-empty set $\{a\in\mathscr{A}\mid a\phi\leqq b\}$.*

LEMMA 2. *In order that the monotone mapping $\phi:\mathscr{A}\to\mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are lattices, be residuated, it is necessary and sufficient that for every b in $\mathscr{B}$ the set $\{a\in\mathscr{A}\mid a\phi\leqq b\}$ be non-empty and contain a greatest element.*

THEOREM 2. *For P in $\mathscr{A}_{nr}$ the following conditions are equivalent*:

(i) $P\in\mathscr{C}(\mathscr{A}_{nr})$.
(ii) $B:P$ *exists for all B in* $\mathscr{A}_{mr}$.
(iii) $B::P$ *exists for all B in* $\mathscr{A}_{nm}$.

Moreover, if $P\in\mathscr{C}(\mathscr{A}_{nr})$ and $B\in\mathscr{A}_{mr}$ ($B\in\mathscr{A}_{nm}$), then $B:P=(B'P^t)'$ ($B::P=(P^tB')'$).

Proof. By Theorem 1 and Lemma 1, (i) implies (ii) and (iii). By Lemma 2 and Theorem 1, (ii) or (iii) implies (i).

2. Motivated by Molinaro [**9**], we define two types of equivalence relations. For P in $\mathscr{C}(\mathscr{A}_{nr})$ define the equivalence relation Ψ_P on $\mathscr{A}_{mr}$ by $A\equiv B\ (\Psi_P)$ if $A:P=B:P$, and define the equivalence relation ${}_P\Psi$ on $\mathscr{A}_{nm}$ by $A\equiv B\ ({}_P\Psi)$ if $A::P=B::P$. For $P\in\mathscr{A}_{rn}$ define the equivalence relation Θ_P on $\mathscr{A}_{nm}$ by $A\equiv B\ (\Theta_P)$ if $PA=PB$, and define the equivalence relation ${}_P\Theta$ on $\mathscr{A}_{mr}$ by $A\equiv B\ ({}_P\Theta)$ if $AP=BP$.

LEMMA 3. *For P in $\mathscr{C}(\mathscr{A}_{nr})$, each class in $\mathscr{A}_{mr}$ ($\mathscr{A}_{nm}$) modulo Ψ_P (${}_P\Psi$) has a smallest element; the smallest element in the class containing A is $(A:P)P$ ($P(A::P)$). For P in $\mathscr{C}(\mathscr{A}_{rn})$, each class in $\mathscr{A}_{nm}$ ($\mathscr{A}_{mr}$) modulo Θ_P (${}_P\Theta$) has a greatest element; the greatest element in the class containing A is $PA::P$ ($AP:P$).*

Proof. Given $P\in\mathscr{C}(\mathscr{A}_{nr})$ and $A\equiv B(\Psi_P)$ in $\mathscr{A}_{mr}$; then $(A:P)P=(B:P)P\leqq B$, i.e., $(A:P)P$ is well defined on the class containing A and is a lower bound for the class. From $(A:P)P=(A:P)P$ we obtain $A:P\leqq(A:P)P:P$. Also, by the definition of right-residual, $[(A:P)P:P]P\leqq(A:P)P\leqq A$, which implies that $(A:P)P:P\leqq A:P$. Hence $(A:P)P\equiv A\,(\Psi_P)$. Given $P\in\mathscr{C}(\mathscr{A}_{rn})$ and $A\equiv B\ (\Theta_P)$ in $\mathscr{A}_{nm}$, it follows that $PA::P=PB::P$. From $PA=PA$ we obtain $A\leqq PA::P$, i.e., $PA::P$ is well defined on the class containing A in an upper bound for the class. Now $PA\leqq P(PA::P)$ by monotonicity of multiplication and $P(PA::P)\leqq PA$ by the definition of left-residual. Hence $PA::P\equiv A\ (\Theta_P)$. The remaining two parts of the lemma follow in a similar manner.

We are now ready to extend a result of Blyth [**3**] for Boolean matrices, to matrices over orthomodular lattices.

LEMMA 4. *For P in $\mathscr{A}_{rn}$, $A\equiv B(\Theta_P)$ in $\mathscr{A}_{nm}\Leftrightarrow A^t\equiv B^t({}_{P^t}\Theta)$ in $\mathscr{A}_{mn}$. For P in $\mathscr{C}(\mathscr{A}_{nr})$, $A\equiv B(\Psi_P)$ in $\mathscr{A}_{mr}\Leftrightarrow A^t\equiv B^t({}_{P^t}\Psi)$ in $\mathscr{A}_{rm}$.*

Proof. The first part is an immediate consequence of $(AP)^t=P^tA^t$. With P in $\mathscr{C}(\mathscr{A}_{nr})$, by Theorem 2 we obtain $A:P=(A'P^t)'=(PA'^t)'^t=(A^t::P^t)^t$. Thus $A\equiv B(\Psi_P)$ in $\mathscr{A}_{mr}\Leftrightarrow(A^t::P^t)^t=(B^t::P^t)^t\Leftrightarrow A^t\equiv B^t({}_{P^t}\Psi)$.

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LEMMA 5. *For P in $\mathscr{C}(\mathscr{A}_{nr})$, $A \equiv B\,(\Theta_{P^t})$ in $\mathscr{A}_{nm} \Leftrightarrow A' \equiv B'\,({}_P\Psi)$ in $\mathscr{A}_{nm}$, and $A \equiv B\,({}_{P^t}\Theta)$ in $\mathscr{A}_{mr} \Leftrightarrow A' \equiv B'\,(\Psi_P)$ in $\mathscr{A}_{mr}$.*

Proof. By Lemma 3, the smallest element in the class containing A modulo ${}_P\Psi$ is $P(A::P) = P(P^tA')'$. The greatest element in the class containing A modulo Θ_{P^t} is $P^tA::P^t = [P(P^tA)']'$. Now

$$A \equiv B\,(\Theta_{P^t}) \Leftrightarrow [P(P^tA)']' = [P(P^tB)']' \Leftrightarrow P(P^tA)' = P(P^tB)' \Leftrightarrow A' \equiv B'\,({}_P\Psi).$$

The remainder of the lemma is proved similarly.

We say that P in $\mathscr{A}_{nr}$ is *left* (*right*) *cancellable* in $\mathscr{A}_{rm}$ ($\mathscr{A}_{mn}$) if $PA = PB$ ($AP = BP$) implies $A = B$ whenever $A, B \in \mathscr{A}_{rm}$ ($A, B \in \mathscr{A}_{mn}$). Note that P is left (right) cancellable if and only if Θ_P (${}_P\Theta$) is the identity relation on $\mathscr{A}_{rm}$ ($\mathscr{A}_{mn}$). E will denote a matrix with $E_{ij} = \delta_{ij}$.

THEOREM 3. *If $P \in \mathscr{C}(\mathscr{A}_{nr})$ and $r \leqq m$ ($n \leqq m$), then the following are equivalent:*
(i) *P is left* (*right*) *cancellable in $\mathscr{A}_{rm}$ ($\mathscr{A}_{mn}$).*
(ii) *There exists $X \in \mathscr{A}_{mn}$ ($Y \in \mathscr{A}_{rm}$) such that $XP = E \in \mathscr{A}_{mr}$ ($PY = E \in \mathscr{A}_{nm}$).*
(iii) *There exists $X \in \mathscr{C}(\mathscr{A}_{mn})$ ($Y \in \mathscr{C}(\mathscr{A}_{rm})$), such that $XP = E \in \mathscr{A}_{mr}$ ($PY = E \in \mathscr{A}_{nm}$).*
(iv) *P is left* (*right*) *cancellable in $\mathscr{C}(\mathscr{A}_{rm})$ ($\mathscr{C}(\mathscr{A}_{mn})$).*

Proof. If P in $\mathscr{C}(\mathscr{A}_{nr})$ is left cancellable in $\mathscr{A}_{rm}$, then Θ_P is the identity relation on $\mathscr{A}_{rm}$. By Lemma 5, ${}_{P^t}\Psi$ is also the identity relation on $\mathscr{A}_{rm}$. The smallest element of the class containing E in $\mathscr{A}_{rm}$ modulo ${}_{P^t}\Psi$ is thus $E = P^t(E::P^t)$. By taking the transpose of each side, we obtain (i) $\Rightarrow$ (ii). Suppose that $X \in \mathscr{A}_{mn}$ and $XP = E$; then $X \leqq E:P$. Now

$$E = XP \leqq (E:P)P \leqq E.$$

By Theorem 2, $E:P = (E'P^t)'$ which is in $\mathscr{C}(\mathscr{A}_{mn})$. For (iii) $\Rightarrow$ (i), let $X \in \mathscr{C}(\mathscr{A}_{mn})$ and $XP = E \in \mathscr{A}_{mr}$. Since two of the three matrices involved are central, (X, P, A) is an associative triple for any A in $\mathscr{A}_{rm}$. Hence $PA = PB$ implies that $EA = EB$, where $E \in \mathscr{A}_{mr}$. If $r \leqq m$, then $EA = EB$ implies that $A = B$. Clearly (i) $\Rightarrow$ (iv). By applying the result (i) $\Rightarrow$ (iii) to matrices over $\mathscr{C}(L)$ we obtain (iv) $\Rightarrow$ (iii).

COROLLARY 1. *If $P \in \mathscr{C}(\mathscr{A}_{nr})$, and if there exists a positive integer m such that $r \leqq m$ ($n \leqq m$) and P is left* (*right*) *cancellable in $\mathscr{A}_{rm}$ ($\mathscr{A}_{mn}$), then P is left* (*right*) *cancellable in $\mathscr{A}_{rs}$ ($\mathscr{A}_{sn}$) for every $r \leqq s$ ($n \leqq s$).*

Proof. Let A be the matrix formed by the first r rows of the matrix described in (iii) of Theorem 3. For any $s \leqq r$, form $A(s)$ by augmenting A to an s rowed matrix whose last $s-r$ rows consist of zeros. Thus $A(s) \in \mathscr{C}(\mathscr{A}_{sn})$ and $A(s)P = E \in \mathscr{A}_{sr}$.

COROLLARY 2. *If $P \in \mathscr{C}(\mathscr{A}_{nn})$, $n \leqq m$, and P is left* (*right*) *cancellable in $\mathscr{A}_{mn}$ ($\mathscr{A}_{mn}$), then $PP^t = P^tP = E$.*

Proof. Let A be the matrix formed by the first n rows of the matrix described in (iii) of Theorem 3. Then $A \in \mathscr{C}(\mathscr{A}_{nn})$ and $AP = E$. The result now follows from a result of Rutherford [**10**, §3].

3. In this section we consider mappings from $\mathscr{A}_{mn}$ into itself which arise from matrix multiplication. Thus for right (left) multiplication by P, we necessarily require that $P \in \mathscr{A}_{nn}$ ($P \in \mathscr{A}_{mm}$). After Foulis [5], for an orthomodular lattice $\mathscr{A}$, define $S(\mathscr{A})$ to be the set of all those monotone mappings $\phi: \mathscr{A} \to \mathscr{A}$ such that there exists at least one, and hence exactly one, monotone mapping $\phi^*: \mathscr{A} \to \mathscr{A}$ with the property that $(a'\phi)'\phi^* \leqq a$ and $(a'\phi^*)'\phi \leqq a$ for every a in $\mathscr{A}$. Foulis shows that, if $\phi \in S(\mathscr{A})$, then ϕ is residuated, and that ϕ^* is given by $a\phi^* = (a'\phi^+)'$. Thus $\phi: A \to AP$ ($\phi: A \to PA$) is in $S(\mathscr{A}_{mn})$ if and only if $P \in \mathscr{C}(\mathscr{A}_{nn})$ ($P \in \mathscr{C}(\mathscr{A}_{mm})$), and in this case ϕ^* is given by right (left) multiplication by P^t. A mapping ϕ on a lattice $\mathscr{A}$ is called a *closure operator* if $a \leqq a\phi$ and $a\phi = (a\phi)\phi$ for all a in $\mathscr{A}$. ϕ is called a *quantifier* on $\mathscr{A}$ if $o\phi = o$, $a \leqq a\phi$, and $(a \wedge b\phi)\phi = a\phi \wedge b\phi$ for all a, b in $\mathscr{A}$.

LEMMA 6. *For $P \in \mathscr{A}_{nn}$ ($P \in \mathscr{A}_{mm}$), $\phi: A \to AP$ ($\phi: A \to PA$) is a closure operator on $\mathscr{A}_{mn}$ if and only if $E \leqq P$, $P = P^2$, and (A, P, P) ((P, P, A)) is an associative triple for all A in $\mathscr{A}_{mn}$.*

Proof. If $E \leqq P$, then $A = AE \leqq AP$. Conversely, $E \leqq E\phi = EP = P$. $A\phi = (A\phi)\phi$ implies that $P = EP = (EP)P = P^2$ and $(AP)P = AP = AP^2$. If $P = P^2$ and (A, P, P) is an associative triple, then $(AP)P = AP^2 = AP$.

COROLLARY. *If $E \leqq P = P^2$ and $P \in \mathscr{C}(\mathscr{A}_{nn})$ ($P \in \mathscr{C}(\mathscr{A}_{mm})$), then $\phi: A \to AP$ ($\phi: A \to PA$) is a closure operator on $\mathscr{A}_{mn}$.*

LEMMA 7. *If $P = P^t \in \mathscr{A}_{nn}$, or if $E \leqq P \in \mathscr{A}_{nn}$, then $P = P^2 \Leftrightarrow P_{ij} \geqq P_{ik} \wedge P_{kj}$ for all $i, j, k = 1, \ldots, n$.*

Proof. Suppose that $P = P^t$ and $P_{ij} \geqq P_{ik} \wedge P_{kj}$. Then $P_{ii} \geqq P_{ik} \wedge P_{ki} = P_{ik}$. Now

$$P_{ij} \geqq (P_{ij} \wedge P_{jj}) \vee \bigvee_{k \neq j}(P_{ik} \wedge P_{kj}) = P_{ij} \vee \bigvee_{k \neq j}(P_{ik} \wedge P_{kj}) \geqq P_{ij},$$

i.e. $P_{ij} = P^2_{ij}$. Conversely, if $P = P^t = P^2$, then $P_{ii} = P_{ii} \vee \bigvee_{n \neq i} P_{ik}$, i.e. $P_{ii} \geqq P_{ik}$. Now

$$P_{ij} = (P_{ij} \vee P_{jj}) \vee \bigvee_{k \neq j}(P_{ik} \wedge P_{kj}) = P_{ij} \vee \bigvee_{k \neq j}(P_{ik} \wedge P_{kj}).$$

Hence $P_{ij} \geqq P_{ik} \wedge P_{kj}$ for all $i, j, k = 1, \ldots, n$. If $P \geqq E$, then $P_{ii} \geqq P_{ik}$ and an obvious modification of the above proof establishes the result.

LEMMA 8. *Given $P \in \mathscr{A}_{nn}$ ($P \in \mathscr{A}_{mm}$), the mapping $A \to AP$ ($A \to PA$) is a quantifier on $\mathscr{A}_{mn}$ if and only if $E \leqq P = P^2 = P^t$, $P \in \mathscr{C}(\mathscr{A}_{nn})$ ($P \in \mathscr{C}(\mathscr{A}_{mm})$), and the columns (rows) of P possess property $\mathscr{D}$ on L. (See* [**1**, §1] *for the definition of property $\mathscr{D}$.)*

Proof. For the sufficiency of the conditions, all that remains is to show that

$$(A \wedge BP)P = AP \wedge BP.$$

By [**1**, Lemma 1], $(A \wedge BP)P \leqq AP \wedge (BP)P = AP \wedge BP$. By Lemma 7, $P_{hk} \geqq P_{hj} \wedge P_{jk}$, and hence, by property $\mathscr{D}$,

$$(AP \wedge BP)_{ij} = \bigvee_k [A_{ik} \wedge P_{kj} \wedge (BP)_{ij}] = \bigvee_k [A_{ik} \wedge P_{kj} \wedge \bigvee_h (B_{ih} \wedge P_{hj} \wedge P_{jk})]$$
$$\leqq \bigvee_k [A_{ik} \wedge P_{kj} \wedge \bigvee_h (B_{ih} \wedge P_{hk})] = [(A \wedge BP)P]_{ij}.$$

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Conversely, if $A \to AP$ is a quantifier on $\mathscr{A}_{mn}$, then, by Janowitz [7, Theorem 2],

$$P = P^2 = P^t \in \mathscr{C}(\mathscr{A}_{nn}).$$

As before, $A \leqq AP$ implies that $E \leqq P$. Let $b \in L$ and let B be such that $B_{ij} = b$ for all $i,j = 1, \ldots, n$. Then $AB \wedge BP = (A \wedge BP)P$ becomes $b \wedge \bigvee_k(A_{ik} \wedge P_{kj}) = \bigvee_k(A_{ik} \wedge P_{kj} \wedge b)$, that is, the columns of P possess property $\mathscr{D}$ on L.

Let $\mathscr{A}$ be a lattice with o and 1, and for a in $\mathscr{A}$ let $\mathscr{A}(o,a) = \{x \in \mathscr{A} \mid x \leqq a\}$. A mapping $\phi: \mathscr{A} \to \mathscr{A}$ is said to be *range closed* if $\phi: \mathscr{A} \to \mathscr{A}(o, 1\phi)$ is a surjective mapping.

For the next lemma we introduce a notation of Rutherford [10]. If P is a matrix with entries in an orthocomplemented lattice, let $\bar{P}$ be the matrix with $\bar{P}_{ij} = P_{ij} \wedge (\bigwedge_{n \neq j} P'_{kj})$ and $\underline{P}$ be the matrix with $\underline{P}_{ij} = P_{ij} \wedge (\bigwedge_{k \neq i} P'_{kj})$.

LEMMA 9. *Given* $P \in \mathscr{C}(\mathscr{A}_{nn})$ $(P \in \mathscr{C}(\mathscr{A}_{mm}))$, *the mapping* $A \to AP$ $(A \to PA)$ *is range closed in* $\mathscr{A}_{mn}$ *if and only if any of the following conditions obtain*:

(i) $(E'P^t)'P = E \wedge IP$ $(P(P^tE')' = E \wedge PI)$.

(ii) $\bigvee_h[P_{hj} \wedge (\bigwedge_{k \neq j} P'_{hk})] = \bigvee_h P_{hj}$ *for all* $j = 1, \ldots, n$, $(\bigvee_h[P_{ih} \wedge (\bigwedge_{k \neq i} P'_{hk})] = \bigvee_h P_{ih}$ *for all* $i = 1, \ldots, m)$.

(iii) $I\bar{P} = IP$ $(\underline{P}I = PI)$, *where* $I_{ij} = 1$ *for all* i,j.

Proof. First we note that (ii) is the assertion $[(E'P^t)'P]_{jj} = [E \wedge IP]_{jj}$ so that (i) $\Rightarrow$ (ii). By [8, Lemma 3.2], $A \to AP$ is range closed if and only if $(A'P^t)'P = A \wedge IP$ for all A in $\mathscr{A}_{mn}$. When $A = E$ one obtains the necessity of (i) and (ii). Conversely, $A \geqq (A'P^t)'P$ and $IP \geqq (A'P^t)'P$ imply that $A \wedge IP \geqq (A'P^t)'P$ for all A in $\mathscr{A}_{mn}$. Since $(A_{ij} \vee P'_{hj}) \wedge P_{hj} = A_{ij} \wedge P_{hj}$, we find that

$$\begin{aligned} [(A'P^t)'P]_{ij} &= \bigvee_h[A_{ij} \wedge P_{hj} \wedge \bigwedge_{k \neq j}(A_{ik} \vee P'_{hk})] \\ &\geqq \bigvee_h[A_{ij} \wedge P_{hj} \wedge \bigwedge_{k \neq j} P'_{hk}] = A_{ij} \wedge \bigvee_h[P_{hj} \wedge \bigwedge_{k \neq j} P'_{hk}] \\ &= A_{ij} \wedge \bigvee_h P_{hj} = (A \wedge IP)_{ij}. \end{aligned}$$

Hence $(A'P^t)'P = A \wedge IP$ for all A in $\mathscr{A}_{mn}$, and (i) $\Rightarrow$ (ii) $\Rightarrow$ $A \to AP$ is range closed. (iii) is of course another way of writing (ii).

COROLLARY. *If* $P \in \mathscr{C}(\mathscr{A}_{nn})$ $(P \in \mathscr{C}(\mathscr{A}_{mm}))$ *and if the elements of each row (column) of* P *form a mutually orthogonal subset of* L, *that is* $P_{ij} \leqq P'_{ik}$ $(P_{ji} \leqq P'_{ki})$ *for all* i,j,k *with* $j \neq k$, *then the mapping* $A \to AP$ $(A \to PA)$ *is range closed.*

LEMMA 10. *Given* $P \in \mathscr{C}(\mathscr{A}_{nn})$ $(P \in \mathscr{C}(\mathscr{A}_{mm}))$. $A \to AP$ $(A \to PA)$ *is range closed in* $\mathscr{A}_{mn}$ *if and only if* $A'P^t = B'P^t \Rightarrow A \wedge IP = B \wedge IP$ $(P^tA' = P^tB' \Rightarrow A \wedge PI = B \wedge PI)$.

Proof. The result follows from [6, Theorem 2].

COROLLARY 1. *If* $P \in \mathscr{C}(\mathscr{A}_{nn})$ $(P \in \mathscr{C}(\mathscr{A}_{mm}))$ *and if* $A \to AP^t$ $(A \to P^tA)$ *is range closed, then, for* $A \geqq (IP^t)'$ $(A \geqq (P^tI)')$, $A \leftrightarrow AP$ $(A \leftrightarrow PA)$ *is a one to one correspondence.*

COROLLARY 2. *Suppose that* $P \in \mathscr{C}(\mathscr{A}_{nn})$ $(P \in \mathscr{C}(\mathscr{A}_{mm}))$, P *is row (column) consistent and* $A \to AP^t$ $(A \to P^tA)$ *is range closed on* $\mathscr{A}_{mn}$; *then* $A \leftrightarrow AP$ $(A \leftrightarrow PA)$ *is a one to one correspondence on* $\mathscr{A}_{mn}$.

Let $\mathscr{A}$ be an orthomodular lattice and let $e \in \mathscr{A}$. Define a mapping ϕ_e by $a\phi_e = (a \vee e') \wedge e$ for a in $\mathscr{A}$. Such mappings are called *Sasaki projections* and are especially interesting members of $S(\mathscr{A})$. Foulis notes that when $\phi = \phi^2 = \phi^* \in S(\mathscr{A})$, ϕ is a Sasaki projection if and only if ϕ is range closed. Thus we have the following:

THEOREM 4. *Let $P \in \mathscr{C}(\mathscr{A}_{nn})$ $(P \in \mathscr{C}(\mathscr{A}_{mm}))$, and let $P = P^2 = P'$. The mapping $A \to AP$ $(A \to PA)$ is a Sasaki projection in $\mathscr{A}_{mn}$ if and only if P is a diagonal matrix, i.e. $P_{ij} = o$ for $i \neq j$.*

Proof. If P is a diagonal matrix, then, by the Corollary to Lemma 9, the mapping $A \to AP$ is range closed and hence is a Sasaki projection. Conversely, by Lemma 7, $P_{ij} \geqq P_{ik} \wedge P_{kj}$ and $P_{jj} \geqq P_{jk}$. Since $P_{hj} \wedge P_{hh} = o$, it follows from Lemma 9 that

$$P_{jk} \leqq P_{jj} = \bigvee_h P_{hj} = \bigvee_h [P_{hj} \wedge \bigwedge_{k \neq j} P'_{hk}] = P_{jj} \wedge \bigvee_{k \neq j} P'_{jk} \leqq P'_{jk} \quad \text{for} \quad j \neq k.$$

Thus $P_{jk} = P_{jk} \wedge P'_{jk} = o$ for $j \neq k$.

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VIRGINIA POLYTECHNIC INSTITUTE
BLACKSBURG, VIRGINIA

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