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ANOTHER NOTE ON SPERNER'S LEMMA

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Let Q be a finite partially ordered (by $\leq$) set with universal bounds O, I. The *height function* h of Q is defined by the rule: $h(x)$ is the maximum length of a chain from O to x. Let $h(I)=n$. Suppose that for each $k\geq 0$, there exist positive integers $a(k)$ and $b(k)$ such that all elements of height k

(i) are covered by $a(k)$ elements of height $k+1$;

(ii) cover $b(k)$ elements of height $k-1$.

Then we call Q a *U-poset*. Call a subset S of a partially ordered set an *antichain* if no two elements of S are comparable.

THEOREM 1. (Sperner [5], Lubell [3], Baker [1]). *Let Q be a U-poset. Let $E(k)$ be the subset of all elements of height k in Q; $e(k)=|E(k)|$; $M=\max_k e(k)$. If S is an antichain of Q, $|S|\leq M$.*

The purpose of this note is to characterize the antichains of cardinality M. Let $T(k)\subset E(k)$, $k>0$. We denote by $T^*(k)$, the set of all x in $E(k)$ satisfying: there exist $y\in T(k)$ and $z\in E(k-1)$ such that x and y cover z. If $T^*(k)=T(k)$, we say that $T(k)$ is *full*.

THEOREM 2. *Let S be a nonempty antichain of a U-poset Q. Let $U(k)=S\cap E(k)$. Let $V(k)=\{x\in E(k):x<y \text{ for some } y\in S\}$. Let $T(k)=U(k)\cup V(k)$. Then $|S|=M$ if and only if the following three conditions are satisfied*:

(a) *$T(k)$ is full for all $k>0$;*

(b) *$U(k)$ is the empty set whenever $e(k)<M$;*

(c) *If $x\in V(k)$ for any $k<n$, then there exists $y\in T(k+1)$ such that $x<y$.*

Proof. The Lubell–Baker proof of Theorem 1 ([3], [1]) goes as follows:

$$1 \geq \sum_{x\in S} p(x) \geq \sum_{x\in S} 1/M = |S|/M$$

where $p(x)$ is the probability that a randomly chosen chain of length n passes through x. Then $|S|=M$ if and only if neither inequality is strict. All chains of length n intersect S if and only if conditions (a) and (c) both hold; thus the first inequality is strict if and only if (a) or (c) fails to hold. The second inequality is strict if and only if (b) fails to hold.

REMARK 1. A finite poset is said to be *graded* if $h(x)=h(y)+1$ whenever x covers y. If one assumes that Q is graded by its height function, then condition (c) may

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be omitted It is easy, however, to construct examples of ungraded U-posets with antichains S such that $0<|S|<M$ even though conditions (a) and (b) are satisfied.

REMARK 2. Let Q be a U-poset for which no $E(k)$ contains any proper full subsets. Then $|S|=M$ if and only if $S=E(k)$ for some k satisfying $e(k)=M$.

REMARK 3. Let Q be a semimodular lattice which is also a U-poset. Then $|S|=M$ if and only if $S=E(k)$ for some k satisfying $e(k)=M$.

Proof. Regard $F(k)=E(k) \cup E(k-1)$ as an undirected bipartite graph (see [4]). Then $T\subset E(k)$ is full if and only if T is the intersection of $E(k)$ with a union of connected components of $F(k)$. Since Q is semimodular and of finite length, Q is graded [2, pp. 39–40]. Also, if x and y cover z, then $x \vee y$ covers x and y. An easy induction on k shows that $F(k)$ is connected for all k. Thus $E(k)$ has no proper full subsets. The result follows from either of Remarks 1, 2.

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