SOME REMARKS ON THE MATHIEU GROUPS

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1. Introduction. In the present note we shall study some properties of the Mathieu groups.

We shall give an invariant characterisation of the 2-Sylow subgroups. The 2-Sylow subgroup of M_{24} is the holomorph of the elementary abelian group of type (1, 1, 1, 1), and for the 2-Sylow subgroups of the other Mathieu groups there are similar characterisations.

As was already known to Frobenius [4], M_{12} is a subgroup of M_{24}. One can easily show that $M_{11} \not\subset M_{23}$. This seems not to be in the literature; however it is a consequence of known theorems as was pointed out by the referee.

Coxeter [2] has given a representation of M_{12} as a matrix group of degree 6 over the Galois field of three elements. This representation bases upon a certain configuration in the five dimensional projective space over the Galois field GF(3). We shall show that Coxeter's configuration also leads to a representation of degree 10.

For the groups M_{11} and M_{12} an abstract definition is due to Coxeter and Moser [3] and Moser [7]. For M_{12} we shall give a slightly different system of defining relations. Then we shall establish an abstract definition for M_{22}. This definition uses a set of defining relations for LF(3, 4) which is a subgroup of M_{22} of index 22.

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2. 2-Sylow subgroups. Generators for the multiply transitive groups we are concerned with have been given by Mathieu [6] and quoted by Carmichael [1]. In the following, we shall give a characterisation for the 2-Sylow subgroups of the Mathieu groups.

The quintuply transitive group M_{12} of degree 12 and order $8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 = 95040$ is generated by the permutations

$$S = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)\,,$$

$$T = (3\ 7\ 11\ 8)(4\ 10\ 5\ 6)\,,$$

and

$$U = (1\ 12)(2\ 11)(3\ 6)(4\ 8)(5\ 9)(7\ 10)$$

[6, p. 35; 1, p. 151]. S and T generate the subgroup M_{11} of order $8 \cdot 9 \cdot 10 \cdot 11 = 7920$, which leaves fixed the symbol 12. The two permutations

$$V = STS^2T^2 = (1\ 5\ 10\ 6\ 2\ 9\ 8\ 3)(4\ 7)$$

and

$$W = (S^{-4}TS^3T^2)^2 = (1\ 8)(2\ 10)(3\ 6)(4\ 7)$$

generate a 2-Sylow subgroup S_2 of M_{11}. S_2 is defined by

$$V^8 = W^2 = E\,, \qquad WVW = V^3\,.$$

This group is $\langle -2, 4 \mid 2 \rangle$ in Coxeter's notation [3, pp. 9, 74, 134]. The relation $V^8 = E$ is redundant. The 2-Sylow subgroup of M_{11} is the group of order 16 which contains a cyclic subgroup of index 2 and whose automorphism group induces the trivial automorphism on its commutator factor group.

The 2-Sylow subgroup of M_{12} is given by the generators V, W and $Z = S^2T^{-1}S^{-2}T^2SUS^{-3}TS^{-2} = (1\ 3\ 10\ 9\ 2\ 6\ 8\ 5)(4\ 12\ 7\ 11)$ and the defining relations

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$$W^2 = (WZ)^2 = (VZ)^2 = E,$$

$$V^4 = Z^4, \quad WVW = V^3, \quad V^2ZV^2 = Z.$$

The group contains an abelian normal subgroup of type (2, 2) and the factor group of type (1, 1) acts as a group of automorphisms upon the normal subgroup.

We now proceed to the quintuply transitive group M_{24} of degree 24 and order $48 \cdot 20 \cdot 21 \cdot 22 \cdot 23 \cdot 24 = 244823040$ generated by the permutations

$A = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23)$,

$B = (3\ 17\ 10\ 7\ 9)(4\ 13\ 14\ 19\ 5)(8\ 18\ 11\ 12\ 23)(15\ 20\ 22\ 21\ 16)$,

$C = (1\ 24)(2\ 23)(3\ 12)(4\ 16)(5\ 18)(6\ 10)(7\ 20)(8\ 14)(9\ 21)$
$(11\ 17)(13\ 22)(15\ 19)$

[6, p. 41-42; 1, p. 164]. A and B generate the quadruply transitive subgroup M_{23} of index 24.

Finally,

$X = (2\ 5\ 8\ 22\ 4\ 14\ 18)(6\ 17\ 21\ 20\ 10\ 16\ 13)(7\ 9\ 12\ 11\ 15\ 23\ 19)$

and

$Y = (3\ 22\ 11\ 12\ 4)(5\ 18\ 20\ 15\ 8)(6\ 7\ 19\ 23\ 21)(9\ 13\ 16\ 10\ 14)$

are generators for the Mathieu group M_{22} of order $48 \cdot 20 \cdot 21 \cdot 22 = 443520$. X and Y yield just the subgroup of $\{A, B\} = M_{23}$ which leaves fixed the symbol 1. It is possible to express X and Y by A and B, but we do not need these expressions here.

The 2-Sylow subgroup of M_{22} is generated by the permutations

$K = (X^{-1}Y)^4 YXY^{-1}X^{-1}$
$= (2\ 12\ 20\ 9\ 13\ 7\ 17\ 10)(3\ 23\ 19\ 14\ 5\ 16\ 18\ 15)(4\ 22)(6\ 21\ 11\ 8)$,

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$M = (YX^3Y^2)^3$
$= (2\ 12)\ (3\ 17)\ (5\ 20)\ (7\ 13)\ (9\ 15)\ (10\ 14)\ (16\ 19)\ (18\ 23)$,

$N = (YX^{-1}Y)^4$
$= (2\ 13)\ (6\ 21)\ (7\ 18)\ (8\ 11)\ (12\ 19)\ (14\ 20)\ (15\ 17)\ (16\ 23)$

and the defining relations

$$K^8 = M^2 = N^2 = (MK)^4 = (MK^4)^2 = (MK^{-1}MK)^2 = E,$$

$$NMN = K^2MK^2, \qquad NKN = MKMK^2.$$

It contains an abelian normal subgroup of type (1, 1, 1, 1). The factor group is dihedral. Take all automorphisms of (1, 1, 1, 1) which leave an element $\neq E$ fixed. Consider the splitting extension of (1, 1, 1, 1) with this group of automorphisms. The 2-Sylow subgroup of M_{22} is the 2-Sylow subgroup of this extension.

Last we consider the 2-Sylow subgroup of M_{24}.

$$L = (XY^2X^{-3})^{-2}ACA^{-1}(XY^2X^{-3})^2$$

and

$$P = (Y^{-1}X^3Y^2X^{-1})^{-1} \cdot CA^{-9}CA^{-9}C \cdot Y^{-1}X^3Y^2X^{-1}$$

lead us to

$Q = LPL = (1\ 8\ 24\ 21)\ (2\ 19\ 23\ 12)\ (3\ 17\ 10\ 15)\ (4\ 11\ 22\ 6)$
$(5\ 20\ 9\ 14)\ (7\ 13\ 18\ 16)$.

The 2-Sylow subgroup of order $2^{10} = 1024$ is defined by

$$K^8 = M^2 = N^2 = Q^4 = (MK)^4 = (QK)^4 = (Q^2N)^2 = E,$$

$$NMN = K^2MK^2, \quad NKN = MKMK^2, \quad KQ^2 = Q^2K,$$

$$QK^2QK^{-2} = E, \quad Q^{-1}MQ = K^{-2}MK^2, \quad QMQ^{-1} = K^2MK^2,$$

$$(QNK^{-1})^2 = E .$$

The 2-Sylow subgroup again contains an abelian normal subgroup of type (1, 1, 1, 1). The factor group is the 2-Sylow subgroup of LF(4, 2) $\simeq \mathfrak{A}_8$. The 2-Sylow subgroup is the 2-Sylow subgroup of the splitting extension of (1, 1, 1, 1) with its group of automorphisms LF(4, 2), i.e. the holomorph of (1, 1, 1, 1).

3. Subgroup theorem. It is due to Frobenius [4], that M_{12} is a subgroup of M_{24}. In fact, one can divide the 24 letters of M_{24} into two sets, each containing 12 letters, such that M_{12} consists of all those permutations of M_{24}, which leave unchanged the two sets (Cf. [9]). Since the order of M_{11} divides the order of M_{22}, it could be possible that M_{11} is a subgroup of M_{22}. The 2-Sylow subgroup of M_{11} is contained in the 2-Sylow subgroup of M_{22}. But in the following we shall show $M_{11} \not\subset M_{22}$.

Assume that the representation of M_{11} on 22 letters is imprimitive. Then there must be two sets of imprimitivity, each containing 11 letters. An element of order 8 in M_{11} would leave at least two letters fixed. But all elements of order 8 in M_{22} leave no letter fixed. So the representation of M_{11} on 22 letters must be primitive. Hence the subgroup of M_{11}, which leaves one letter fixed, must be maximal. It is of order $2^3 . 3^2 . 5 = 360$. But there is no maximal subgroup of this order in M_{11}. Hence $M_{11} \not\subset M_{22}$.

If $M_{11} \subset M_{23}$, then M_{11} must be transitive on 23 letters which is impossible. So we have

THEOREM: M_{11} is not a subgroup of M_{23}.

This completely settles the problem of how the Mathieu groups are contained in each other (see fig. 1).

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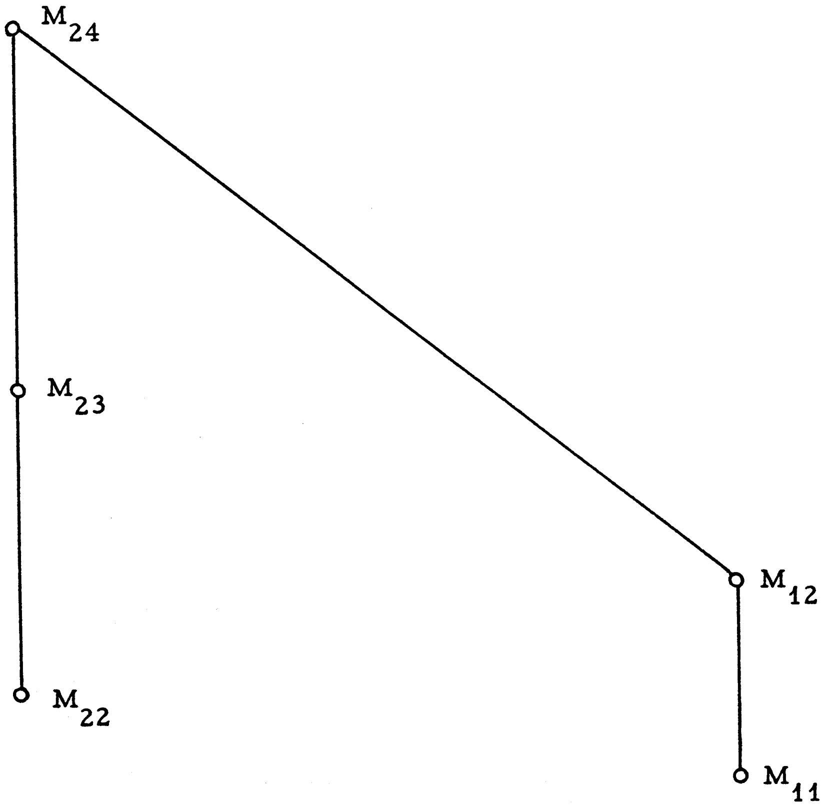

Fig. 1

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Remark: As the referee kindly pointed out, the fact that $M_{11} \not\subset M_{23}$ is also an immediate consequence of two known theorems:

a) Every multiply transitive group is a primitive group [1, p. 160].

b) Netto's Theorem. If a transitive group of degree n contains a circular permutation of prime order $q < \frac{2}{3} n$, then the group is either non-primitive or it contains the alternating group A_n. (E. Netto, The Theory of Substitutions, tr F.N. Cole, Ann Arbor, 1892.)

4. Matrix representations of M_{12}. Coxeter [2] has given a matrix representation of M_{12} of degree 6 over the Galois field of 3 elements. It consists of all collineations which leave fixed a certain configuration of 12 points in PG(5, 3), namely

1:	(1, 0, 0, 0, 0, 0)	7:	(0, 1, -1, -1, 1, 1)
2:	(0, 1, 0, 0, 0, 0)	8:	(1, 0, 1, -1, -1, 1)
3:	(0, 0, 1, 0, 0, 0)	9:	(-1, 1, 0, 1, -1, 1)
4:	(0, 0, 0, 1, 0, 0)	10:	(-1, -1, 1, 0, 1, 1)
5:	(0, 0, 0, 0, 1, 0)	11:	(1, -1, -1, 1, 0, 1)
12:	(0, 0, 0, 0, 0, 1)	6:	(-1, -1, -1, -1, -1, 0)

We would like to remark that Coxeter's configuration also leads to a representation of degree 10 over GF(3).

Consider all quadrics which contain the twelve points of Coxeter's configuration. A simple calculation yields that there are precisely 10 such quadrics, namely

$$Q_1 : x_1x_2 - x_3x_5 + x_4x_6 = 0$$

$$Q_2 : x_1x_3 - x_2x_6 - x_4x_5 = 0$$

$$Q_3 : x_1x_4 - x_2x_3 - x_5x_6 = 0$$

$$Q_4 : x_1x_5 - x_2x_4 + x_3x_6 = 0$$

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$$Q_5 : x_1x_6 - x_2x_5 + x_3x_4 = 0$$

$$Q_6 : x_2x_3 + x_2x_5 + x_3x_4 - x_5x_6 = 0$$

$$Q_7 : x_2x_3 - x_2x_4 - x_3x_6 - x_5x_6 = 0$$

$$Q_8 : x_2x_3 + x_2x_6 - x_4x_5 - x_5x_6 = 0$$

$$Q_9 : x_2x_3 - x_3x_5 - x_4x_6 - x_5x_6 = 0$$

$$Q_{10} : x_2x_3 + x_3x_4 + x_3x_6 + x_4x_5 + x_4x_6 = 0$$

A collineation of PG(5, 3) leaving invariant Coxeter's configuration also induces a collineation of the space EG(10, 3) spanned by the quadrics.

The following involutions A, B, C generate M_{12}, as can be seen by observing that $\{A, B, C\}$ contains the Sylow subgroups of M_{12}.

$$A = (1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12),$$

$$B = (1\ 12)(2\ 9)(3\ 11)(4\ 8)(5\ 10)(6, 7),$$

$$C = (1\ 8)(2\ 12)(3\ 4)(5\ 9)(6\ 7)(10\ 11).$$

The corresponding matrices of Coxeter's representation are

$$A = \begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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These matrices induce the following collineations in the space of quadrics:

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 & -1 & 0 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

5. Defining Relations for M_{11}, M_{12} and M_{22}. Abstract definitions for M_{11} and M_{12} were given by Coxeter and Moser [3] and Moser [7] respectively. We will now establish another abstract definition for M_{11} and M_{12}.

Using the generators from sec. 2, M_{11} is defined by

$$S^{11} = T^4 = (ST^2)^3 = (S^4T^2S^{-5}T^2)^2 = E,$$

$$(S^{-4}T^{-1})^3 = S^{-1}TS^{-2}T, \quad S^{-5}T^2S^2T = (S^3T^{-1}ST)^{-1}.$$

It is easy to verify this by the Todd-Coxeter enumeration method, enumerating the cosets of LF(2, 11) in M_{11}. S and $R = T^2$ satisfy Miller's system of defining relations for LF(2, 11):

$$S^{11} = R^2 = (SR)^3 = (S^4RS^{-5}R)^2 = E$$

[3, p. 139].

For the Mathieu group M_{12} we give the following set of defining relations:

$$S^{11} = T^4 = U^2 = (ST^2)^3 = (S^4T^2S^{-5}T^2)^2 = E,$$

$$(S^{-1}UT)^3 = (SU)^3 = E,$$

$$(S^{-4}T^{-1})^3 = S^{-1}TS^{-2}T, \quad S^{-5}T^2S^2T = (S^3T^{-1}ST)^{-1},$$

$$(S^{-1}TS^2T)^2 = UTU, \quad (STUS^{-4})^2 = S^2T^2UT.$$

The proof is by enumeration of the cosets of M_{11} in M_{12}.

(We would like to remark, that our relation $(STUS^{-4})^2 = S^2T^2UT$ can replace the relation $US^2T^{-1}S^4U = S^{-1}T^2S^3T^2S^4TS^5$ in Moser's abstract definition for M_{12}.)

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We now proceed to the Mathieu group M_{22}. Generators for it have been given in sec. 2. As is well known (Cf. [9]), LF(3,4) is the subgroup of M_{22} which leaves fixed one of the 22 letters. LF(3,4) is generated by Y and $D = (XY^{-1}X)^2$ and defined by

$$Y^5 = D^3 = (YD)^4 = (Y^{-1}DY^{-1}D^{-1}YD)^3 = E ,$$

$$DY^2D\cdot Y^{-2}\cdot(DY^2D)^{-1}\cdot Y^2 = YDY^{-1}D^{-1} .$$

This is proven by enumeration of the cosets of $\{Y\}$ in LF(3,4). The enumeration of the 4032 cosets was carried out by an electronic computer.+)

Using this abstract definition for LF(3,4), we finally obtain a system of defining relations for M_{22}:

$$X^7 = Y^5 = D^3 = (XY)^2 = (DY)^4 = (DX)^4 = (Y^{-1}DY^{-1}D^{-1}YD)^3 = E ,$$

$$D = (XY^{-1}X)^2, \quad DY^2D\cdot Y^{-2}\cdot(DY^2D)^{-1}\cdot Y^2 = YDY^{-1}D^{-1} ,$$

$$X^{-2}YX^3 = Y^2D^{-1}YDY^{-2}DY^2, \quad X^2Y^2X^{-3}YX^2Y^{-1}X^2 = Y^2D^{-1}YDY^{-1}D^{-1}YDY^2.$$

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