# MUTUAL CONTRADICTION OF TWO SELF-CONSISTENT ABSTRACTIONS

KATUZI ONO

A vast class of abstractions are proved self-contradictory by Russell-type paradoxes in the sense that the negation of any one of them can be proved tautologically.[1] On the other hand, there are a vast class of abstractions, each being self-consistent. A simple criterion for abstractions to be self-consistent (a sufficient condition) can be given. However, even a fairly restricted class of abstractions, each satisfying the criterion to be self-consistent, may contradict to each other.[2]

In this short note, I would like to notice that we can give a simple criterion for abstractions to be self-consistent (in (2)) but we can also give a simple example pair consisting of self-consistent abstractions satisfying the criterion and contradicting to each other (in (3)). Section (1) serves for preparation.

## (1) *TF*-invariance

Let us assume any set-theoretical system having only one primitive notion MEMBERSHIP $\in$ and standing on the lower classical predicate logic or possibly on a weaker logic such as the intuitionistic predicate logic.

If we evaluate every elementary sentence, naturally a sentence of the form $x\in y$, as TRUE, then we can evaluate every sentence of the system as TRUE or as FALSE definitely even when some free variables may occur in the sentence. I call this truth-value evaluation *T*-EVALUATION, shortly *TEV*. Likewise, we obtain another truth-value evaluation by evaluating every elementary sentence as FALSE. This is called *F*-EVALUATION, shortly *FEV*.

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[1] In a joint work with my younger colleague M. OHTA, I have given a sufficient condition for abstractions to be self-contradictory. Known Russell-type paradoxes satisfy the condition. The work will be published in the near future.

[2] QUINE and HINTIKKA gave an example of a pair of mutually contradictory propositions, the one being a deformation of abstraction and the other being the natural assumption that there are at least two distinct objects. See QUINE, W. V., 'On Frege's way out', Mind, N. S. **64** (1955), 145–159, and HINTIKKA, K. J. J., 'Vicious circle principle and the paradoxes', J. Symb. Log., **22** (1957), 245–249.

Any relation ($n$-ary; $n=0, 1, \dots$) is called *TF*-INVARIANT if and only if it takes a common definite truth-value for the both evaluations, *TEV* and *FEV*.

There are simple examples of *TF*-invariant relations. Namely, $x \subseteq y$ defined as usual by $(s)(s \in x \rightarrow s \in y)$ is surely *TF*-invariant, and any relation of the form $\mathfrak{A}(x, y, \dots, z) \wedge x \not\equiv y$ is also *TF*-invariant. Furthermore, any relation expressible exclusively in terms of *TF*-invariant relations is also *TF*-invariant.

It should be noticed that some relation, not *TF*-invariant itself, can be expressed by a *TF*-invariant relations under some assumptions. For example $\emptyset(x)$ defined by $\neg(\exists s)s \in x$ is not *TF*-invariant, but it can be expressed by the *TF*-invariant relation $(y)x \subseteq y$ as far as $(\exists x)\emptyset(x)$ is assumed.

### (2) A criterion for abstractions to be self-consistent

Let us call any abstraction

(A) $$(\exists p)(x)(x \in p \equiv \mathfrak{A}(x))$$

PROPER if and only if its kernel $\mathfrak{A}(x)$ is *TF*-invariant.

*Any proper abstraction is self-consistent.* For, any proper abstraction of the form (A) satisfies *TEV* if $\mathfrak{A}(x)$ is evaluated as TRUE by *TEV* and *FEV*, and it satisfies *FEV* if $\mathfrak{A}(x)$ is evaluated as FALSE by the both evaluations. Anyway, any proper abstraction satisfies one of these evaluations, *TEV* or *FEV*, so it must be self-consistent.

Now, for any set $\mathfrak{S}$ of relations, let us call any abstraction of the form (A) an $\mathfrak{S}$-ABSTRACTION if and only if $\mathfrak{A}(x)$ is expressed exclusively in terms of the relations in $\mathfrak{S}$.

If $\mathfrak{S}$ is a set of exclusively *TF*-invariant relations, any $\mathfrak{S}$-abstraction is surely proper, so it is self-consistent.

### (3) Mutual contradiction of a fairly restricted class of proper abstractions

If we restrict $\mathfrak{S}$ to a very small set of *TF*-invariant relations, the class of $\mathfrak{S}$-abstractions becomes very much limited.[3] However, even if we restrict $\mathfrak{S}$

[3] I have suggested to study theoretical systems starting exclusively from $\mathfrak{S}$-abstractions for limited sets $\mathfrak{S}$ of relations ('On a restricted abstraction principle', spoken at (*Continued*)

to the unit set of simple $TF$-invariant relation $\underline{\in}$ defined by

$$x \underline{\in} y \overset{\leftharpoondown}{=} (x \notin x \wedge x \in y),$$

we can still find out a pair of $\mathfrak{S}$-abstractions which contradict to each other.

Namely, the unary relation $x \notin x$ which is no $TF$-invariant relation itself can be expressed in terms of $\underline{\in}$ as $(\exists y) x \underline{\in} y$ by making use of the $\mathfrak{S}$-abstraction

$$(\exists p)(x)(x \in p \equiv (x \underline{\in} x \rightarrow x \underline{\in} x)).$$

Accordingly, the $\mathfrak{S}$-abstraction

$$(\exists p)(x)(x \in p \equiv (\exists y) x \underline{\in} y)$$

contradicts with the preceding $\mathfrak{S}$-abstraction, because the latter $\mathfrak{S}$-abstraction is equivalent to the paradoxical abstraction $(\exists p)(x)(x \in p \equiv x \notin x)$ if the former $\mathfrak{S}$-abstraction is assumed.

*Mathematical Institute,*
*Nagoya University*

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the 1965 Annual Meeting of The Mathematical Society of Japan held at Waseda University on May 21, 1965.). Especially, I have suggested to study $\mathfrak{S}_0$-abstractions for the pair set $\mathfrak{S}_0$ consisting of two relations $\subseteq$ and $\underset{0}{\in}$ defined by

$$x \subseteq y \overset{\leftharpoondown}{\equiv} (s)(s \in x \rightarrow s \in y) \qquad \text{and} \qquad x \underset{0}{\in} y \overset{\leftharpoondown}{\equiv} x \in y \notin x,$$

because I could show self-consistency of every $\mathfrak{S}_0$-abstraction and I could also develop a set theory starting exclusively from $\mathfrak{S}_0$-abstractions.

The result of the present paper has been a byproduct of my unsuccessful struggle to find out a suitable system $\mathfrak{S}$ of relations that makes every $\mathfrak{S}$-abstraction self-consistent and enables to develop a set theory safely starting exclusively from $\mathfrak{S}$-abstractions. The example system $\mathfrak{S}$ given in the present paper may be one of the simplest systems which make each $\mathfrak{S}$-abstraction self-consistent and also make a certain set of $\mathfrak{S}$-abstractions mutually contradictory. Recently, the system $\mathfrak{S}_0$ above mentioned has been also proved to belong to the same category by Y. INOUE, one of my younger colleagues.