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SOME RESULTS ON GENERALIZED LOTOTSKY SUMMABILITY

BY

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ABSTRACT. The (F, d_n) method is investigated with respect to perfectness, strong regularity, and summing bounded divergent sequences. In the process the columns of the inverse matrix are characterized in terms of $\{d_n\}$.

1. **Introduction**. We wish to investigate perfectness, strong regularity, and the summation of bounded divergent sequences for the Jakimovski methods (F, d_n). These methods were introduced by Jakimovski in [2] as a generalization of Lototsky summability.

DEFINITION 1.1. *The* (F, d_n) *method is defined by the triangular matrix* $A = (a_{nk})$ *which has* $a_{oo} = 1$, $a_{ok} = 0$ *when* $k > 0$ *and*

$$\prod_{j=1}^{n} \frac{z + d_j}{1 + d_j} = \sum_{k=0}^{n} a_{nk} z^k, \; n \geq 1. \tag{1.2}$$

Here $\{d_n\}_1^\infty$ *is an arbitrary complex sequence with* $d_n \neq -1$.

For convenience we will denote $\Pi_{j=1}^n (1 + d_j)$ *by* $(1 + d_n)!$

If $A^{-1} = (b_{nk})$, *then explicit formulas for* a_{nk} *and* b_{nk} *are given by (see* [2]*)*

$$a_{nk} = \frac{1}{(1 + d_n)!} \sum_{1 \leq j_1 < \cdots < j_{n-k} \leq n} d_{j_1} \ldots d_{j_{n-k}} \tag{1.3}$$

where $k \leq n$ *and the sum is defined to be* 1 *when* $k = n$, *and*

$$b_{nk} = (1 + d_k)!(-1)^{n-k}[d_1^n, \ldots, d_{k+1}^n] \tag{1.4}$$

where $k \leq n$ *and we take* $(1 + d_o)! \equiv 1$.

We note that the formula (1.4) *and the proof given in* [2, §8] *when the* d_j*'s are pairwise different remain valid even when some of the* d_j*'s are equal, if the definition of a divided difference for multiple knots is used (see* [3]*).*

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In the next section we will use the fact that the divided difference of a function or a sequence is a linear functional and that the divided difference $[f(d_1), \ldots, f(d_{m+1})]$ is a linear combination, with non-zero coefficients depending on $d_1, \ldots, d_{m+1}$ only, of $f(d_j), f'(d_j), \ldots, f^{(r_j - 1)}(d_j)$ where r_j is the number of times the number d_j appears in the sequence $d_1, \ldots, d_{m+1}$ and in the linear combination d_j ranges over all the different values in the sequence $d_1, \ldots, d_{m+1}$.

2. Perfectness and (F, d_n)

DEFINITION 2.1. *Let A be a conservative triangle. Then A is* perfect *if the convergent sequences c are dense in c_A, the summability field of A, with respect to the usual norm topology on c_A.*

Before relating perfectness to (F, d_n), we give two results characterizing the columns of the inverse of the (F, d_n) matrix.

THEOREM 2.2. *For the (F, d_n) method with associated matrix $A = (a_{nk})$ the following are equivalent.*

(i) $|d_n| < 1$ for each n.
(ii) A^{-1} has null columns.
(iii) A^{-1} has convergent columns.

PROOF. Let $A^{-1} = (b_{nk})$. From the remarks at the end of the previous section, for $n \geq 1$ and a fixed $k \geq 0$

$$b_{nk} = (-1)^{n-k}(1 + d_k)![d_1^n, \ldots, d_{k+1}^n]$$

is a finite linear combination, with non-zero coefficients depending on $d_1, \ldots, d_{k+1}$ only, of terms of the form $n(n - 1) \ldots (n - r + 1)d_j^{n-r}$ where d_j ranges over all the different values in the sequence $d_1, \ldots, d_{k+1}$ and $r < r_j$. The proof follows by considering first $b_{n,0}$ then $b_{n,1}$ and so on.

THEOREM 2.3. *For the (F, d_n) method with associated matrix $A = (a_{nk})$ we have that $|d_n| \leq 1$ for each n and the d_n's of modulus one are distinct if and only if A^{-1} has bounded columns.*

PROOF. Let $A^{-1} = (b_{nk})$. If d_j appears only once among $d_1, \ldots, d_{k+1}$ then its only contribution to the sum defining b_{nk} is of the form of a constant times d_j^n (which is a geometric sequence). Otherwise d_j contributes additional terms which are of the form of a constant times each one of the following: $nd_j^{n-1}, n(n - 1)d_j^{n-2}, \ldots,$. The proof follows again by considering $b_{n,0}$ then $b_{n,1}$ and so on.

From this last result and Lemma 1 of [8] we obtain a sufficient condition for (F, d_n) to be perfect.

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THEOREM 2.4. *Let A be the matrix associated with a regular* (F, d_n) *method. If* $|d_n| \leq 1$ *for each n and the* d_n*'s of modulus one are distinct, then A is perfect.*

We note that the converse is not true, e.g.,

$$d_1 = d_2 = 1 \text{ and } d_n = 0 \text{ for } n \geq 3 \text{ or } d_n = p > 1,\ n = 1, 2, \ldots$$

are each regular and perfect.

3. Strong Regularity for (F, d_n)

DEFINITION 3.1. *A bounded sequence* $x = \{x_k\}$ *is said to be almost convergent to s, its generalized limit, if each Banach limit (see [5] p. 58) of x is s. We denote the class of almost convergent sequences by f.*

DEFINITION 3.2. *A matrix* $A = (a_{nk})$ *is strongly regular if it sums every* $x \in f$ *to the value to which it is almost convergent.*

In [5] p. 62 we have the following characterization of strong regularity:

THEOREM 3.3. *A regular matrix* $A = (a_{nk})$ *is strongly regular if and only if*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = 0.$$

As noted in [5], p. 65, the matrices $A = (a_{nk})$ *satisfying* $\max_{0 \leq k \leq n} |a_{nk}| \to 0$ $(n \to \infty)$ *form a wider class than the strongly regular matrices. However, when restricted to the* (F, d_n) *matrices with* $d_n \geq 0$ *for each n, we have*

THEOREM 3.4. *Let* (F, d_n) *with corresponding matrix* $A = (a_{nk})$ *be regular with* $d_n \geq 0$ *for each n. Then the following are equivalent.*

(i) (F, d_n) is strongly regular.

(ii) $\max_{0 \leq k \leq n} |a_{nk}| \to 0$ $(n \to \infty)$.

(iii) $\sum_{n=1}^{\infty} \frac{d_n}{(1 + d_n)^2} = \infty$.

PROOF. (iii) $\Rightarrow$ (i). This is due to Groetsch in [1]. (i) $\Rightarrow$ (ii). This follows from Theorem 3.3 and the remark following it. (ii) $\Rightarrow$ (iii).

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Assume

$$\text{(3.5)} \qquad \sum_{n=1}^{\infty} \frac{d_n}{(1+d_n)^2} < \infty.$$

From Lemma 2.2 of [6] we have

(3.6) An (F, d_n) method with $d_n \geq 0$ for each n is regular if and only if

$$\sum_{n=1}^{\infty} \frac{1}{1+d_n} = \infty.$$

From (ii) and (1.3) we have $a_{nn} = 1/(1+d_n)! \to 0$ $(n \to \infty)$, i.e., $(1+d_n)! \to \infty$ $(n \to \infty)$ and thus

$$\text{(3.7)} \qquad \sum_{n=1}^{\infty} d_n = \infty.$$

We have $\overline{\lim}_{n\to\infty} d_n = \infty$. Otherwise, there would exist an $L > 0$ such that $d_n \leq L$ for each n. Thus, for each n, $d_n/(1+d_n)^2 \geq d_n/(1+L)^2$, which along with our assumption (3.5) implies $\Sigma_{n=1}^{\infty} d_n < \infty$. This contradicts (3.7). Similarly, $\underline{\lim}\, d_n = 0$. If not, then there would exist an $\epsilon > 0$ such that $d_n \geq \epsilon$ for n sufficiently large. The function $x/1+x$ is increasing for $x > 0$ hence

$$\frac{d_n}{(1+d_n)^2} \geq \frac{1}{1+d_n} \cdot \frac{\epsilon}{1+\epsilon}$$

which forces

$$\sum_{n=1}^{\infty} \frac{1}{1+d_n} < \infty.$$

This contradicts (3.6).

Consider the subsequences of n given by

$$\{\nu_i \mid d_{\nu_i} \leq 1\} \text{ and } \{l_i \mid d_{l_i} > 1\}.$$

Then

$$\sum_{n=1}^{\infty} \frac{d_n}{(1+d_n)^2} = \sum_{i=1}^{\infty} \frac{d_{\nu_i}}{(1+d_{\nu_i})^2} + \sum_{i=1}^{\infty} \frac{d_{l_i}}{(1+d_{l_i})^2}$$

Since each series converges by our assumption in (3.5), we have

$$\text{(3.8)} \qquad \sum_{i=1}^{\infty} d_{\nu_i} < \infty \text{ and } \sum_{i=1}^{\infty} \frac{1}{1+d_{l_i}} < \infty$$

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by the same arguments used to determine $\overline{\lim}_{n\to\infty} d_n$ and $\underline{\lim}_{n\to\infty} d_n$. Now let $m = m(n)$ be the number of d_j's, $1 \le j \le n$, such that $d_j > 1$. We may suppose, by (3.7), that n is sufficiently large such that $m \ge 1$. Consider the nth row of $A = (a_{nk})$. By (1.3) with $k = n - m$,

$$a_{n,n-m} = \frac{1}{(1 + d_n)!} \sum_{1 \le j_1 < j_2 < \cdots < j_m \le n} d_{j_1} d_{j_2} \ldots d_{j_m}$$

$$\ge \frac{\prod_{i=1}^{m} \left(1 - \frac{1}{1 + d_{l_i}}\right)}{\prod_{i=1}^{n-m} (1 + d_{\nu_i})}. \tag{3.9}$$

But by (3.8) $\Pi_{i=1}^{\infty}(1 + d_{\nu_i})$ converges and hence is bounded, say $\Pi_{i=1}^{N}(1 + d_{\nu_i}) \le M$ for all N. Also $\Pi_{i=1}^{\infty}(1 - 1/1 + d_{l_i})$ converges by (3.8). These imply from (3.9) that

$$a_{n,n-m} \ge \frac{\prod_{i=1}^{\infty} \left(1 - \frac{1}{1 + d_{l_i}}\right)}{M} = \text{const} > 0.$$

Thus $\max_{0 \le k \le n} |a_{nk}| \not\to 0_{(n\to\infty)}$.
This contradicts hypothesis (ii). Hence

$$\sum_{n=1}^{\infty} \frac{d_n}{(1 + d_n)^2} = \infty.$$

4. **Summation of Bounded Divergent Sequences for (F, d_n).** Applying the corollary on p. 505 of [7] and Theorem 2.3 we have

THEOREM 4.1. *Let (F, d_n) be regular with $|d_n| \le 1$ for each n and such that the d_n's of modulus one are distinct. Then (F, d_n) is either Mercerian or sums a bounded divergent sequence.*

From this result we have the following special cases.

COROLLARY 4.2. *Under the same hypotheses as in Theorem 4.1, if at least one d_n is such that $|d_n| = 1$, then (F, d_n) sums a bounded divergent sequence.*

PROOF. This follows from Theorem 2.1 (iv) of [4].

COROLLARY 4.3. *If (F, d_n) is regular, non-Mercerian, and sums no bounded divergent sequences, then*
(i) $|d_n| \ne 1$ for each n, and
(ii) $|d_n| > 1$ for some n.

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PROOF. (i) If some d_n has modulus one, say d_N, then the (F, d_n') method given by $d_n' = d_n$ for $n \neq 1$ and N, $d_1' = d_N$, and $d_N' = d_1$, sums the columns of its own inverse. But the first column of the inverse for (F, d_n') is by (1.4) $b_{no}' = (-d_N)^n$, a bounded divergent sequence. It is easily seen from (1.2) that interchanging two d_j's (or any finite number for that matter) results in essentially the same (F, d_n) matrix, i.e., for sufficiently large n, $a_{nk} = a_{nk}'$ for each k. Therefore (F, d_n) sums a bounded divergent sequence.

(ii) This now follows from Theorem 4.1 and (i).

If we restrict $\{d_n\}$ to $d_n \geq 0$ for each n, then, since f clearly contains bounded divergent sequences, from Theorem 3.4 we have

COROLLARY 4.4. *Let (F, d_n) be regular with $d_n \geq 0$ for each n. If (F, d_n) sums no bounded divergent sequences, then*

$$\sum_{n=1}^{\infty} \frac{d_n}{(1 + d_n)^2} < \infty.$$

From this last result and an argument in the proof of Theorem 3.4, it follows that $\underline{\lim}\, d_n = 0$. In particular then, $d_n \not\to \infty$ $(n \to \infty)$. Thus we have

COROLLARY 4.5. *If (F, d_n) is regular with $d_n \geq 0$ for each n and $d_n \to \infty$ $(n \to \infty)$, then (F, d_n) sums a bounded divergent sequence.*

The conclusion of Corollary 4.4 yields

COROLLARY 4.6. *If (F, d_n) is regular with $d_n \geq 0$ for each n and sums no bounded divergent sequences, then d_n is either a null sequence (in fact is in l) or has exactly* 0 *and* ∞ *as limit points.*

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