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THE SPHERICITY OF HIGHER DIMENSIONAL KNOTS

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In 1956 C. D. Papakyriakopoulos showed [**5**] that the complement C of a 1-sphere S^1 tamely imbedded in a 3-sphere S^3 is *aspherical*; that is, that for all $i \geqq 2$, $\pi_i(C) = 0$. In this note we show that for $n \geqq 2$ the complement C of an n-sphere S^n smoothly imbedded in S^{n+2} is aspherical only if the fundamental group of C is infinite cyclic. Combined with results of J. Stallings [**6**] or of J. Levine [**3**], this implies that if the complement of an S^n smoothly imbedded in S^{n+2} is aspherical, $n \geqq 4$, then S^n is topologically unknotted in S^{n+2}.

The above result is obtained as a corollary of our main result, which is of a more technical nature.

A local coefficient system $\mathscr{G}$ with abelian group G over a space X is determined by and determines an action of $\mathbf{Z}(\pi_1(X))$, the integral group ring of $\pi_1(X)$, on G (see [**7**] for details). Each n-manifold M has a unique local coefficient system $\mathscr{O}_M$ with group $\mathbf{Z}$, the *orientation bundle* of M. It is given by local, integral, n-dimensional homology at each point of M with action defined by paths (homotopies) in the manifold.

THEOREM. *Let M be a compact, connected n-manifold with non-empty, connected boundary B. Let I denote the image of the homomorphism $\pi_1(B) \to \pi_1(M)$ induced by inclusion and $\mathscr{O}_M$ denote the orientation bundle of M. If*

(a) *$\pi_1(B) \to \pi_1(M)$ is not onto, and*

(b) *$H_{n-1}(I; \mathscr{O}_M) \to H_{n-1}(\pi_1(M); \mathscr{O}_M)$ is injective.*

then M is not aspherical.

Proof. We shall assume (a) and (b) and also that M is aspherical. Let $G = \pi_1(B)$ and $H = \pi_1(M)$. Since

$$H_0(M; \mathbf{Z}(H)) \cong \mathbf{Z}(H)/I(H) \cdot \mathbf{Z}(H) \cong \mathbf{Z},$$

and

$$H_0(B; \mathbf{Z}(H)) \cong \mathbf{Z}(H)/I(G) \cdot \mathbf{Z}(H) \cong \bigoplus_d \mathbf{Z},$$

where d is the number of cosets of I in H ($d > 1$ by (a)), $H_1(M, B; \mathbf{Z}(H)) \neq 0$. By duality,

$$H^{n-1}(M; \mathbf{Z}(\mathrm{H}) \otimes \mathscr{O}_M) \cong H_1(M, B; \mathbf{Z}(H)).$$

Thus,

$$H^{n-1}(M; \mathbf{Z}(H) \otimes \mathscr{O}_M) \neq 0.$$

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By the Seifert–van Kampen Theorem the fundamental group of the union of two copies of M adjoined along B, $M_1 \cup_B M_2$, is given by the pushout diagram

$$\begin{array}{ccc} G & \longrightarrow & H \\ \downarrow & & \downarrow \\ H & \longrightarrow & \pi_1\left(M_1 \underset{B}{\cup} M_2\right) \end{array}$$

Since the image of G in H is I,

$$\begin{array}{ccc} I & \longrightarrow & H \\ \downarrow & & \downarrow \\ H & \longrightarrow & \pi_1\left(M_1 \underset{B}{\cup} M_2\right) \end{array}$$

is also a pushout diagram. Thus,

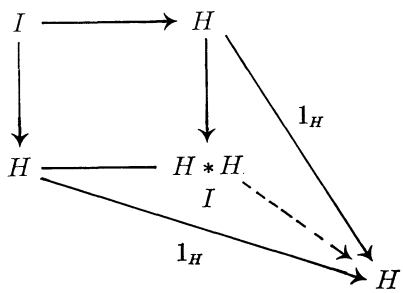
$$\pi_1\left(M_1 \underset{B}{\cup} M_2\right) \cong H \underset{I}{*} H,$$

the free product of H with itself amalgamated along I. There is consequently a map

$$j: M_1 \underset{B}{\cup} M_2 \rightarrow K\left(H \underset{I}{*} H, 1\right),$$

obtained by attaching cells to $M_1 \cup_B M_2$ in dimensions three and higher, with $\pi_1(j)$ an isomorphism. Thus, $H_1(j)$ is an isomorphism with any local coefficients.

Let $\bar{f}$: $H *_I H \rightarrow H$ be the folding homomorphism induced by the diagram



and let f: $K(H *_I H, 1) \rightarrow K(H, 1)$ be a map such that $\bar{f} = \pi_1(f)$. Since M is a $K(H, 1)$, we have the diagram

$$M \xrightarrow{i_1} M_1 \underset{B}{\cup} M_2 \xrightarrow{j} K\left(H \underset{I}{*} H, 1\right) \xrightarrow{f} M$$

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for which the composition induces an isomorphism of fundamental groups. Since M is a $K(H, 1)$, this implies that the composition is a homotopy equivalence.

Let $x \in H^{n-1}(H *_I H; f^*(\mathbf{Z}(H) \otimes \mathscr{O}_M))$ be the image under f^* of a non-zero class in $H^{n-1}(M; \mathbf{Z}(H) \otimes \mathscr{O}_M)$. Both x and j^*x are non-zero since $i_1{}^*j^*x$ is non-zero. Hence,

$$0 \neq j^*x \cap \mu_{M_1 \cup_B M_2} \in H_1(M_1 \cup_B M_2; j^*f^*(\mathbf{Z}(H) \otimes \mathscr{O}_M) \otimes \mathscr{O}_{M_1 \cup_B M_2}),$$

where $\mu_{M_1 \cup_B M_2} \in H_n(M_1 \cup_B M_2; \mathscr{O}_{M_1 \cup_B M_2})$ is the fundamental class of the closed manifold $M_1 \cup_B M_2$.

Since $\pi_1(j)$ is an isomorphism, there is a local coefficient system $\mathscr{O}$ with group $\mathbf{Z}$ over $K(H *_I H, 1)$ such that $j^*\mathscr{O} = \mathscr{O}_{M_1 \cup_B M_2}$. (In fact $f^*\mathscr{O}_M = \mathscr{O}$, but we do not need this.) Since $H_1(j)$ is an isomorphism with any local coefficient system,

$$0 \neq j_*(j^*x \cap \mu_{M_1 \cup_B M_2}) = x \cap j_*\mu_{M_1 \cup_B M_2}.$$

The class $j_*\mu_{M_1 \cup_B M_2}$ is an element of the group $H_n(H *_I H; \mathscr{O})$, and we shall reach a contradiction by showing this group is trivial.

Since $j^*\mathscr{O} = \mathscr{O}_{M_1 \cup_B M_2}$ and $\mathscr{O}_{M_1 \cup_B M_2}|M_i$ is $\mathscr{O}_{M_i}$ for $i = 1, 2$, each of the homomorphisms

$$H \xrightarrow{k_i} H \underset{I}{*} H$$

in the diagram

$$\begin{array}{ccc} I & \longrightarrow & H \\ \big\downarrow & & \big\downarrow k_1 \\ H & \xrightarrow{k_2} & H \underset{I}{*} H \end{array}$$

induces from $\mathscr{O}$ a $\mathbf{Z}(H)$ action on $\mathbf{Z}$ which describes the local coefficient system $\mathscr{O}_M$.

By the Mayer-Vietoris sequence for free products of groups with amalgamation [**8**], we have the exact sequence

$$H_n(H; \mathscr{O}_M) \oplus H_n(H; \mathscr{O}_M) \xrightarrow{k_*} H_n\left(H \underset{I}{*} H; \mathscr{O}\right) \xrightarrow{\partial_*} H_{n-1}(I; \mathscr{O}_M)$$
$$\xrightarrow{i_*} H_{n-1}(H; \mathscr{O}_M) \oplus H_{n-1}(H; \mathscr{O}_M).$$

The Hypothesis (b) implies that i_* is an injection. Also,

$$H_n(H; \mathscr{O}_M) \cong H_n(M; \mathscr{O}_M) = 0$$

by Poincare duality. Thus,

$$H_n\left(H \underset{I}{*} H; \mathscr{O}\right) = 0.$$

COROLLARY 1. *Let S^n be a smoothly imbedded n-sphere in S^{n+2}, $n \geqq 2$. If the complement of S^n in S^{n+2} is aspherical, then its fundamental group is infinite cyclic. If moreover $n \geqq 4$, then S^n is unknotted.*

Proof. A normal tube about S^n is homeomorphic to $S^n \times D^2$ (see [**4**]). The closure of the complement of this tube in S^{n+2} (which has the homotopy type of $S^{n+2} - S^n$) is then a compact, connected $(n + 2)$-manifold M with connected boundary $B = S^n \times S^1$. In the notation of the proof of the theorem, $G = \pi_1(B) \cong \mathbf{Z}$, and we shall show $I \cong \mathbf{Z}$.

Let $1 \neq \alpha \in \pi_1(B)$ and $\bar{\alpha}$ be the Hurewicz image of α in $H_1(B; \mathbf{Z})$. Then $\bar{\alpha} \neq 0$, but $\bar{\alpha}$ is a bounding cycle in the normal tube. If $\bar{\alpha}$ were also a boundary in M, by the Mayer–Vietoris sequence there would be a non-zero 2-dimensional integral homology class in S^{n+2}. Thus, α is not null-homotopic in M; i.e., $\pi_1(B) \rightarrow \pi_1(M)$ is injective.

Thus, $I \cong \mathbf{Z}$ and $H_{n+1}(I; \mathscr{O}_M) = 0$, and hypothesis (b) of the theorem is satisfied. If M is aspherical, then hypothesis (a) of the theorem is not satisfied. Hence, $\mathbf{Z} \cong \pi_1(B) \rightarrow \pi_1(M)$ is an isomorphism.

The second conclusion of the corollary is an immediate consequence of results of J. Levine [**3**] or J. Stallings [**6**].

We note that previously D. B. A. Epstein proved this result for spun knots [**1**]. Also, we note this responds to Probelm 37 of R. H. Fox [**2**]. Finally, it is clear that for the first conclusion of the corollary, S^{n+2} can be replaced by more general manifolds.

Comment. As stated in the introduction, the conclusion of Corollary 1 is false for $n = 1$. In that case $B = S^1 \times S^1$ and it can be shown that if S^1 is knotted, then $\pi_1(B) \rightarrow \pi_1(M)$ is injective. The 3-manifold M is orientable; $\mathscr{O}_M$ is the non-twisted coefficient system. Also, M is a $K(\pi_1(M), 1)$ by [**5**], and by Alexander duality $H_2(M; \mathbf{Z}) = 0$. Hence, $H_2(I; \mathscr{O}_M) \cong \mathbf{Z}$ and $H_2(\pi_1(M); \mathscr{O}_M) = 0$.

Our argument does not apply in this case since the Hypothesis (b) of the theorem is not satisfied.

COROLLARY 2. *Let M be a compact connected aspherical manifold of dimension $n \geqq a + 2$ where a = cohomological dimension of $\pi = \pi_1(M)$. Then $B = \partial M$ is connected and $\pi_1(B) \rightarrow \pi_1(M)$ is onto.*

Proof. If B were empty M would have non-zero homology with local coefficients in a dimension n exceeding a; but $H_i(M; \mathscr{B}) \cong H_i(\pi; \mathscr{B})$ since M is aspherical. If B were disconnected, then

$$0 \neq H_1(M, B; \mathbf{Z}) \cong H^{n-1}(M; \mathscr{O}_M).$$

Since $n - 1$ also exceeds the dimension of π, this is impossible.

Since I is a subgroup of π, its cohomological dimension is bounded by a and is hence less than $n - 1$. Thus, $H_{n-1}(I; \mathscr{O}_M) = 0$ and Hypothesis (b) of the theorem is satisfied.

Since M is aspherical, it follows that Hypothesis (a) of the theorem is not satisfied. But the negation of Hypothesis (a) is the conclusion of this corollary.

Comment. Let M be a compact, connected n-manifold with connected, non-empty boundary B imbedded in a closed, simply connected n-manifold. Let N be the closure of the complement of M. By the Seifert–van Kampen Theorem, the diagram

$$\begin{array}{ccc} \pi_1(B) & \longrightarrow & \pi_1(M) \\ \downarrow & & \downarrow \\ \pi_1(N) & \longrightarrow & (1) \end{array}$$

is a pushout. Thus, the normal closure in $\pi_1(M)$ of the image of the homomorphism

$$\pi_1(B) \to \pi_1(M)$$

is all of $\pi_1(M)$.

References

1. D. B. A. Epstein, *Linking spheres*, Proc. Cambridge Philos. Soc. *56* (1960), 215–219.
2. R. H. Fox, *Some problems in knot theory*, *Topology of 3-manifolds*, (Prentice-Hall, 1962, p. 168–176).
3. J. Levine, *Unknotting spheres in codimension two*, Topology *4* (1965), 9–16.
4. W. S. Massey, *On the normal bundle of a sphere imbedded in Euclidean space*, Proc. Amer. Math. Soc. *10* (1959), 959–964.
5. C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. *66* (1957), 1–26.
6. J. Stallings, *On topologically unknotted spheres*, Ann. of Math. *77* (1963), 490–503..
7. N. E. Steenrod, *Homology with local coefficients*, Ann. of Math. *44* (1943), 610–627.
8. R. G. Swan, *Groups of cohomological dimension one*, J. Algebra *12* (1969), 586–610.

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