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The multiplicator of finite nilpotent groups

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Let G be a group and M a G-module; then $d(G)$ denotes the minimal number of generators of G and $d_G(M)$ the minimal number of generators over ZG of M. For G a finite nilpotent group let $G = F/R$, F free, be a presentation for G; then it is shown that

$$d(R/[F, R]) = d_G(R/[R, R]) ,$$

that is

$$d(G) + d(M(G)) = d_G(R/R') ,$$

where $M(G)$ denotes the Schur multiplicator of G.

1. Introduction

If a finite group G is generated by n elements and defined by m relations between them then G has a presentation

$$G = \{x_1, \ldots, x_n | R_1, \ldots, R_m\} .$$

Clearly $m \geq n$ and the value $n - m$ is said to be the deficiency of the given presentation. The deficiency of G, denoted $\text{def}(G)$, is the maximum of the deficiencies of all the finite presentations of G.

It is implicit in J. Schur [3] that the minimal number of generators of the Schur "multiplicator", as an abelian group, is less than or equal to $-\text{def}(G)$. B.H. Neumann [2] asks whether a finite group with trivial

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multiplicator has deficiency zero; R.G. Swan [4] answers this question by giving a family of finite soluble groups with trivial multiplicator and negative deficiency. However the question is still unanswered in the case of finite nilpotent groups.

In this paper we apply a theorem of R.G. Swan [4] to show that if G is a finite nilpotent group generated by n elements such that the Schur multiplicator is minimally generated by r elements then G has a presentation

$$G = F/R = \{x_1, \ldots, x_n | R_1, \ldots, R_{n+r}, S_1, \ldots, S_t\}$$

where F has free generators $\{x_1, \ldots, x_n\}$ and R is the smallest normal subgroup of F containing the defining relations $R_1, \ldots, R_{n+r}$, $S_1, \ldots, S_t$ such that $S_1, \ldots, S_t$ belong to R', the commutator subgroup of R.

2. The Lyndon resolution

Let G be a finite group, then we construct a sequence of matrices with elements in ZG as follows,

$$M^o = \begin{pmatrix} x_1 - 1 \\ \ldots \\ x_{\alpha_1} - 1 \end{pmatrix} \text{, a column matrix}$$

where $x_1, \ldots, x_{\alpha_1}$ is a set of elements generating G.

Given M^{r-1}, let M^r be any matrix whose row space spans (over ZG) all vectors v such that

$$v.M^{r-1} = 0 ,$$

that is the row space of M^r is a set of vectors

$$v_1, \ldots, v_{\alpha_{r+1}} \text{ such that if } v.M^{r-1} = 0 \text{ then } v = \sum_{i=1}^{\alpha_{r+1}} y_i v_i , \quad y_i \in ZG .$$

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Since G is finite we may choose α_r finite for all r and the $\alpha_{r+1} \times \alpha_r$ matrix M^r is said to be the r-th incidence matrix for G. Let F_r be a ZG module free on α_r generators, then

$$\xrightarrow{M^r} F_r \xrightarrow{M^{r-1}} \ldots \to F_1 \to ZG \to Z \to 0$$

is a free ZG-resolution of Z due to R.C. Lyndon [1] called the Lyndon resolution.

We state without proof the following two lemmas implicit in Lyndon [1],

LEMMA 2.1. *Let G be a finite group. If $\{x_1, \ldots, x_n | R_1, \ldots, R_m\}$ is a presentation for G, then we may take the first incidence matrix, M^1, to be the matrix*

$$M^1 = \left(\gamma\left(\partial R_i/\partial x_j\right)\right) ,$$

where γ is the natural homomorphism of F onto G and $\partial R_i/\partial x_j$ denotes the Fox derivative of R_i with respect to x_j.

Conversely corresponding to any M^1, there exists a presentation $\{x_1, \ldots, x_n | R_1, \ldots, R_m\}$ for G such that

$$M^1 = \left(\gamma\left(\partial R_i/\partial x_j\right)\right) . \qquad //$$

LEMMA 2.2. *Let G be a finite group with presentation*

$$G = F/R = \{x_1, \ldots, x_n | R_1, \ldots, R_m\}$$

and

$$M^1 = \left(\gamma\left(\partial R_i/\partial x_j\right)\right) ,$$

then R/R' is equivalent as a ZG module to $\bar{R}$ where $\bar{R}$ is the submodule generated by the row space of M^1. The equivalence mapping is defined by ϕ where

$$\phi(rR') = \gamma\left(\partial r/\partial x_1, \ldots, \partial r/\partial x_n\right) . \qquad //$$

Let $\tau : ZG \to Z$ be the homomorphism induced by $\tau(g) = 1$, for all g belonging to G, then we have

THEOREM 2.3. *Let G be a finite group, then we may choose a presentation for G such that*

$$G = \{x_1, \ldots, x_n | R_1, \ldots, R_m\} ,$$

$$M^1 = \left[\gamma\left(\partial R_i / \partial x_j\right)\right] ,$$

$$\tau(M^1) = \begin{pmatrix} M_n \\ \\ 0 \end{pmatrix} ,$$

where M_n is a non-singular $n \times n$ integral matrix, and

$$\tau(M^2) = \begin{pmatrix} 0 & D_{m-n} \\ \\ 0 & 0 \end{pmatrix} ,$$

where D_{m-n} is a non-singular diagonal $(m-n) \times (m-n)$ integral matrix, $D\left(z_1, \ldots, z_{m-n}\right)$, such that

$$z_i | z_{i+1} , \quad i = 1, \ldots, m-n-1 .$$

Proof. Clearly we can carry out elementary row operations on M^1 and M^2. Thus M^1 may be put in the required form. With M^1 in this form then the first n columns of $\tau(M^2)$ are zero, so that column operations are then induced on the non-zero columns of $\tau(M^2)$ by carrying out row operations on the zero rows of $\tau(M^1)$. //

COROLLARY 2.4. *Let Z_p be a trivial ZG-module, then*

(i) $\dim H^2\left(G, Z_p\right) = m - s - \operatorname{rank} M_n$;

(ii) $\dim H^1\left(G, Z_p\right) = \operatorname{nullity} M_n$;

(iii) $\dim H^0\left(G, Z_p\right) = 1$,

where M_n is considered as a matrix with entries in Z_p and s is the number of z_i in the set $\{z_1, \ldots, z_{m-n}\}$ prime to p. //

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COROLLARY 2.5. *The minimal number of generators of the multiplicator of* G *is equal to* $m - n - t$ *where* t *is the number of times* 1 *occurs in the set* $\{z_1, \ldots, z_{m-n}\}$. //

COROLLARY 2.6. *Let* G *be a finite group such that the minimal number of generators of the multiplicator of* G *is* r ; *then* $r+1 \geq \dim H^2(G, Z_p) - \dim H^1(G, Z_p) + \dim H^0(G, Z_p)$, *for all trivial* ZG *modules* Z_p . //

3. A theorem of Swan

The following theorem is due to R.G. Swan [4], Theorem (5.1). The proof will only be outlined to the extent we wish to use it.

THEOREM 3.1. *Let* G *be a finite group of order* g . *Let* $f_0, f_1, \ldots$ *be given integers. Then there is a free resolution of* Z *over* ZG

$$\ldots \to F_2 \to F_1 \to F_0 \to Z \to 0$$

with each F_i *free on* f_i *generators, if and only if the following two conditions are satisfied:*

(1) for all primes $p|g$ *(and one other if* $G = 1$*) and all simple* Z_pG*-modules* M , *we have*

$$(\dim M)(f_n - f_{n-1} + \ldots) \geq \dim H^n(G, M) - \dim H^{n-1}(G, M) + \ldots$$

for all n ;

(2) if G *has periodic cohomology with (minimal) period* q , *then for every* n *such that* $n \equiv -1 (\mathrm{mod}\ q)$ *and* G *has no periodic free resolution of period* $n+1$, *we must have* $f_n - f_{n-1} + \ldots \geq 1$.

Proof. The theorem is proved by supposing we have an exact sequence

$$F_n \to F_{n-1} \to \ldots \to F_0 \to Z \to 0$$

where F_i is ZG-free on f_i generators; then if the f_i satisfy the

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conditions of the theorem we extend the exact sequence to

$$F_{n+1} \to F_n \to \dots \to F_0 \to Z \to 0$$

where F_{n+1} is free on f_{n+1} generators. //

4. Applications

LEMMA 4.1. *Let G be a finite nilpotent group such that the minimal number of generators of the multiplicator of G is r ; then*

$$r + 1 \geq \dim H^2(G, Z_p) - \dim H^1(G, Z_p) + \dim H^0(G, Z_p) .$$

Proof. The lemma was proved for Z_p trivial in Corollary 2.6. For Z_p not trivial then $H^i(G, Z_p) = 0$. //

LEMMA 4.2. *Let G be a finite nilpotent group such that the minimal number of generators of the multiplicator of G is r ; then*

$$(\dim M)(r+1) \geq \dim H^2(G, M) - \dim H^1(G, M) + \dim H^0(G, M)$$

for all simple Z_pG-modules M .

Proof. M simple implies M is an elementary abelian group of exponent p and order p^r . The case $r = 1$ was treated in the previous lemma so we may assume $r > 1$.

For K a normal subgroup of G let M^K be the maximal trivial Z_pK-submodule of M . Also let S_p be the Sylow p subgroup of G , whence $G = S_p \times S$. We consider the various cases:

(i) $M^G = M$ then $r = 1$, hence $M^G = 0$;

(ii) $M^S = 0$ then $H^i(G, M) = 0$ for all $i \geq 0$;

(iii) $M^S \neq 0$ then $A = \left\langle gm \mid g \in S_p, m \in M^S \right\rangle$ is a submodule of M whence $M = A = M^S$;

(iv) $M = M^S$ then $r = 1$. //

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LEMMA 4.3. *Let G be a finite nilpotent group generated by n elements $\{x_1, \ldots, x_n\}$ such that the multiplicator of G is generated by r elements; then there exists an exact sequence*

$$F_2 \xrightarrow{\alpha} F_1 \xrightarrow{M^0} F_0 \xrightarrow{\tau} Z \to 0$$

where F_2 is free on $n + r$ generators, F_1 free on n generators, F_0 free on 1 generator and M^0 is given in matrix form by

$$M^0 = \begin{pmatrix} x_1 - 1 \\ \cdots \\ x_n - 1 \end{pmatrix} .$$

Proof. This follows from the Lyndon resolution, Theorem 3.1 and Lemma 4.2. //

THEOREM 4.4. *Let G be a finite nilpotent group generated by n elements such that the multiplicator of G is generated by r elements; then G has a presentation*

$$G = \{x_1, \ldots, x_n \mid R_1, \ldots, R_{n+r}, S_1, \ldots, S_t\}$$

where S_i belong to R' for $i = 1, \ldots, t$.

Proof. Writing α of Lemma 4.3 in matrix terms and using the fact that $H_1(G, ZG) = 0$ then α is a possible M^1 for the Lyndon resolution and hence by Lemmas 2.1 and 2.2 the result follows. //

COROLLARY 4.5. *Let G be a finite nilpotent group, where n equals the minimal number of generators of the multiplicator of G, then there exists a group K with deficiency $-n$ such that G is the maximal soluble factor group of K.* //

It would be of interest to know the form of the presentation given by Theorem 4.4 for some of the well known three generator p groups with trivial multiplicator; for example, let

$$G = \left\{a, b, c \mid b^{-1}ab = a^{1+p}, c^{-1}bc = b^{1+p}, a^{-1}ca = c^{1+p}, a^{p^3} = b^{p^3} = c^{p^3} = 1\right\} .$$

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Then G has trivial multiplicator for $p \geq 5$; however actual calculation of a presentation of the form given by Theorem 4.4 with $r = 0$ seems very difficult.

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