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CARDINAL INTERPOLATION AND GENERALIZED EXPONENTIAL EULER SPLINES

A. SHARMA AND J. TZIMBALARIO

1. Introduction. Let $\mathscr{S}_n$ denote the class of cardinal splines $S(x)$ of degree n ($n \geqq 1$) having their knots at the integer points of the real axis. We assume that the knots are simple so that $S(x) \in C^{n-1}(-\infty, \infty)$. Recently Schoenberg [**3**] has studied cardinal splines $S(x) \in \mathscr{S}_n$ such that $S(x)$ interpolates the exponential function t^x at the integers and

$$(1.1) \quad S(x + 1) = tS(x)$$

for some fixed t and for all real x. Schoenberg has shown that if $t \neq 0$ or 1 and if t is not a zero of the Euler-Frobenius polynomial $\Pi_n(x)$, then the exponential Euler spline always exists and is unique. Besides giving a simple method for obtaining the explicit form of $S(x)$, he also shows that as $n \to \infty$, $S(x)$ converges to t^x if t is not negative. He shows by an example that if $t = -e$, then $S(x)$ does not converge to t^x. These results have been extended by Greville, Schoenberg and Sharma (G.S.S.) [**2**] who replace (1.1) by the functional equation

$$(1.2) \quad \sum_{j=0}^{k} a_j S(x + j) = 0, \quad a_0 \cdot a_k \neq 0$$

where $S(x)$ interpolates at the integers a given function $f(x)$ which is a solution of the functional equation

$$(1.3) \quad \sum_{j=0}^{k} a_j f(x + j) = 0.$$

The purpose of this note is to generalize the exponential Euler splines of Schoenberg in a different direction. We shall consider the functional equation

$$(1.4) \quad S(x + 1) - tS(x) = S^*(x)$$

where $S^*(x) \in S_n$ is a given cardinal spline. By a suitable choice of $S^*(x)$ in (1.4), we are led to rediscover some of the results of G.S.S. [**2**].

In § 2 we obtain a solution of (1.4) in terms of B-splines. This representation however is not enough for a study of the convergence problem. In § 3 we define generalized exponential Euler polynomials and give a generating function for the polynomial component of the spline $S(x)$ in (0, 1) when the corresponding polynomial restriction of $S^*(x)$ on [0, 1] is given by a generating

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function. § 4 generalizes the extremum property of the polynomials introduced in § 3. Lastly in § 5 we come to the study of the convergence problem.

2. B-splines. We shall need the forward B-spline which is denoted by

$$(2.1)\qquad Q(x) = \frac{1}{n!}\left\{x_+^n - \binom{n+1}{1}(x-1)_+^n + \ldots + (-1)^{n+1}(x-n-1)_+^n\right\}$$

where $x_+ = \max(0, x)$. It is known that $Q(x) = Q(n+1-x)$, $Q(x) > 0$ for $0 < x < n+1$ and that $Q(x) = 0$ elsewhere. Schoenberg has shown [**3**] that the Euler-Frobenius polynomial

$$(2.2)\qquad \Pi_n(t) = \sum_{j=0}^{n-1} Q(j+1)t^j$$

is a reciprocal polynomial and that it has only simple negative roots. We shall denote these roots by $\lambda_1, \ldots, \lambda_{n-1}$ with

$$(2.3)\qquad \lambda_{n-1} < \lambda_{n-2} < \ldots < \lambda_2 < \lambda_1 < 0.$$

We shall need a property of the B-splines which we state as

LEMMA 1. [**3**; **4**]. *Every $S(x) \in \mathscr{S}_n$ has a unique representation of the form*

$$(2.4)\qquad S(x) = \sum_{-\infty}^{\infty} c_j Q(x-j)$$

with constant coefficients c_j. Conversely for every sequence $\{c_j\}$, (2.4) *defines an element of $\mathscr{S}_n$.*

Problem. Given a spline $S^*(x) \in \mathscr{S}_n$, find a spline $S(x) \in \mathscr{S}_n$ such that

$$(2.5)\qquad S(x+1) - tS(x) = S^*(x), \quad t \neq 0, 1$$

$$(2.5a)\qquad S(0) = B \text{ (a given constant).}$$

If $S^*(x) = 0$ and $B = 1$, the problem has been completely solved by Schoenberg [**3**; **4**]. We shall show later that by a suitable choice of $S^*(x)$, one can bring the generalization of Schoenberg's result as discussed in [**2**] within the scope of the above problem.

We shall first solve this problem in a generic way and later approach it from a different angle. We prove

THEOREM 1. *Suppose $S^*(x)$ is given by*

$$(2.6)\qquad S^*(x) = \sum_{-\infty}^{\infty} \alpha_j Q(x-j).$$

Then the general solution $S(x) \in \mathscr{S}_n$ of the functional equation (2.5) *is given by*

$$(2.7)\qquad S(x) = a_0 \sum_{-\infty}^{\infty} t^j Q(x-j) + \Lambda(x), \quad t \neq 0$$

where a_0 *is an arbitrary constant and*

$$(2.8)\qquad \Lambda(x) = \sum_{j=1}^{\infty} t^j Q(x-j)\left(\sum_{\nu=0}^{j-1} \alpha_\nu t^{-\nu-1}\right) - \sum_{j=-\infty}^{-1} t^j Q(x-j)\left(\sum_{\nu=-1}^{j} \alpha_\nu t^{-\nu-1}\right).$$

Moreover, if $t \neq 0, 1, \lambda_1, \ldots, \lambda_{n-1}$ *then there is a unique constant* a_0 *such that* $S(x)$ *satisfies* (2.5a).

Proof. If $S(x) = \sum_{-\infty}^{\infty} c_j Q(x-j)$, then from (2.5), we have

$$\sum_{-\infty}^{\infty} c_{j+1}Q(x-j) - t\sum_{-\infty}^{\infty} c_j Q(x-j) = \sum_{-\infty}^{\infty} \alpha_j Q(x-j)$$

so that from Lemma 1, we have

$$(2.9)\qquad c_{j+1} - tc_j = \alpha_j, \quad j = 0, 1, 2, \ldots.$$

In order to solve (2.9), we set $c_j = a_j t^j$ so that from (2.9) we have

$$a_{j+1} - a_j = \alpha_j t^{-j-1}$$

whence

$$a_j = a_0 + \sum_{\nu=0}^{j-1} \alpha_\nu t^{-\nu-1}; \quad j \geqslant 1$$

$$a_{-j} = a_0 - \sum_{\nu=1}^{j} \alpha_{-\nu} t^{\nu-1}, \quad j \geqslant 1.$$

This leads immediately to the formulae (2.7) and (2.8).

Since $\sum_{-\infty}^{\infty} t^j Q(j) = c\Pi_n(t)$, where $\Pi_n(t)$ is the Euler-Frobenius polynomial of (2.2), it is possible to determine a_0 uniquely in (2.7) so that (2.5a) is satisfied if and only if t is not a zero of $\Pi_n(x)$.

3. Generalized exponential Euler polynomials. In order to study the spline $S(x)$ satisfying (2.5), the solution given by (2.7) and (2.8) is not enough. For later use we shall study the restriction of $S(x)$ to the interval $[0, 1]$ and thereby obtain an explicit expression which will be used for the study of the convergence problem as n grows larger.

Denote by $P_n(x)$ the restriction of $S^*(x)$ to $[0, 1]$, where

$$(3.1)\qquad P_n(x) = p_0 x^n + \binom{n}{1} p_1 x^{n-1} + \ldots + p_n.$$

LEMMA 2. *If* $t \neq 1$, *there is a unique monic polynomial* $B_n(x; t)$ *of degree* n *such that*

$$(3.2)\qquad B_n^{(\nu)}(1; t) - tB_n^{(\nu)}(0; t) = P_n^{(\nu)}(0), \quad \nu = 0, 1, \ldots, n-1.$$

Proof. Set

$$B_n(x, t) = x^n + \binom{n}{1} b_1 x^{n-1} + \ldots + b_n.$$

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Then the condition (3.2) leads to the following system of linear equations

$$(3.3) \qquad 1 + \binom{\nu}{1} b_1 + \ldots + b_\nu = t b_\nu + p_\nu, \quad \nu = 1, \ldots, n.$$

Here we have n equations in the n unknowns $b_1, \ldots, b_n$ with a non-singular determinant since $t \neq 1$. Hence they have a unique solution which completes the proof of the lemma.

If $P_n(x) = 0$ in the above lemma, we get the polynomials $A_n(x; t) = x^n + \binom{n}{1} a_1(t) x^{n-1} + \ldots + a_n(t)$, where

$$(3.4) \qquad \frac{t-1}{t-e^z} e^{xz} = \sum_{n=0}^{\infty} A_n(x; t) \frac{z^n}{n!}.$$

These polynomials have been studied by Schoenberg [**4**] and are called *exponential Euler polynomials*. We can now give another solution to the problem (2.5), (2.5a) different from Theorem 1. If $P_n(x)$ denotes the restriction to $[0, 1]$ of the given spline $S^*(x)$ in (2.5), we have

THEOREM 2. *If* $t \neq 1, \lambda_1, \ldots, \lambda_{n-1}$ *where* $\lambda_1, \ldots, \lambda_{n-1}$ *are the zeros of* $\Pi_n(x)$, *then the spline* $S_n(x; t)$ *which satisfies* (2.5) *and* (2.5a) *may be defined on* $[0, 1]$ *by*

$$(3.5) \qquad S_n(x; t) = B_n(x; t) + \frac{A_n(x; t)}{A_n(0; t)} \{B - B_n(0, t)\}$$

where $B_n(x; t)$ *is a monic polynomial determined by* (3.2).

The extension of $S_n(x; t)$ to the whole real line can be accomplished by means of the functional equation

$$(3.6) \qquad S_n(x+1; t) - t S_n(x; t) = S^*(x).$$

Proof. The solution of (2.5) and (2.5a) is obviously the sum of a particular solution of this system and a solution of the homogeneous equation $S(x + 1) - tS(x) = 0$. By Lemma 2, $B_n(x; t)$ is clearly the restriction of a particular solution of (2.5). The right side in (3.5) obviously satisfies (2.5a). Since $A_n(0; t) = (t-1)^{-n} \Pi_n(t)$ (see [**3**, p. 392]) occurs in the denominator, t cannot be equal to a zero of $\Pi_n(t)$, i.e. $t \neq \lambda_1, \ldots, \lambda_{n-1}$. This completes the proof of Theorem 2.

We now make a special choice of the polynomials $P_n(x)$. Let $\{P_n(x, t)\}$ be a sequence of Appell polynomials given by the generating function

$$(3.7) \qquad G(z, t) e^{zx} = \sum_{n=0}^{\infty} P_n(x, t) \frac{z^n}{n!}.$$

If we form the monic polynomials $B_n(x; t)$ of Lemma 2 with respect to $P_n(x, t)$ we can give their generating function. More precisely we have

LEMMA 3. *If* $\{P_n(x; t)\}_0^\infty$ *is determined by* (3.7), *then the sequence of monic*

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polynomials $B_n(x; t)$ *of Lemma* 2 *is given by the generating function*

$$(3.8) \quad \frac{1 - t - p_0 + G(z, t)}{e^z - t} e^{xz} = \sum_{n=0}^{\infty} B_n(x; t) \frac{z^n}{n!}.$$

Proof. It follows from (3.2) that

$$B_n(x + 1; t) - tB_n(x, t) = P_n(x; t) + (1 - t - p_0)x^n.$$

Multiplying both sides by $z^n/n!$ and summing on n, we have

$$(3.9) \quad F(x + 1, t, z) - tF(x, t, z) = G(z, t)e^{xz} + (1 - t - p_0)e^{xz}$$

where

$$F(x, t, z) = \sum_{n=0}^{\infty} B_n(x, t) \frac{z^n}{n!}.$$

In the equations (3.3), if $p_0, p_1, \ldots, p_n$ form a section of an infinite sequence, then $b_1, b_2, \ldots, b_n$ also form a section of an infinite sequence so that the polynomials $\{B_n(x, t)\}$ also form an Appell sequence. Hence $F(x, t, z) = e^{xz}f(z, t)$. Then from (3.6), we have

$$(3.10) \quad f(z, t) = \frac{1 - t - p_0 + G(z, t)}{e^z - t}.$$

which proves (3.8).

Example 1. Suppose $S^*(x)$ denotes the exponential Euler spline of order r whose restriction to [0, 1] is a monic polynomial $A_{n,r}(x: t)$. Then we know [**2**] that

$$(3.11) \quad \frac{(1 - t)^r}{(e^z - t)^r} e^{xz} = \sum_{n=0}^{\infty} A_{n,r}(x; t) \frac{z^n}{n!}.$$

Then in Lemma 3, we have $P_n(x) = A_{n,r}(x: t)$, $p_0 = (1 - t)$, $G(z, t) = (1 - t)^r/(e^z - t)^r$, so that from (3.8) we have

$$\frac{(1 - t)^{r+1}}{(e^z - t)^{r+1}} e^{xz} = \sum_{n=0}^{\infty} B_n(x, t) \frac{z^n}{n!}$$

whence $B_n(x; t) = A_{n,r+1}(x; t)$ is a monic polynomial which is the restriction of a spline $S_n(x: t)$ satisfying

$$S_n(x + 1; t) - tS_n(x; t) = (1 - t)S^*(x).$$

Example 2. Suppose $S^*(x)$ is a spline of degree n and order r which interpolates the data $\left\{\binom{\nu}{r} t^\nu\right\}$ at the integers, and if $S_n(x; t)$ satisfies

$$S_n(x + 1; t) - tS_n(x; t) = S^*(x)$$

with $S_n(0; t) = 0$, then $S_n(x; t)$ coincides with $S_{n,r+1}^*(x; t)$ in the notation of

G.S.S. [**2**] and interpolates the data $\left\{\binom{\nu}{r+1}t^{\nu-1}\right\}$ at the integers. That its restriction to [0, 1] is the polynomial

$$A_{n,r+1}(\mathrm{x};t) - \frac{A_{n,r+1}(0;t)}{A_n(0;t)}A_n(x;t)$$

follows from Example 1.

4. An extremal property of $B_n(x;t)$, $(t > 1)$**.** By Lemma 2, $B_n(x;t)$ is the unique monic polynomial satisfying (3.2) where $P_n(x)$ is a *given* polynomial. We consider the class of functions $f(x)$ such that (i) $f(x) \in C^{n-1}[0, 1]$, (ii) $f^{(n-1)}$ satisfies a Lipschitz condition and

$$(4.1)\qquad \begin{cases} f(1) - f(0) \geqq B_n(1;t) - B_n(0;t) \\ f^{(\nu)}(1) - tf^{(\nu)}(0) \geqq B_n^{(\nu)}(1;t) - tB_n^{(\nu)}(0;t),\ \nu = 0, 1, \dots, n-1. \end{cases}$$

The last condition in (4.1) can also be rewritten as

$$f^{(\nu)}(1) - tf^{(\nu)}(0) \geqq P_n^{(\nu)}(0), \quad \nu = 0, 1, \dots, n-1.$$

We shall denote this class of functions by $\mathscr{F}(P_n)$. We then formulate

THEOREM 3. *If $t > 1$, the polynomial $B_n(x;t)$ is the unique element $\in \mathscr{F}(P_n)$ which minimizes the norm*

$$||f^{(n)}|| = \operatorname*{ess\,sup}_{0 \leqq x \leqq 1} |f^{(n)}(x)|,$$

giving it its least norm

$$||B_n^{(n)}|| = n!$$

Remark. If $P_n(x) = 0$, the monic polynomial $B_n(x;t)$ reduces to $A_n(x;t)$ which has been studied by Schoenberg [**3**]. In this case the conditions (4.1) become

$$(4.1a)\qquad \begin{cases} f(1) - f(0) \geqq (t-1)A_n(0;t) \\ f^{(\nu)}(1) - tf^{(\nu)}(0) \geqq 0, \quad \nu = 0, 1, \dots, n-1. \end{cases}$$

These conditions define a class of functions which is slightly larger than the class F_n of Schoenberg except for a change of scale.

Example. If $P_n(x) = A_{n,r-1}(x;t)$ $(r \geqq 1)$, the exponential Euler polynomial of higher order, then the monic polynomials $B_n(x;t)$ reduce to $A_{n,r}(x;t)$ which are given by the generating function (3.11). We also have

$$B_n^{(\nu)}(1;t) - tB_n^{(\nu)}(0;t) = (1-t)A_{n,r-1}^{(\nu)}(0;t), \quad \nu = 0, 1, \dots, n-1.$$

The class of functions $f(x)$ in the above theorem is then denoted by $\mathscr{F}(A_{n,r-1})$

and satisfy

$$\text{(4.1}b\text{)}\quad \begin{cases} f(1) - f(0) \geqq (t-1)\{A_{n,r}(0;t) - A_{n,r-1}(0;t)\} \\ f^{(\nu)}(1) - tf^{(\nu)}(0) \geqq \dfrac{n!(1-t)}{(n-\nu)!} A_{n-\nu,r-1}(0;t), \quad \nu = 0, 1, \ldots, n-1. \end{cases}$$

Thus the theorem asserts that the exponential Euler polynomials of order r minimize $||f^{(n)}||$ over all functions $f \in \mathscr{F}(A_{n,r-1})$.

Proof. The proof follows the same lines as that of Schoenberg [**3**] with minor modifications.

5. Convergence problem. Suppose $\{S_n^*(x)\}$ is a sequence of cardinal splines where $S_n^*(x) \in \mathscr{S}_n$ $(n = 1, 2, 3, \ldots)$. Then the sequence of functional equations

$$\text{(5.1)}\quad S_n(x+1;t) - tS_n(x;t) = S_n^*(x), \quad S_n(0;t) = B$$

with $S_n(x;\ t) \in \mathscr{S}_n$, gives rise to the sequence of cardinal splines $S_n(x;\ t)$. Assuming that $S_n^*(x)$ interpolates a given function $f(x)$ at the integers and also converges to the function $f(x)$ as $n \to \infty$, we seek to investigate the convergence of $S_n(x;\ t)$ as $n \to \infty$. In particular, if $S_n^*(x) \equiv 0$, and $B = 1$, we come back to the case treated by Schoenberg who showed that the cardinal spline $S_{n,0}(x;t)$ satisfying

$$\text{(5.2)}\quad S_{n,0}(x+1;t) - tS_{n,0}(x;t) = 0, \quad S_{n,0}(0;t) = 0$$

interpolates the function t^x and converges to t^x as $n \to \infty$ when t is non-negative and $\neq 1$.

In order to study the general case, we shall need some lemmas.

LEMMA 4. [**2**]. *Suppose $S_{n,\nu}^*(x;\ t)$, $t > 0(\neq 1)$ is the exponential Euler spline of order ν which interpolates the function $\binom{x}{\nu} t^x$ at the integers. Then for $\nu =$ 0, 1, 2, ..., the following inequality is valid:*

$$\text{(5.3)}\quad \left| S_{n,\nu}^*(x;t) - \binom{x}{\nu} t^x \right| \leqq M_\nu(x) t^x (n+1)^\nu \gamma^n$$

where

$$\gamma = \max\left(\left|\frac{t_0}{t_1}\right|, \left|\frac{t_0}{t_{-1}}\right| \right), \quad t_k = \log t + 2k\pi i,$$

and $M_\nu(x)$ is given by the recursion formula:

$$\text{(5.4)}\quad M_\nu(x) = \frac{|x|}{\nu} M_{\nu-1}(x-1) + \frac{2}{\nu} \sum_{l=0}^{\nu-1} \left(\frac{t^l}{|t-1|^{l+1}} + 2 \right) M_{\nu-1-l}(x-1)$$

and $M_0 = M$ is a constant independent of n.

For a proof of this lemma we refer to [**2**].

LEMMA 5. *For* $t > 0, t \neq 1$, *the functions* $M_\nu(x)$ *in* (5.4) *of the preceding lemma satisfy the following inequality:*

$$(5.5) \qquad M_{\nu+1}(x) \leqq M(C_1(t))^{\nu+1} \prod_{k=0}^{\nu} \left(\frac{|x-\nu+k|}{k+1} + \frac{2(1+2t)}{t}\right), \quad \nu \geqq -1$$

where

$$(5.6) \qquad C_1(t) = \max\left\{1, \frac{t}{|t-1|}\right\}$$

Proof. We shall prove this formula by induction on ν. For $\nu = 0$, we know that $M_0(x) = M$ is a constant [**2**; **3**]. Suppose the formula is true for $\nu = 0, 1, \ldots, r$. Consider the case when

$$(5.6) \qquad \left|\frac{t}{t-1}\right| \leqq 1.$$

Then

$$\frac{t^l}{|t-1|^{l+1}} \leqq \frac{1}{t},$$

so that from (5.4) and the induction hypothesis we have

$$M_{r+1}(x) \leqq M \prod_{k=0}^{r-1} \left(\frac{|x-r+k|}{k+1} + \frac{2(1+2t)}{t}\right)$$
$$\times \left\{\frac{|x|}{r+1} + \frac{2}{r+1} \sum_{k=0}^{r} \left(\frac{1}{t} + 2\right)\right\} = M \prod_{k=0}^{r} \left(\frac{|x-r+k|}{k+1} + \frac{2(1+2t)}{t}\right).$$

If

$$\left|\frac{t}{t-1}\right| > 1,$$

then

$$C_1(t) = \left|\frac{t}{t-1}\right|$$

and we have from (5.4) again,

$$M_{r+1}(x) \leqq \frac{|x|}{r+1} M_r(x-1) + \frac{2}{r+1} \sum_{l=0}^{r} \frac{t^{l+1}}{(t-1)^{l+1}} \left(\frac{2t+1}{t}\right)$$
$$\times M_{r-l}(x-1) \leqq M \prod_{k=0}^{r-1} \left(\frac{|x-r+k|}{k+1} + \frac{2(1+2t)}{t}\right)$$
$$\times \left\{\frac{|x|}{r+1} \left|\frac{t}{t-1}\right|^{r+1} + \frac{2(2t+1)}{(r+1)t} \sum_{l=0}^{r} \left|\frac{t}{t-1}\right|^{l+1} \left|\frac{t}{t-1}\right|^{r-l}\right\}$$
$$\leqq M(C_1(t))^{r+1} \prod_{k=0}^{r} \left(\frac{|x-r+k|}{k+1} + \frac{2(1+2t)}{t}\right).$$

This completes the proof of the lemma.

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LEMMA 6. *Let $\{a_\nu\}_0^\infty$ be a sequence of constants such that*

$$(5.7)\qquad |a_\nu| \leqq M \frac{(c_2(t))^\nu}{(\nu!)^2}, \quad \nu = 0, 1, 2, \ldots$$

where

$$c_2(t) < \frac{t \log \frac{1}{\gamma}}{4C_1(t)(1 + 2t)}.$$

If $t > 0, \neq 1$, then $f(x) \equiv \sum_{\nu=0}^\infty a_\nu \binom{x}{\nu} t^x$ and $S_n^*(x; t) \equiv \sum_0^\infty a_\nu S_{n,\nu}^*(x; t)$ converge for all x. Moreover $S_n^*(x; t)$ is a spline of degree n which interpolates $f(x)$ at the integers and

$$\lim_{n\to\infty} S_n^*(x; t) = f(x)$$

for all x.

Proof. From a well-known theorem [**1**, p. 137 Theorem 5] about the abscissa of convergence of the Newton series, the series for $f(x)$ converges for all finite x.

In order to prove our assertion for $S_n^*(x; t)$, it is enough to show that

$$(5.8)\qquad \Delta_n(x) \equiv \sum_{\nu=0}^\infty |a_\nu| \left| S_{n,\nu}^*(x; t) - \binom{x}{\nu} t^x \right| \to 0 \text{ as } n \to \infty$$

for any finite x. For integral x, we know that $\Delta_n(x)$ is zero because $S_{n,\nu}^*(x; t)$ is the exponential Euler spline of higher order and interpolates $\binom{x}{\nu} t^x$ at the integers.

Using Lemma 5, elementary but tedious calculations show that $\Delta_n(x) \to 0$ as $n \to \infty$.

THEOREM 4. *Suppose $\{a_\nu\}$ is a sequence of numbers as required in Lemma 6. Let $f(x)$ be the function given by*

$$(5.9)\qquad f(x) = \sum_{\nu=0}^\infty a_\nu \binom{x}{\nu} t^x, \quad (t > 0, t \neq 1).$$

Let $S_n^*(x\colon t)$ be a sequence of cardinal splines of degree n which interpolate the given function $f(x)$ at the integers with the representation

$$(5.10)\qquad S_n^*(x; t) = \sum_{\nu=0}^\infty a_\nu S_{n,\nu}^*(x; t).$$

If $S_n(x; t) \in \mathscr{S}_n$ is a sequence of cardinal splines which satisfy the functional equation

$$(5.11)\qquad S_n(x + 1; t) - tS_n(x; t) = S_n^*(x; t), \quad S_n(0; t) = 0,$$

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then $S_n(x; t)$ *interpolates the function* $F(x)$ *at the integers where*

$$(5.12) \quad F(x) = t^{x-1} \sum_{k=0}^{\infty} a_k \binom{x}{k+1}.$$

Moreover as $n \to \infty$, *we have*

$$(5.13) \quad \lim_{n \to \infty} S_n(x; t) = F(x)$$

for every finite x.

Consider the difference equation

$$S_n(x + 1) - tS_n(x) = S_{n,k}^*(x; t), \quad (t > 0, t \neq 1).$$

Then the unique spline $S_n(x) \in \mathscr{S}_n$ satisfying $S_n(0) = 0$ is given by

$$S_n(x) = t^{-1}S_{n,k+1}^*(x; t)$$

because of Lemma 3. Then because of (5.10), the solution of (5.11) is given by

$$S_n(x; t) = \sum_{\nu=0}^{\infty} a_\nu t^{-1} S_{n,\nu+1}^*(x; t)$$

which is convergent by Lemmas 4, 5 and 6 and converges to the function $F(x)$ given by (5.12).

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University of Alberta,
Edmonton, Alberta