A SIMPLE CLOSED CURVE IS THE ONLY HOMOGENEOUS BOUNDED PLANE CONTINUUM THAT CONTAINS AN ARC

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1. Introduction. One of the unsolved problems of plane topology is the following:

Question. What are the homogeneous bounded plane continua?

A search for the answer has been punctuated by some erroneous results. For a history of the problem see **(6)**.

The following examples of bounded homogeneous plane continua are known: a point; a simple closed curve; a pseudo arc **(2, 12)**; and a circle of pseudo arcs **(6)**. Are there others?

The only one of the above examples that contains an arc is a simple closed curve. In this paper we show that there are no other such examples. We list some previous results that point in this direction. Mazurkiewicz showed **(11)** that the simple closed curve is the only non-degenerate homogeneous bounded plane continuum that is locally connected. Cohen showed **(8)** that the simple closed curve is the only homogeneous bounded plane continuum that contains a simple closed curve. Cohen showed **(8)** that the simple closed curve is the only non-degenerate homogeneous bounded plane continuum that is arcwise connected.

In this paper we prove the following theorem:

THEOREM 1. *The simple closed curve is the only homogeneous bounded plane continuum that contains an arc.*

Theorem 1 is proved by listing certain properties possessed by any homogeneous bounded plane continuum that contains an arc but is not a simple closed curve (these properties with their consequences are listed in §§ 2, 3, 5, 6, and 10) and then showing (Theorem 6) that no homogeneous bounded plane continuum could have one of these properties. The proof of Theorem 1 is completed in § 10.

In this paper, all sets are assumed to be metric. For the most part we will deal with planar sets but since some of the results apply to more general metric spaces, we do not suppose that sets discussed are planar unless this is stated. We recall some definitions, related results, and related questions.

A set is *homogeneous* if for each pair of its points p, q there is a homeomorphism of the set onto itself that takes p to q.

An ϵ collection is a collection each of whose elements is of diameter no more than ϵ.

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An ϵ chain is a finite ordered ϵ collection of open sets $d_1, d_2, \ldots, d_n$ such that d_i intersects d_j if and only if i is adjacent to j.

A compact continuum X is snake-like if for each positive number ϵ, X can be covered by an ϵ chain. It is known **(5)** that the only non-degenerate homogeneous snake-like continuum is a pseudo arc.

It is convenient to associate with any open covering G a 1-complex $C(G)$, called the 1-nerve of G, such that there is a $1 - 1$ correspondence between the elements of G and the vertices of $C(G)$ and two elements of G intersect if and only if the corresponding vertices of $C(G)$ are joined by a 1-simplex in $C(G)$. Note that the 1-nerve of an ϵ chain is topologically an arc.

A compact continuum X is tree-like if for each positive number ϵ there is an ϵ collection G of open sets covering X such that the 1-nerve of G contains no simple closed curve. Each 1-dimensional compact plane continuum that does not separate the plane is tree-like **(3)**.

Question. Is there a homogeneous tree-like continuum that contains an arc?

Jones has shown **(10)** that each homogeneous tree-like compact continuum is indecomposable. Perhaps each is hereditarily indecomposable.

A compact continuum X is circle-like if it is not snake-like but for each positive number ϵ there is an ϵ collection G of open sets irreducibly covering X such that the 1-nerve of G is topologically a circle. A simple closed curve and a circle of pseudo arcs are examples of circle-like homogeneous planar continua. Example 2 of **(4)** is not known to be non-homogeneous. A solenoid is an example of a compact homogeneous continuum that contains an arc. However, the simple closed curve is the only solenoid that is planar.

A solenoid may be defined as the intersection of a sequence of tori $T_1, T_2, \ldots,$ such that T_{i+1} runs smoothly around inside T_i n_i times longitudinally without folding back and T_i has cross diameter of less than $1/i$. The sequence $n_1, n_2, \ldots,$ determines the topology of the solenoid. If it is $1, 1, \ldots,$ after some place, the solenoid is a circle. If it is $2, 2, \ldots,$ the solenoid is the dyadic solenoid.

There is no loss of generality in supposing that each integer in the sequence $n_1, n_2, \ldots,$ used in defining a solenoid is prime, for if n_i is not prime, it may be replaced in the sequence by its prime factors. The order of the elements of the sequence does not affect the topology of the solenoid—that is, if $m_1, m_2, \ldots,$ is a reordering of $n_1, n_2, \ldots,$ the solenoids determined by the sequences are topologically equivalent. Also, the first few terms of the sequence does not affect the topology of the solenoid. Hence, solenoids determined by the sequences of primes $n_1^1, n_2^1, \ldots,$ and $n_1^2, n_2^2, \ldots,$ are topologically equivalent if it is possible to remove a finite number of elements from each so that each prime greater than 1 occurs the same number of times in each of the remainders. Perhaps the converse of this is true.

Another way of describing a solenoid is to consider a unit circle C in the plane with centre at the origin and a sequence of maps $f_1, f_2, \ldots,$ of C onto itself so that in polar co-ordinates

$$f_i(1, \theta) = (1, n_i\theta).$$

This solenoid is the inverse limit of the circles and the f_i's and consists of all points $p_1 \times p_2 \times p_3 \times \ldots$ of the Cartesian product $C \times C \times C \times \ldots$ such that for each i, $p_i = f_i(p_{i+1})$.

We show in Theorem 9 that if a circle-like homogeneous continuum contains an arc, it is a solenoid. Although a solenoid may not be planar, it is locally planar. Anderson has shown **(1)** that the only 1-dimensional locally connected continuum that is not locally planar at any point is the Menger universal curve. It is homogeneous **(1)**.

Question. Are solenoids and the Menger universal curve the only homogeneous 1-dimensional compact continua that contain arcs?*

2. Some elementary properties of M. In §§ 2 and 3 we suppose that M is a homogeneous, non-degenerate, bounded plane continuum that contains an arc but is not a simple closed curve. Our plan is to list enough of the properties of such an assumed M to show that it cannot exist. This section lists some elementary properties of M.

Property 1. *M is not locally connected.* Mazurkiewicz **(11)** showed that the simple closed curve is the only non-degenerate homogeneous bounded locally connected plane continuum.

Property 2. *M is not connected im kleinen at any point.* A set X is connected *im kleinen* at a point x if for each neighbourhood U of x there is a neighbourhood N of x such that $N \cdot X$ lies in the component of $U \cdot X$ containing x. If M were connected *im kleinen* at one point, it would follow from the homogeneity of X that it is connected *im kleinen* at every point. Since a continuum is locally connected at each point if it is connected *im kleinen* at each point, Property 1 implies Property 2.

Property 3. *M contains an open set U with uncountably many components.* Property 3 follows from Property 2 and the following theorem.

THEOREM 2. *If a complete metric space fails to be connected im kleinen at each point of a dense G_δ set, it contains an open set U with uncountably many components.*

Proof. Suppose X is a complete metric space that fails to be connected *im kleinen* at each point of a dense G_δ set Y and Y is the intersection of the open sets $U_1, U_2 \ldots$.

*At the 1959 Summer Meeting of the American Mathematical Society J. H. Case presented an abstract announcing another such continuum.

Assume that X fails to contain an open set U with uncountably many components. Let V_1 be an open subset of U_1 of diameter less than $\frac{1}{2}$. It follows from the Baire Category Theorem that V_1 contains an open subset V_2 such that

$$\bar{V}_2 \subset V_1 \cdot U_2,$$
diameter $V_2 < 1/2^2$, and
V_2 lies in a component of V_1.

Similarly, there is an open set V_3 satisfying

$$\bar{V}_3 \subset V_2 \cdot U_3,$$
diameter $V_3 < 1/2^3$,
V_3 lies in a component of V_2.

If one continues to get $V_4, V_5, \ldots,$ one finds that X is connected *im kleinen* at $\bar{V}_1 \cdot \bar{V}_2 \ldots$. This contradicts the fact that $\bar{V}_1 \cdot \bar{V}_2 \ldots$ is a point of Y.

Grace **(9)** has given an example of a compact metric continuum that fails to be locally connected anywhere but which contains no open subset with uncountably many components. This shows that Theorem 2 cannot be weakened by replacing the property of not being connected *im kleinen* with the property of not being locally connected.

Property 4. *M contains no simple triod.* A simple triod is the sum of three arcs such that the intersection of any two of them is the same point p. If M contained a simple triod, it would follow from the homogeneity of M that each component of U of Property 3 would contain a simple triod. This would violate the fact that the plane does not contain uncountably many mutually exclusive simple triods. See Theorem 4.

Property 5. *M contains no simple closed curve.* Cohen showed **(8)** that if a bounded homogeneous plane continuum contains a simple closed curve, it is one.

3. Arc components of M. An arc component of a set X is a subset of X maximal with respect to the property that each pair of points of the subset belongs to an arc in X. In this section we show that the closure of each arc component of the assumed homogeneous bounded plane continuum M is homogeneous. In doing this we find it convenient to work with only certain parts of the arc components. These parts are called rays and are defined as follows.

Suppose p and q are two points of the same arc component of M. The sum of all arcs in M that have p as end point and contain q is called a *ray* starting at p. One may note that this ray differs from an ordinary ray of the plane in that it is neither straight nor closed. However, it has a starting point and is the image of an ordinary ray under a $1 - 1$ continuous transformation.

Property 6. *Each ray in M is the sum of a countable number of arcs.* Let p be the starting point of a ray R and $\{p_i\}$ be a countable dense subset of R. Then R is the sum of the arcs $pp_1, pp_2 \ldots$. If there were a point r of R not in any pp_i, we would consider the arc pr. It follows from the homogeneity of M that r is the interior point of an arc so pr may be extended to an arc ps so that r is contained in the interior of ps. Since M contains no simple triod, each p_i belongs to the arc pr. However, $\{p_i\}$ would not be dense in R since no p_i is near s.

Property 7. *For each point p of an arc component A of M, A is the sum of two rays R_1, R_2 starting at p such that $R_1 \cdot R_2 = p$.* It follows from the homogeneity of M that p is an interior point of an arc ab. It follows from the fact that M contains no simple triod (Property 4) that A is the sum of two rays starting at p and going through a, b respectively. Since M contains no simple closed curve (Property 5), these rays intersect only at p.

Property 8. *M has uncountably many arc components.* If M had only countably many arc components, it would follow from Properties 6 and 7 that M is the sum of a countable collection of arcs. It would then follow from the Baire Category Theorem that one of these arcs contains an open subset of M. The homogeneity of M would then imply that M is a 1-manifold. However, the simple closed curve is the only compact connected 1-manifold.

Property 9. *If R is a ray of M and p is a point of $\bar{R}$, one of the rays starting at p lies in $\bar{R}$.* If neither of the rays starting at p lies in $\bar{R}$, p belongs to an arc ab in M such that neither a nor b is a point of $\bar{R}$. It follows from Property 8 and the homogeneity of M that there is an uncountable family $\{h_\alpha\}$ of homeomorphisms of $(ab + \bar{R})$ into M such that if $\alpha \neq \beta$, $h_\alpha(ab)$, $h_\beta(ab)$ belong to different arc components of M.

It follows from Theorem 3 of § 4 that there is an arc A_0 of the collection $\{h_\alpha(ab)\}$ with two sequences $A_1, A_3, \ldots$, and $A_2, A_4, \ldots$, of arcs of $\{h_\alpha(ab)\}$ converging homeomorphically to A_0 from opposite sides. For convenience we suppose $A_0 = ab$. Some two of the arcs A_{2i}, A_{2i+1} near A_0 would separate some point of $\bar{R}$ from A_0 in $\bar{R}$ and hence two points p, q of R from each other in R. But then the arc pq in R would cross either A_{2i} or A_{2i+1} and violate Property 4.

Property 10. *If $\bar{R}_1$ is the closure of a ray of M, it contains a continuum $\bar{R}$ that is irreducible with respect to being the closure of a ray.* Let $D_1, D_2, \ldots$, be a countable basis of open sets for the plane and $R_1, R_2, \ldots$, be a sequence of rays such that

1. R_{i+1} is a ray in $\bar{R}_i$ missing D_i if any ray in $\bar{R}_i$ misses D_i; $R_{i+1} = R_i$ if each ray in $\bar{R}_i$ intersects D_i.

If p is a point common to the elements of the decreasing sequence $\bar{R}_1, \bar{R}_2, \bar{R}_3, \ldots$, it follows from Property 9 that one of the rays R starting at p lies in infinitely many of the $\bar{R}_i$'s. Hence it lies in $\bar{R}_1 \cdot \bar{R}_2 \cdot \bar{R}_3 \ldots$. If R' is a ray

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in $\bar{R}$, it follows from Condition 1 above that R' intersects each D_i that R intersects. Hence $\bar{R}' = \bar{R}$.

Property 11. *If R is a ray in an arc component A of M, $\bar{R} = \bar{A}$.* Assume p is a point in $A - \bar{R}$. It follows from Property 10 and the homogeneity of M that p is the starting point of a ray R' whose closure is irreducible with respect to being the closure of a ray. Then $\bar{R}'$ does not contain R and there is a point q of $R - \bar{R}'$. Let R'' be a ray starting at q whose closure is irreducible with respect to being the closure of a ray. Since neither of the rays R', R'' contains the other and the starting point q of R'' does not belong to R', the rays R', R'' do not intersect. Either some point of pq belongs to both $\bar{R}'$ and $\bar{R}''$ or some point of pq belongs to neither. We show that in either case, the assumption that $\bar{R} \neq \bar{A}$ has led to a contradiction.

If some point r of the arc pq fails to belong to $\bar{R}' + \bar{R}''$, there is no ray starting at r whose closure is irreducible with respect to being the closure of a ray. This violates Property 10 and the homogeneity of M.

If some point r of pq belongs to both $\bar{R}'$ and $\bar{R}''$, there are two mutually exclusive rays in A each missing r and such that r belongs to the closure of each. This violates the homogeneity of M since there are not two mutually exclusive rays in A each missing q such that q belongs to the closure of each.

Property 12. *If the closures of two arc components of M intersect, the closures are equal.* Suppose A_1, A_2 are two arc components whose closures contain the point p. Let A_p be the arc component of M containing p. It follows from Property 9 that one of the rays starting at p lies in $\bar{A}_1$ and from Property 11 that A_p lies in $\bar{A}_1$. Similarly A_p lies in $\bar{A}_2$. It follows from Property 10, Property 11, and the homogeneity of M that the closure of each arc component of M is irreducible with respect to being the closure of an arc component. Hence, $\bar{A}_p = \bar{A}_1 = \bar{A}_2$.

Property 13. *The closure of each arc component A of M is homogeneous.* We show that $\bar{A}$ is homogeneous by showing that if p is a point of A and q is a point of $\bar{A}$, there is a homeomorphism of $\bar{A}$ onto itself taking p to q. The homogeneity of M implies that there is a homeomorphism h of M onto itself taking p to q. Since such a homeomorphism takes arc components onto arc components, it follows from Property 12 that $h(\bar{A}) = \bar{A}$.

If q and r are points of $\bar{A} - A$ and one wishes a homeomorphism of $\bar{A}$ onto itself taking q onto r, one could use the preceding paragraph to show that there is a homeomorphism h_1 of $\bar{A}$ onto itself taking q to p and a homeomorphism h_2 of $\bar{A}$ onto itself taking p to r. The required homeomorphism is h_2h_1.

4. Collections of arcs in the plane. In this section we digress from our consideration of homogeneity to consider collections of arcs in the plane.

Theorem 3 is used in establishing Properties 9 and 15 but is of interest aside from these applications.

We recall the following notions concerning the abutting of arcs in the plane E^2 Suppose ab, cd, and ef are arcs in E^2 such that $ab \cdot cd = c$ and $ab \cdot ef = e$ are interior points of ab. Then cd and ef are said to *abut on opposite sides* of ab if there is a homeomorphism of E^2 onto itself that takes ab onto a horizontal segment and cd, ef onto vertical segments which lie except for their points of contact with ab on opposite sides of the line containing ab.

A sequence of arcs $A_1, A_2, \ldots$, is said to *converge homeomorphically* to an arc A_∞ if for each positive number ϵ there is an integer n such that if $n < i$, there is a homeomorphism of A_i onto A_∞ that moves no point more than ϵ.

Suppose ab, cd, ef are arcs such that cd and ef abut on ab from opposite sides. A sequence of arcs $A_1, A_2, \ldots$, converging homeomorphically to ab is said to *converge homeomorphically from the cd side of ab* if none of the arcs intersect ab and all but possibly a finite number of these arcs intersect cd. Two sequences or arcs converging homeomorphically to ab are said to *converge homeomorphically from opposite sides* if one of the sequences converges from the cd side of ab and the other from the ef side of ab.

THEOREM 3. *If W is an uncountable collection of mutually exclusive arcs in E^2, then there is an element w of W and two sequences of elements of W converging homeomorphically to w from opposite sides.*

This result follows as a corollary of the following result which has a more cumbersome statement.

THEOREM 3′. *Each uncountable collection of mutually exclusive arcs in E^2 has a countable subcollection W' such that each element w_0 of $W - W'$ has the following property:*

For each pair of arcs cd, ef abutting on w_0 from opposite sides and each positive number ϵ there are uncountable subcollections W_1, W_2 of $W - W'$ such that

1. *each element of W_1 intersects cd,*
2. *each element of W_2 intersects ef, and*
3. *for each element w of $W_1 + W_2$ there is a homeomorphism of w onto w_0 that moves no point by more than ϵ.*

Proof of Theorem 3′. Let W' be the collection of all elements w of W with the property that there is an arc cd abutting on w from one side and a positive number ϵ such that no uncountable subcollection W_1 satisfies Conditions 1 and 3 of the statement of Theorem 3′. Theorem 3′ is established by showing that the collection W′ does not have uncountably many elements. Assume W' is uncountable.

For each element w_α of W' let v_α be an arc abutting on w_α from one side and ϵ_α be a positive number such that

4. v_α intersects only a countable number of elements w of W' such that there is a homeomorphism of w_α onto w that moves no point by more than ϵ_α.

Let ϵ' be a positive number so small that for an uncountable subcollection W'' of W', ϵ' will serve as the ϵ_α for each element w_α of W''.

Suppose T is a triod which is the sum of an arc ab and an arc cd abutting on ab from one side. For each element w_α of W'' let h_α be a homeomorphism of $ab + cd = T$ onto $w_\alpha + v_\alpha$ that takes ab onto w_α. Let ρ denote the ordinary distance function for the plane. The homeomorphisms h_α may be regarded as points of a function space metrized as follows:

$$D(h_\alpha, h_\beta) = \max_{t \epsilon T} \rho(h_\alpha(t), h_\beta(t)).$$

Then $\{h_\alpha\}$ is a separable metric space and some element h_0 of it is a limit point of an uncountable order (each neighbourhood of h_0 contains uncountably many points of $\{h_\alpha\}$).

Let H be the set of all elements of $\{h_\alpha\}$ within $\frac{1}{2}\epsilon'$ of h_0 and W''' be the set of all elements of W'' that are images of ab under an element of H. We note that if w_1, w_2 are two elements of W''' then there is a homeomorphism of w_1 onto w_2 that moves no point by more than ϵ'.

For convenience we suppose that $h_0(ab) = w_0$ is the horizontal diameter of a unit circle C with centre at the origin and $h_0(cd) = v_0$ is a vertical radius of C which extends upward. Also, we suppose $\epsilon' < 1$.

Since each element of H is within $\frac{1}{2}\epsilon'$ of h_0, each element of W''' intersects the y-axis. Let p_α be the highest point where w_α intersects this axis. Let p_γ be one of these p_α's which has uncountably many other p_α's above it. But then v_γ intersects all of the w_α's such that p_α lies above p_γ. This contradicts the definition of v_γ given in Condition 4. The assumption that W' was uncountable led to this contradiction.

THEOREM 4. *Suppose $B, B_1, B_2, \ldots,$ is a sequence of mutually exclusive arcs in E^2 such that $B_1, B_3, \ldots,$ and $B_2, B_4, \ldots,$ converge homeomorphically to B from opposite sides. If C is a continuum intersecting each B_i but neither end of B and h is a homeomorphism of $C + B + B_1 + B_2 + \ldots$ into E^2, then $h(B_1), h(B_3), \ldots,$ and $h(B_2), h(B_4), \ldots,$ converges homeomorphically to $h(B)$ from opposite sides.*

Proof. The proof is divided into two steps.

Step 1. *C contains two subcontinua C_1, C_2 such that C_1 intersects all but possibly a finite number of the odd B_i's but no even B_i and C_2 intersects all but possibly a finite number of the even B_i's but no odd B_i.* With no loss of generality we suppose that B is the horizontal interval ab, that each odd B_i intersects the perpendicular bisector of ab at a point above ab and each even B_i intersects this perpendicular bisector at a point below C. The two continua C_1, C_2 that we describe will lie except for their intersections with ab on opposite sides of the line containing ab.

Let ϵ be a positive number so small that neither a nor b lies within ϵ of C. Let K_1, K_2 be circles with centres at a, b respectively with radii equal to

$\frac{1}{2}\epsilon$. Since we can throw away a finite number of B_i's, we suppose that for each B_i there is a homeomorphism of B_i onto B that moves no point by more than $\frac{1}{2}\epsilon$.

Let X_i be an arc of B_i irreducible from K_1 and K_2 and Y_i be the arc from a to b obtained by adding to X_i a radius of K_1 and a radius of K_2. Each of $Y_1, Y_3, \ldots$, lies except for a, b above ab. We suppose that the ordering is such that Y_{2i+1} is above (except at a, b) Y_{2i+3}.

Let D be the disc bounded by $ab + y_1$. If each arc in D from a to b intersects C, it follows from the unicoherence of D that some component C_1 of $D \cdot C$ separates a from b in D. This continuum C_1 intersects each Y_{2i+1} and is the C_1 promised in Step 1. If there is an arc in D from a to b that misses C, there is such an arc Z which intersects ab only at a, b. Let D' be the disc bounded by $ab + Z$ and X_{2i+1} be an arc on the interior of D'. Any component C_1 of $D' \cdot C$ that intersects X_{2i+1} intersects each X_{2j+1} $(j \geqslant i)$ and this C_1 will serve as the C_1 promised by Step 1. The continuum C_2 is obtained in a similar fashion.

Step 2. *If cd and ef are arcs abutting on $h(B)$ from opposite sides and infinitely many of the $h(B_{2i+1})$'s intersect cd, all but a finite number of the $h(B_{2i})$'s intersect ef.* We suppose with no loss of generality that $h(B) = ab$ is a horizontal segment, *cd* is a vertical segment pointing upward from ab, and *ef* is a vertical segment pointing downward from ab.

Let ϵ be a positive number so small that neither a nor b is within ϵ of $cd + ef + h(C_1) + h(C_2)$. Following Step 1, we let K_1, K_2 be circles with centres at a, b respectively and radii equal to $\frac{1}{2}\epsilon$. Since we can disregard any finite collection of the $h(B_i)$'s, we suppose with no loss of generality that there is a homeomorphism of each $h(B_i)$ onto $h(B)$ that moves no point by more than $\frac{1}{2}\epsilon$.

We let X_i be a subarc of $h(B_i)$ irreducible from K_1 to K_2 and Y_i be the arc from a to b obtained by adding to X_i radii of K_1 and K_2 respectively. Since infinitely many of the $h(B_{2i+1})$'s intersect *cd*, infinitely many of the Y_{2i+1}'s lie above ab (except for a, b).

Suppose Y_1 lies above ab (except for a, b) and let D be the disc bounded by $Y_1 + ab$. Since each point of $\operatorname{Int} D$ is separated from ab by a Y_{2i+1} and each Y_{2i+1} misses C_2, C_2 does not intersect the interior of D. Since no X_{2i} lies interior to D (each intersects C_2), all but a finite number of these X_{2i}'s lie below ab. Hence, all but a finite number of the X_{2i}'s (and hence the $h(B_{2i})$'s) intersect *ef*.

Since all but a finite number of the $h(B_i)$'s intersect $cd + ef$ we suppose with no loss of generality that infinitely many $h(B_{2i+1})$'s intersect *cd*. It follows from Step 2 that all but a finite number of the $h(B_{2i})$'s intersect *ef* and by a repetition of Step 2 that all but a finite number of the $h(B_{2i+1})$'s intersect *cd*.

It is known that the plane does not contain uncountably many mutually exclusive triods (**3**, Theorem 5, p. 254). In extending this result to higher dimensions it is convenient to think of a simple triod as having a topological 1-simplex as base and having a feeler sticking out from an interior point of this base. The following theorem is a strengthening of this result concerning triods in the plane.

THEOREM 5. *Suppose W is an uncountable collection of simple triods in the plane such that each of these triods has a designated base and feeler. If no two of the bases of the elements of W intersect, some feeler intersects uncountably many bases.*

Proof. If the bases are mutually exclusive, it follows from Theorem 3′ that there is a base b_0 with uncountably many bases arbitrarily close on either side of b_0. The feeler from b_0 would intersect uncountably many of these nearby bases.

5. The reduced continuum M'. In this section we return to a study of the assumed homogeneous bounded plane continuum M studied in §§ 2 and 3 which contains an arc but is not a simple closed curve. It follows from Properties 5 and 13 that if there is such an M, there is a continuum M' that is the closure of one of its arc components. We list some properties that such an M' would need to possess in order to show that there is no such M' and hence no M. In §§ 5, 6, 10, we use M' to denote a homogeneous bounded plane continuum one of whose arc components is dense in M' but which is not a simple closed curve.

Property 14. *If C is a non-degenerate subcontinuum of M' that is not an arc, C intersects uncountably many arc components of M'.* This is true by Property 8 if $C = M'$ so we suppose C is a proper subcontinuum of M'. Let p be a point of $M' - C$ and A be the arc component of M' containing p. Since each ray is dense in M', there is a sequence of points $p_1, p_{-1}, p_2, p_{-2}, \ldots$, of $A - C$ such that A is the sum of the arcs $p_i p_{i+1}$ and no two of the $p_i p_{i+1}$'s intersect except possibly at an end point of each. If one considers the intersections of these arcs $p_i p_{i+1}$ with C, one finds that $A \cdot C$ is the sum of a countable collection of mutually exclusive closed sets. Since no continuum is the sum of a countable number more than one of mutually exclusive closed point sets, C intersects uncountably many arc components of M'.

Property 15. *Each non-degenerate proper subcontinuum* of *M' is an arc.* If M' contains a non-degenerate proper subcontinuum C that is not an arc, it follows from Property 14 and the fact that each ray is dense in M' that M' contains an uncountable collection of mutually exclusive arcs each of which intersects C but no one of which has an end on C. It follows from Theorem 3 of § 4 that there is one of these arcs B that has two sequences of arcs $B_1, B_3, \ldots$, and $B_2, B_4, \ldots$, of the arcs converging homeomorphically to B from opposite sides. It follows from Theorem 4 of § 4 that under no homeomorphism h of

$C + B + B_1 + B_2 \ldots$ into the plane is the image of any interior point of B accessible from the complement of $h(C + B + B_1 + B_2 + \ldots)$. This violates the homogeneity of M' since some points of it are accessible from the complement of M'.

Property 16. *M' is indecomposable.* If M' were the sum of two proper subcontinua, it would follow from Property 15 that these subcontinua were arcs. The only homogeneous continuum that is the sum of two arcs is a simple closed curve.

6. Continua each of whose proper subcontinua is an arc. A solenoid is a non-degenerate homogeneous compact continuum each of whose proper subcontinua is an arc. Other examples are not at hand. We note that Property 15 shows that M' has this property. The following question is related to the last two given in § 1.

Question. Are solenoids the only non-degenerate homogeneous compact continua each of whose proper subcontinua is an arc?

Theorems 7 and 9 answer this question in the affirmative for the cases of tree-like and circle-like continua.

In developing the following property, we use merely the fact that each proper subcontinuum of M' is an arc (Property 15) rather than the facts that M' is homogeneous and lies in the plane.

Property 17. *For each positive number ϵ and each arc xy in M' there is an ϵ-chain $d_1, d_2, \ldots, d_n$ covering xy such that x, y belong to d_1, d_n respectively and $M' \cdot Bd \sum d_i \subset \bar{d}_1 + \bar{d}_n$.* Let $e_1, e_2, \ldots, e_n$ be an ϵ-chain covering xy such that x, y belong to e_1, e_n respectively. There are open sets O_1, O_2 in e_1, e_2 respectively such that xy is an arc component of $M' - (O_1 + O_2)$. It follows from Property 15 that xy is a component of $M' - (O_1 + O_2)$.

Since no component of $M' - (O_1 + O_2)$ intersects both xy and $M' - \sum e_i$, then $M' - (O_1 + O_2)$ is contained in two mutually exclusive open sets A, B such that $xy \subset A$, $M' - \sum e_i \subset B$ (see **13**, Theorem 35, p. 21). The link d_i of the chain $d_1, d_2, \ldots, d_n$ is defined to be $e_i \cdot (A + O_1 + O_2)$.

Since

$$M' \cdot \sum \bar{d}_i = M' \cdot A + M' \cdot (B + O_1 + O_2) \cdot \sum \bar{d}_i \subset \sum d_i + (\bar{O}_1 + \bar{O}_2),$$

one finds on subtracting $\sum d_i$ fromthe ends of the above inequality that

$$M' \cdot Bd \sum d_i \subset \bar{O}_1 + \bar{O}_2 \subset \bar{d}_1 + \bar{d}_n.$$

Property 18. *For each positive number ϵ there is a positive number δ such that if ab is an arc in M' such that $\rho(a, b) < \delta$, then either diameter $ab < \epsilon$ or ab is ϵ dense in M'.* Assume that there is no such δ. Then for each integer i there is an arc $a_i b_i$ in M' such that

$\rho(a_i, b_i) < 1/i$,
diameter $a_i b_i > \epsilon$,
$a_i b_i$ is not ϵ dense in M'.

Some subsequence of $a_1b_1, a_2b_2, \ldots$, converges to a non-degenerate proper subcontinuum. Hence it is an arc. Some subarc of this arc is the limit of a folded sequence of arcs in M' (each in an a_ib_i). The assumption that there is no δ leads to the contradiction of Theorem 6 of the next section.

Property 19. *If a point p of M' is accessible from a component U of $E^2 - M'$, each point of any arc in M' containing p is accessible from the same side that p is.* Let xy be an arc containing the two points p, q on its interior. With no loss of generality we suppose that xy is horizontal, q is between p and y, and rp is a vertical interval lying except for p below xy and in U. We show that q is accessible from U from below.

Let

$$\epsilon = \min \rho(r, p), \qquad \rho(x, p), \qquad \tfrac{1}{2}\rho(q, y),$$

δ be the positive number promised by Property 18, with $\delta < \epsilon$, and D be a δ-chain covering xy and satisfying the conditions of Property 17. We use D^* to denote the sum of the links of D. If q is not accessible from U from below, there is a point s in $M' \cdot D^*$ which is beneath the point q. Let ab be an arc in $M' \cdot D^*$ containing s such that ab lies below xy and each end of ab is in an end link of D. The arc rp prevents either a or b from being in the end link of D containing x so $\rho(a, b) < \delta$. Also, ab is not ϵ dense in M' since it is not near x. However, diameter $ab > \epsilon$ since $\frac{1}{2}\rho(q, y) \geqslant \epsilon$. The assumption that q was not accessible from U from below led to a contradiction of Property 18.

7. Folded sequences of arcs. A solenoid is an example of a homogeneous continuum each of whose proper subcontinua is an arc. The arcs in this continuum seem to run in a parallel fashion and not to "zig-zag" or "fold back." We find from Theorem 6 that such folding is impossible in a homogeneous continuum each of whose proper subcontinua is an arc. No use is made of the fact that the continuum lies in the plane.

Suppose $a_1b_1, a_2b_2, \ldots$, is a sequence of arcs converging (not necessarily homeomorphically) to an arc xy. The sequence is called a *folded sequence* converging to xy if $a_1, b_1, a_2, b_2, \ldots$, converges to x.

THEOREM 6. *Suppose X is a homogeneous compact continuum each of whose proper subcontinua is an arc. Then no folded sequence of arcs in X converges to an arc.*

Proof. Assume $a_1b_1, a_2b_2, \ldots$, is a sequence of arcs converging to an arc xy such that $a_1, b_1, a_2, b_2, \ldots$, converges to x. If $\epsilon < \frac{1}{2}\rho(x, y)$, y has the following property:

Property (y, ϵ). For each positive integer n there is a $1/n$ chain E such that X intersects each link of E, the distance between y and the first link of D is less than $1/n$, the distance between the end links of E is more than ϵ, and $X \cdot Bd\, E^*$ lies in the closure of the last link of E.

We can obtain such an E as follows.

1. Let D be a $1/n$ chain covering xy such that the first link of D contains y, the last link contains x, and for each other link d, $\bar{d} \cdot X \subset D^*$. The existence of such a D is guaranteed by the proof given in Property 17 of the last section.

2. Let a_ib_i be an arc of the folded sequence converging to xy such that each of a_i, b_i lies in the last link of D, some point of a_ib_i lies in the first link of D, and D covers a_ib_i.

3. Let D' be a chain covering a_ib_i such that D' refines D, the first and last link of D' lie in the last link of D, some link of D' lies in the first link of D, and $X \cdot Bd\, D'^*$ lies in the sum of the closures of the two end links of D'.

Then E can be formed as follows. The jth link of E is the sum of the elements of D' in the jth link of D.

Let $X(\epsilon)$ be the set of all points y of X such that y has Property (y, ϵ). Then $X(\epsilon)$ is closed and it follows from the homogeneity of X that each point of X belongs to some $X(\epsilon)$ for some ϵ. It follows from the Baire Category Theorem that there is an integer m and an open set U such that $U \subset X(1/m)$.

We obtain a sequence of chains $E_1, E_2, \ldots$, such that

1. E_i is a $1/i$ chain such that the distance between its end links is more than $1/m$,

2. $X \cdot Bd\, E_i^*$ lies in the closure of the last link of E_i, and

3. the first link e_1^1 of E_1 lies in U, and the closure of the first link e_1^{i+1} of each E_{i+1} lies in the first link of E_i.

The intersection of the e_1^i's is a point that cannot be the interior point of any arc in X. This contradiction results from the false assumption that there is a folded sequence of arcs in X converging to an arc.

The following result gives an immediate application of Theorem 6. The result is not needed in the proof of Theorem 1 but it can be used in lieu of Property 19 in finishing the proof of Theorem 1 for the case where M does not separate the plane, since each 1-dimensional bounded plane continuum that does not separate the plane is tree-like **(3)**.

THEOREM 7. *There exists no non-degenerate, homogeneous, tree-like continuum each of whose proper subcontinua is an arc.*

Proof. Assume X is a non-degenerate, homogeneous, tree-like continuum each of whose proper subcontinua is an arc. It follows from **(10)** that X is indecomposable. Let D_i be a $1/i$ tree-chain covering X and a_ib_i be an arc in X such that both ends of a_ib_i lie in the same link of D_i and

$$\text{diameter } X/4 - 1/i < \text{diameter } a_ib_i < \text{diameter } X/2.$$

Such an arc a_ib_i may be found by considering an arc of large diameter in X both of whose ends lie in the same link of D_i, reducing this arc by throwing away the part of it in this link and considering one of the larger components of the remainder, reducing the component in a similar fashion, . . . , and stopping this reduction when an arc of the required diameter is found.

Some subsequence of $a_1b_1, a_2b_2, \ldots,$ converges to a proper subcontinuum of X. It follows from the hypothesis that this subcontinuum is an arc ab. However, there is a folded sequence of arcs (each in one of the a_ib_i's) converging to a subarc of ab. This contradicts Theorem 6.

8. A nearly homogeneous example. Consider the example Y shown in Figure 1. At a glance it might appear to be homogeneous. The example Y intersects the x axis in a Cantor set and is the sum of semicircles with ends on this Cantor set. Also, the example may be obtained by starting with a punctured disc with three holes and digging canals into the disc from the four complementary domains of the punctured disc.

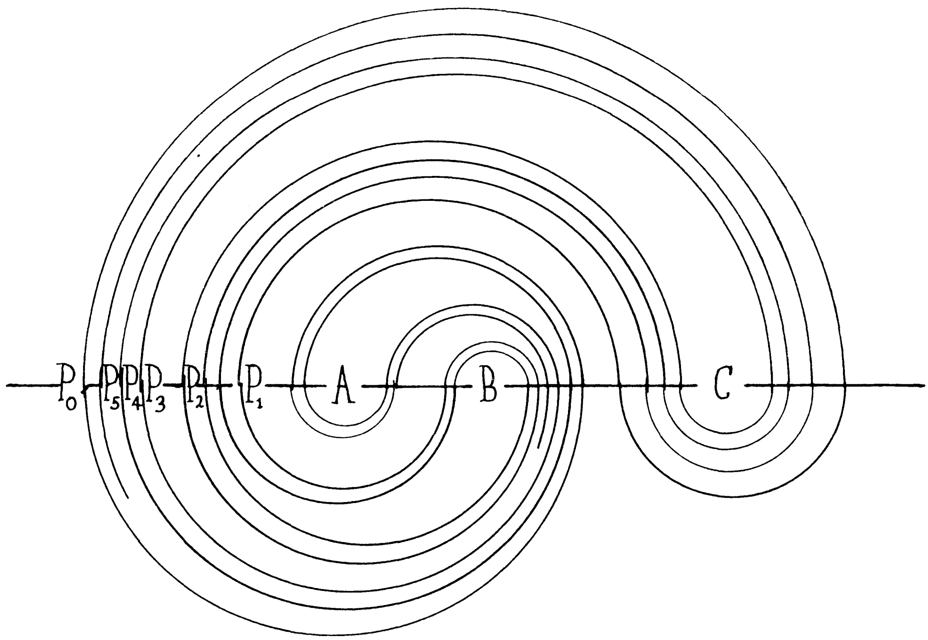

FIGURE 1

The canal from the unbounded complementary domain may be defined in terms of its right bank as follows. Let p_0 be the point furthest to the left of Y and consider the ray R starting at p_0, going along the upper semicircle of X, then along the lower right semicircle, and then down the right bank of the canal leading from the unbounded complementary domain of Y. Let p_1 be

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the first point on R which is between p_0 and A on the x axis, p_2 be the first point of R that is between p_0 and p_1 on the x axis, and in general, p_{i+1} is the first point of R that is between p_0 and p_i on the x axis.

As viewed from C, p_0p_1 circles C. However, it circles neither A nor B when viewed from A, B respectively. Furthermore, p_0p_2 circles B and C but not A, p_0p_3 circles B, p_0p_4 circles A and B, p_0p_5 circles A, p_0p_6 circles A and C, p_0p_7 circles C.... The other canals from A, B, C run between the canals from the unbounded complementary domains and each canal is dense in Y. Figure 1 does not show the banks of the canals from A, B, C but shows only an arc on the outer bank. This arc does not separate the plane. There are points of X not shown in the figure that are nearer A, B, C than any point on the outer bank. These points keep the complementary domains containing A, B, C from running into each other.

We could write the equation of Y by giving functions f, g such that $(x, f(x))$ are abscissas of the ends of semicircles of Y in the upper half-plane and $(x, g(x))$ are the abscissas of the ends of semicircles of Y in the lower half-plane. However, we shall not do this since we are interested in Y's topological properties rather than its equation.

Example Y is locally homogeneous—that is, for each pair of points p, q of Y there are arbitrarily small homeomorphic open subsets N_p, N_q containing p, q respectively. In fact, the open subsets may be taken to be homeomorphic with the Cartesian product of a Cantor set with an open interval.

Also, example Y is nearly homogeneous—if p and q are points of Y, then for each open subset U of Y containing q there is a homeomorphism of Y onto itself taking p into U. One may see that this is true since each arc component is dense in Y and each arc lies in an open subset homeomorphic with the Cartesian product of a Cantor set and an open interval. See **(7)** for a discussion of various types of homogeneity.

However, M is not homogeneous. If it were, for each positive number ϵ and each point p there would be a homeomorphism h of Y onto itself such that h moves no point by more than ϵ and p, $h(p)$ belong to different composants of Y. (See Theorem 8 of § 9.) Suppose that ϵ is taken to be less than the distance across the canal leading from the unbounded complementary domain at a wide point and p is taken to be the highest point of Y. There is a canal leading from the outside that locally separates p from $h(p)$ in the plane. As p is moved parallel to this canal and in the direction of its wide spot, the canal continues to separate the moving p from the corresponding $h(p)$. However, as the canal widens, it is no longer possible for p to be within ϵ of its image under h.

This intuitive reason of why Y is not homogeneous is refined in § 10 to establish Property 20 and finish the proof of Theorem 1.

9. Homeomorphisms near the identity. In indicating why the nearly homogeneous Example Y of § 8 is not homogeneous, we made use of the fact that if Y were homogeneous there would be a homeomorphism of Y onto itself that does not move any point far but which takes one arc component of Y onto another. We formalize this in the following theorem.

THEOREM 8. *If p is a point of a homogeneous, compact, indecomposable, non-degenerate continuum X, then there is a sequence of homeomorphisms h_1, $h_2, \ldots$, converging to the identity such that no two $h_i^{-1}(p)$'s belong to the same composant of X.*

Proof. Let $\{x_\alpha\}$ be an uncountable collection of points all belonging to different composants of X and h_α be a homeomorphism of X onto itself that takes x_α to p.

If the collection of homeomorphisms $\{h_\alpha\}$ is metrized by the distance function

$$D(h_\alpha, h_\beta) = \max_{x \epsilon X} \rho(h_\alpha(x), h_\beta(x)),$$

the collection $\{h_\alpha\}$ becomes an uncountable subset of a separable metric function space and some sequence $h_1', h_2', \ldots$, of elements of $\{h_\alpha\}$ converges to an element h_0' of $\{h_\alpha\}$. Then

$$h_i = h_i' h_0'^{-1}.$$

Since $x_1, x_2, \ldots$, belong to different composants of X, $h_0'(x_1), h_0'(x_2), \ldots$, also belong to different composants. These are the $h_i^{-1}(p)$'s.

10. M' contains a folded sequence of arcs. In this section we complete the proof of Theorem 1. We showed in §§ 2 and 3 that if there is a homogeneous bounded plane continuum M that contains an arc, there is one such M' each of whose proper subcontinua is an arc. Theorem 6 showed that no such M' contains a folded sequence of arcs converging to an arc. Finally, we show that there is no such M' except a circle, for if there were, it would have the following property.

Property 20. *M' contains a folded sequence of arcs converging to an arc.* Let a_0a_6 be an arc in M' which is accessible from a component of $E^2 - M'$. With no loss of generality we suppose that a_0a_6 is horizontal and $a_1, a_2, \ldots, a_5$ are points of a_0a_6 such that

$$\text{abscissa } a_i = i \quad (i = 0, 1, \ldots, 6).$$

We suppose furthermore that a_0a_6 is accessible from $E^2 - M$ from below. It follows from the methods used in establishing Property 19 that there is a positive number ϵ_1 such that

$$\text{no point of } M' \text{ below } a_0a_6 \text{ is within } \epsilon_1 \text{ of } a_1a_5.$$

Assume M' contains no folded sequence of arcs converging to an arc. Then there is a positive number ϵ_2 such that if D is an ϵ_2-chain covering a_1a_5 with

a_1 in one end link of D and a_5 in the other, then each arc of M' being covered by D and having both ends in the same link of D is of diameter less than $\frac{1}{2}$. Note that $\epsilon_2 \leqslant \frac{1}{2}$. Let D be such an ϵ_2 chain covering a_1a_5 satisfying conditions of Property 17.

Let rs be an arc such that rs lies above a_1a_5; rs misses M'; rs is irreducible from the vertical line containing a_1 to the vertical line containing a_5; the vertical segments ra_1, sa_5 lie in end links of D, and each point between rs and a_1a_5 lies in a link of D. We find that there is such an rs as follows. Cover a_0a_6 by a chain of small mesh satisfying the conditions of Property 17, consider an accessible point of M' above a_3 and in one link of this chain, and note from Property 17 and Theorem 6 that this point lies in an accessible arc in M' slightly above a_0a_6 and with ends near the ends of a_0a_6. It follows from Property 19 that there is an arc in the complement of M' slightly to one side of this first arc. It is this second arc that contains rs.

Let K be the topological disc bounded by a_1a_5, a_1r, rs, and a_5s. We note that if p is a point of $M'\cdot K$ that is above a_2a_4, then the closure of the component of $M'\cdot \text{Int}\, K$ containing p is an arc irreducible from a_1r to a_5s. If it were not, an arc being covered by D and having diameter more than $\frac{1}{2}$ would have ends in the same link of D.

Let

$$\epsilon_3 = \min\,(\epsilon_1, \rho(rs, M')).$$

It follows from Property 18 that there is a positive number ϵ_4 such that if ab is an arc in M' with $\rho(a, b) < \epsilon_4$, then either

diameter $ab < \epsilon_3$, or
ab is ϵ_3 dense in M'.

Let A be the arc component of M' containing a_3. It follows from Theorem 8 that there is a homeomorphism h of M' onto itself that moves no point by more than ϵ_4 and which takes a_3 into a point of $M' - A$. Then $h(a_3)$ is a point of K and lies above a_2a_4. Also, $h(a_3)$ lies on an arc in $M'\cdot K$ that is irreducible from a_1r to a_5s.

Since A is dense in M', there is an arc xy in $A\cdot K$ such that xy is irreducible from a_1r to a_5s and xy separates $h(a_3)$ from a_1a_5 in K. By considering points slightly above a_3 we find that xy has the following property.

Special Separating Property. The arc xy separates two points of $K\cdot(M' - A)$ from each other in K such that the first of the points is above a_3 and the other is the image of the first under h.

Let $x_1x_2x_3\ldots x_{2n}$ be the arc in A such that $x_1x_2 = xy$, $x_{2n-1}x_{2n} = a_1a_5$, and x_1x_2, x_3x_4, x_5x_6, $\ldots$, $x_{2n-1}x_{2n}$ are the closures of the components of $x_1x_2x_3\ldots x_{2n}\cdot$ Int K that are irreducible from a_1r to a_5s. Then x_1x_2 has the special separating property but $x_{2n-1}x_{2n}$ does not.

We now show that if $x_{2i-1}x_{2i}$ has the special separating property, then so does $x_{2i+1}x_{2i+2}$. The resulting contradiction arises as a result of consequences

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of our false assumption that there is an M' that contains no folded sequence of arcs converging to an arc.

Suppose p, $h(p)$ are points of $K \cdot (M' - A)$ such that p is above a_3 and $x_{2i-1}x_{2i}$ separates p from $h(p)$ in K. For convenience we suppose that $x_{2i+1}x_{2i+2}$ is below $x_{2i-1}x_{2i}$ and $h(p)$ is above $x_{2i-1}x_{2i}$. (Other cases are handled with similar arguments to that given in this case.) Then $x_{2i+1}x_{2i+2}$ separates p from $h(p)$ in K unless p is above $x_{2i+1}x_{2i+2}$, so we suppose p is between $x_{2i-1}x_{2i}$ and $x_{2i+1}x_{2i+2}$. Our proof now breaks down into two cases.

Case 1. *If* x_{2i}, x_{2i+1} *belong to the same one of* a_1r, a_5s, (see Figure 2). Let tu be the closure of the component of $M' \cdot \text{Int } K$ containing p. It is an arc irreducible from a_1r to a_5s. We suppose u belongs to the vertical line through a_1 containing x_{2i} and x_{2i+1}.

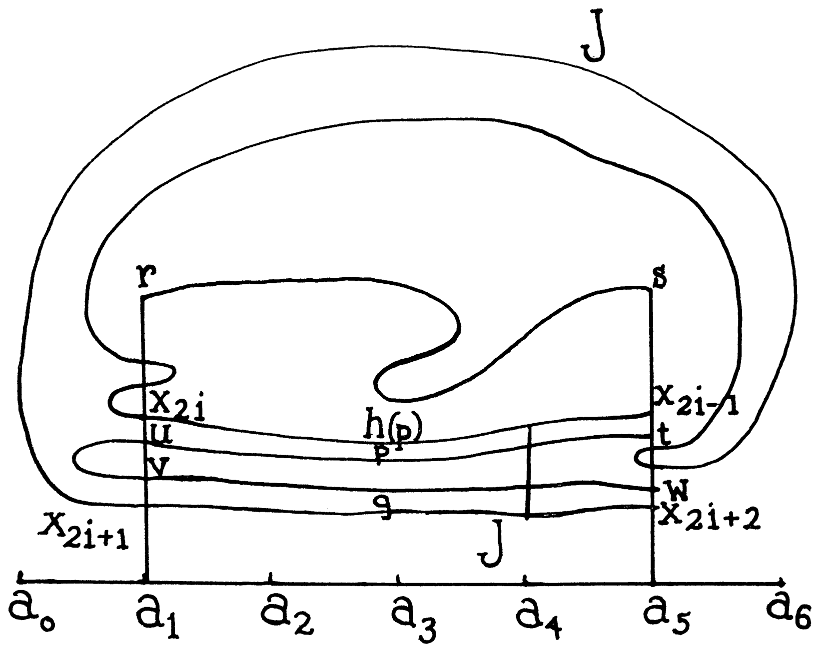

FIGURE 2

Suppose a point moves in an arc in M' through p, past u, and (vw) is the next component of $M' \cdot \text{Int } K$ it meets whose closure vw is an arc irreducible from a_1r to a_5s. Let q be a point of vw directly above a_3. It follows from the Jordan curve theorem that q lies between $x_{2i-1}x_{2i}$ and $x_{2i+1}x_{2i+2}$. Also, q is below tu and $\rho(q, x_{2i-1}x_{2i}) \geqslant \epsilon_4$ or else the arc $tuvw$ contains an arc ab such that

$$\rho(a, b) < \epsilon_4,$$
$$\text{diameter } ab > \epsilon_3, \text{ and}$$
$$\rho(ab, a_4) > \epsilon_3.$$

If q were above tu, we would take $p = a$ and let b be a point of vw between p and $x_{2i-1}x_{2i}$. If q were below tu and $\rho(q, x_{2i-1}x_{2i}) < \epsilon_4$, we would take q to be b and a to be a point of tu between q and $x_{2i-1}x_{2i}$. Since the existence of such an arc ab violates the definition of ϵ_4, we suppose that

$$\rho(q, x_{2i-1}x_{2i}) \geqslant \epsilon_4.$$

We now show that $x_{2i+1}x_{2i+2}$ separates q from $h(q)$ in K. Note that q is above $x_{2i+1}x_{2i+2}$.

Consider the simple closed curve J that is the sum of a vertical interval in K above a_4 and an arc in $x_{2i-1}x_{2i+2}$ that contains $x_{2i}x_{2i+1}$. Note that no point of the arc pq in M is within ϵ_4 of this vertical part of J above a_4. As a point moves from p to q, the image of the point under h does not intersect J. Hence, $h(q)$ is either above $x_{2i-1}x_{2i}$ or below $x_{2i+1}x_{2i+2}$. It is not above $x_{2i-1}x_{2i}$ because $\rho(q, x_{2i-1}x_{2i}) > \epsilon_4$ and $\rho(q, h(q)) < \epsilon_4$. Hence $x_{2i+1}x_{2i+2}$ has the special separation property.

Case 2. *If x_{2i} and x_{2i+1} belong to different vertical lines, see Figure* 3. We suppose x_{2i}, u, x_{2i+2} belong to a_5s and define vw and q as in Case 1.

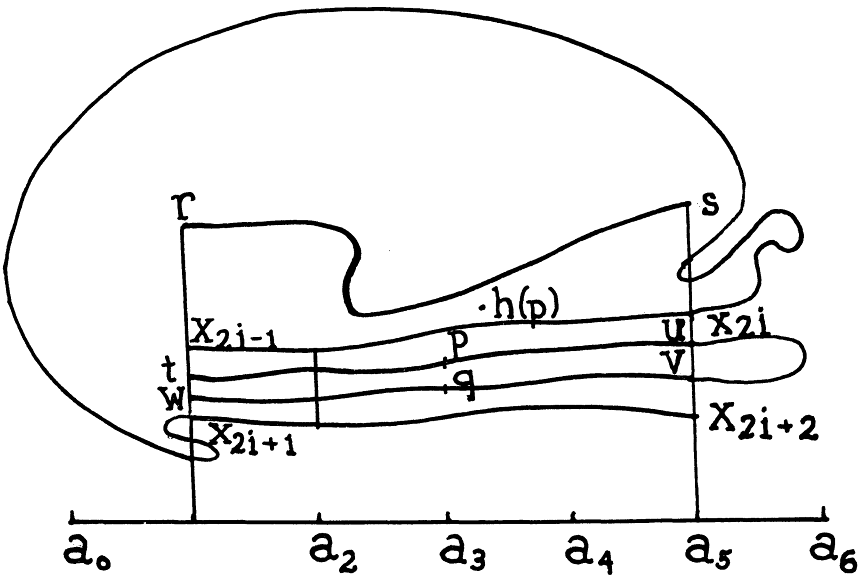

FIGURE 3

If v is below $x_{2i+1}x_{2i+2}$, q is below $x_{2i+1}x_{2i+2}$ and $h(q)$ is above.

If v is above $x_{2i+1}x_{2i+2}$ it is between the points x_{2i} and x_{2i+2}, q is between $x_{2i-1}x_{2i}$ and $x_{2i+1}x_{2i+2}$, and $h(q)$ is above $x_{2i-1}x_{2i}$. Since each of p, q is within ϵ_4 of $x_{2i-1}x_{2i}$, it follows as in Case 1 that $tuvw$ contains an arc ab such that

$\rho(a, b) < \epsilon_4$,
diameter $ab > \epsilon_3$,
$\rho(ab, a_2) > \epsilon_3$.

These conditions violate the definition of ϵ_4.

We note that in establishing Property 20 we used properties of the plane or 2-sphere and not just properties of an arbitrary 2-manifold alone. If this could be by-passed, one might get an affirmative answer to the following question.

Question. Suppose X is a 1-dimensional homogeneous compact continuum that contains an arc and lies on a compact 2-manifold. Is X necessarily a simple closed curve?

11. Circle-like continua and tree-like continua. Solenoids and a circle of pseudo-arcs are the known examples of homogeneous circle-like continua. Since each proper subcontinuum of a circle-like continuum is snake-like and each homogeneous non-degenerate snake-like continuum is a pseudo-arc, one might suspect that the answer to the following is in the affirmative.

Question. Does each homogeneous circle-like continuum other than a solenoid contain a pseudo-arc? We do not provide an answer.

THEOREM 9. *Each homogeneous circle-like continuum that contains an arc is a solenoid.*

Indication of proof. This theorem is much easier to establish than Theorem 1 but the same method of attack may be used.

By using rays as in § 3, it may be shown that the homogeneous circle-like continuum X contains a non-degenerate subcontinuum X' such that each proper subcontinuum of X is an arc. In proving the counterpart of Property 9, we cannot use Theorem 3 (which is a theorem about the plane) to show that $ab + \bar{R}$ cannot lie in X, but instead we use the fact that each proper subcontinuum of X is snake-like to prove this.

We may as well suppose that $X' = X$, for if it is not, it is snake-like, it is a pseudo-arc **(5)**, and it contains no arc.

We finish the indication of proof of Theorem 9 by showing that there is a sequence of circular chains (open coverings whose 1-nerves are simple closed curves) $D_1, D_2, \ldots$, covering X such that

1. D_{i+1} is a refinement of D_i,
2. D_{i+1} circles around $D_i n_i$ times without any folding back, and
3. the mesh of D_{i+1} is less than $1/2^i$ times the distance between any two non-adjacent elements of D_i.

It is then only a matter of getting an open covering of a similar kind of the solenoid which is the intersection of the tori $T_1, T_2, \ldots$, where T_{i+1}

winds about T_i n_i times and use the two coverings to get a homeomorphism of X onto the solenoid. (D_{i+1} is said to fold back in D_i if D_i contains two adjacent links d_x, d_y, and D_{i+1} contains a subchain E such that each link of E lies in either d_x or d_y, each end link of E intersects $d_x - d_y$ but not each link of E lies in d_x.)

Suppose that D_i has already been obtained and it is such that there is a positive number ϵ such that if D' is any circular chain of mesh less than ϵ covering X, then D' refines D_i and circles about D_i without any folding back. We show how to get D_{i+1}, With no loss of generality we suppose that $\epsilon < 1/2^i$ times the distance between any two non-adjacent elements of D_i.

We apply Theorem 6 to show that no folded sequence of arcs in X converges to an arc. Hence, there is a δ such that if ab is an arc of diameter greater than $\epsilon/14$, no δ chain D'' covers ab in such a way that both a, b lie in the same link of D''. We suppose $\delta < \epsilon/14$.

Let D be a δ circular chain covering X with its links ordered $d_1, d_2, \ldots, d_n$. Let $d_{n_1} = d_1$; d_{n_2} is the first link of D whose distance from d_{n_1} is more than $\epsilon/14$; d_{n_3} is the next link of D after d_{n_2} whose distance from d_{n_2} is more than $\epsilon/14 \ldots$. Let $d_{n_{2r-1}}$ or $d_{n_{2r}}$ be the last such link obtained.

The first link of D_{i+1} is the sum of the links between d_{n_1} and d_{n_4} inclusive; the next link of D_{i+1} is the sum of the links of D between d_{n_3} and d_{n_6} inclusive; $\ldots$; and the last link of D_{i+1} is the sum of the links between $d_{n_{2r-1}}$ and d_{n_2} inclusive (this link contains d_n and d_1). Each link of D_{i+1} other than the last is of diameter less than $4\epsilon/14 + 7\delta$ and the last is of diameter less than $5\epsilon/14 + 9\delta$. In either case, D is of mesh less than ϵ. If D' is a refinement of D_{i+1} of mesh less than δ, D' circles about D_{i+1} without any folding back.

A triodic continuum is the sum of three continua A, B, C such that $A \cdot B = A \cdot C = B \cdot C$ is a proper subcontinuum of each of A, B, C. Theorem 7 did not provide an answer as to whether or not each homogeneous tree-like continuum fails to contain an arc. Our methods do not give this because we fail to prove the counterpart of Properties 4 and 9.

THEOREM 10. *A homogeneous tree-like continuum contains no arc if it contains no triodic continuum.*

To establish Theorem 10 we use the hypothesis that the continuum contains no triodic continuum to establish the counterpart of Property 9. Property 15 then follows and reduces Theorem 10 to Theorem 7.

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