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NONINCREASING DEPTH FUNCTIONS OF MONOMIAL IDEALS

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Abstract. Given a nonincreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ such that (i) $f(k) - f(k+1) \leq 1$ for all $k \geq 1$ and (ii) if $a = f(1)$ and $b = \lim_{k\to\infty} f(k)$, then $|f^{-1}(a)| \leq |f^{-1}(a-1)| \leq \cdots \leq |f^{-1}(b+1)|$, a system of generators of a monomial ideal $I \subset K[x_1, \ldots, x_n]$ for which depth $S/I^k = f(k)$ for all $k \geq 1$ is explicitly described. Furthermore, we give a characterization of triplets of integers (n, d, r) with $n > 0, d \geq 0$ and $r > 0$ with the properties that there exists a monomial ideal $I \subset S = K[x_1, \ldots, x_n]$ for which $\lim_{k\to\infty}$ depth $S/I^k = d$ and dstab$(I) = r$, where dstab(I) is the smallest integer $k_0 \geq 1$ with depth $S/I^{k_0} =$ depth $S/I^{k_0+1} =$ depth $S/I^{k_0+2} = \cdots$.

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1. Introduction. The study on depth of powers of ideals, which originated in **[4]**, has been achieved by many authors in the last decade. Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K and $I \subset S$ a homogeneous ideal. The numerical function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ defined by $f(k) =$ depth S/I^k is called the *depth function* of I. It is known [**1**] that $f(k) =$ depth S/I^k is constant for $k \gg 0$. We call $\lim_{k\to\infty} f(k)$ the *limit depth* of I. The smallest integer $k_0 \geq 1$ for which $f(k_0) = f(k_0+1) = f(k_0+2) = \cdots$ is said to be the *depth stability number* of I and is denoted by dstab(I).

An exciting conjecture [**4**, p. 549] is that *any* convergent function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ can be the depth function of a homogeneous ideal. In [**4**, Theorem 4.1], given a bounded nondecreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$, a system of generators of a monomial ideal I for which depth $S/I^k = f(k)$ for all $k \geq 1$ is explicitly described. In [**3**, Theorem 4.9], it is shown that, given a nonincreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$, there exists a monomial ideal Q for which depth $S/Q^k = f(k)$ for all $k \geq 1$. Unlike the proof of [**4**, Theorem 4.1], since the proof of [**3**, Theorem 4.9] relies on induction on $\lim_{k\to\infty} f(k)$, no explicit description of a system of generators of a monomial ideal Q is provided.

Our original motivation to organize this paper was to find an explicit description of a system of generators of a monomial ideal Q of [**3**, Theorem 4.9]. However, there seems to be a gap in the proof of [**3**, Theorem 4.9]. In the proof of this theorem, the authors use an inductive method for giving a monomial ideal which has the desired depth function. However, unfortunately, their inductive method does not work in the case that the limit depth is 0. In fact, it cannot be valid for the nonincreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ with $f(1) = f(2) = 2$ and $f(3) = f(4) = \cdots = 0$ from their inductive method.

In the present paper, given a nonincreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ such that

- $f(k) - f(k+1) \leq 1$ for all $k \geq 1$;
- if $a = f(1)$ and $b = \lim_{k\to\infty} f(k)$, then

$$|f^{-1}(a)| \leq |f^{-1}(a-1)| \leq \cdots \leq |f^{-1}(b+1)|,$$

a system of generators of a monomial ideal I for which depth $S/I^k = f(k)$ for all $k \geq 1$ is explicitly described (Theorem 2.1). The statement of Theorem 1.1 is the exact one that [**3**, Theorem 4.9] would correctly prove. Furthermore, we give a characterization of triplets of integers (n, d, r) with $n > 0$, $d \geq 0$ and $r > 0$ with the properties that there exists a monomial ideal $I \subset S = K[x_1, \ldots, x_n]$ for which $\lim_{k\to\infty}$ depth $S/I^k = d$ and $\operatorname{dstab}(I) = r$ (Theorem 3.1).

2. Nonincreasing depth functions. Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K with each $\deg x_i = 1$.

In this section, we show the following theorem.

THEOREM 2.1. *Given a nonincreasing function* $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ *such that*

- $f(k) - f(k+1) \leq 1$ *for all* $k \geq 1$*;*
- *if* $a = f(1)$ *and* $b = \lim_{k\to\infty} f(k)$*, then*

$$|f^{-1}(a)| \leq |f^{-1}(a-1)| \leq \cdots \leq |f^{-1}(b+1)|,$$

there is a monomial ideal I for which depth $S/I^k = f(k)$ *for all* $k \geq 1$.

At first, we prepare some lemmas to prove Theorem 2.1.

LEMMA 2.2 [**6**, Corollary 5.11]. *Let I be a monomial ideal in S. Then, for any integer $k \geq 1$, we have*

$$\operatorname{depth} I^{k-1}/I^k = \min\{\operatorname{depth} I^{k-1}, \operatorname{depth} I^k - 1\}.$$

LEMMA 2.3. *Let I be a monomial ideal in S. Then, the following are equivalent:*

(a) depth S/I^k *is nonincreasing.*
(b) depth I^{k-1}/I^k *is nonincreasing.*

Moreover, when this is the case, depth S/I^k = depth I^{k-1}/I^k *for any* $k \geq 1$.

Proof. Set $f(k) = \operatorname{depth} S/I^k$ and $g(k) = \operatorname{depth} I^{k-1}/I^k$. Since we obtain depth I^k = depth $S/I^k + 1$ for any $k \geq 1$, by Lemma 2.2, it is obvious that

$$g(k) = \min\{f(k-1) + 1, f(k)\}, k = 1, 2, \ldots.$$

Hence, we know that if $f(k)$ is nonincreasing, then we have $g(k) = f(k)$.

On the other hand, we assume that $g(k)$ is nonincreasing. If $f(t) = g(t)$ for an integer $t \geq 1$, then we have $f(t+1) = g(t+1)$. Since $f(1) = g(1)$, it follows that for any integer $k \geq 1$, $f(k) = g(k)$. □

LEMMA 2.4. *Set $A = K[x_1, \ldots, x_{n'}]$ and $B = K[x_{n'+1}, \ldots, x_n]$, and we let I, J are monomial ideals in A and B. Then, for any integer $t \geq 1$, we have*

$$\operatorname{depth}(I+J)^{t-1}/(I+J)^t = \min_{\substack{i+j=t+1 \\ i,j\geq 1}}\{\operatorname{depth} I^{i-1}/I^i + \operatorname{depth} J^{j-1}/J^j\}.$$

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Proof. It follows by combining [**3**, Theorem 3.3 (i)] and [**6**, Theorem 1.1]. □

The following proposition is important in this paper.

Proposition 2.5. *Let $t \geq 2$ be an integer, and we set a monomial ideal $I = (x^t, xy^{t-2}z, y^{t-1}z)$ in $B = K[x, y, z]$. Then*

$$\operatorname{depth} B/I^n = \begin{cases} 1, & \text{if } n \leq t-1, \\ 0, & \text{if } n \geq t. \end{cases}$$

Proof. First of all, for each integer $n \geq t$, we show that depth $B/I^n = 0$. For this purpose, we find a monomial belonging to $(I^n : \mathfrak{m}) \setminus I^n$, where $\mathfrak{m} = (x, y, z)$. We claim that the monomial $u = x^{tn-t^2+t}y^{t^2-2t}z^{t-1}$ belongs to $(I^n : \mathfrak{m}) \setminus I^n$. Indeed, each generator of I^n forms

$$w(a, b, c) := (x^t)^a(xy^{t-2}z)^b(y^{t-1}z)^c = x^{ta+b}y^{(t-2)b+(t-1)c}z^{b+c},$$

where $a + b + c = n$ and $a, b, c \geq 0$. Then, we have

$$xu | w(n - t + 1, 1, t - 2),$$

$$yu | w(n - t + 1, 0, t - 1),$$

$$zu | w(n - t, t, 0).$$

Thus, $u \in (I^n : \mathfrak{m})$. While the degree of u is less than that of generators in I^n. Hence, we obtain $u \notin I^n$.

Next, we show that pd $I^n = 1$ for all $1 \leq n \leq t - 1$. In order to prove this, we use the theory of *Buchberger graphs*. Let $m_1, \ldots, m_s$ be the generators of I^n. The Buchberger graph Buch(I^n) has vertices $1, \ldots, s$ and an edge (i, j) whenever there is no monomial m_k such that m_k divides lcm(m_i, m_j) and the degree of m_k is different from lcm(m_i, m_j) in every variable that occurs in lcm(m_i, m_j). Then, it is known that the syzygy module syz(I^n) is generated by syzygies

$$\sigma_{ij} = \frac{\operatorname{lcm}(m_i, m_j)}{m_i}\mathbf{e}_i - \frac{\operatorname{lcm}(m_i, m_j)}{m_j}\mathbf{e}_j$$

corresponding to edges (i, j) in Buch(I^n) [**5**, Proposition 3.5].

Let $G(I^n) := \{w(a, b, c) = x^{ta+b}y^{(t-2)b+(t-1)c}z^{b+c} \mid a, b, c \geq 0, a + b + c = n\}$ be the set of generators of I^n. We introduce the following lexicographic order $<$ on $G(I^n)$. Let $w(a, b, c), w(a', b', c') \in G(I^n)$. Then, we define

- $w(a', b', c') < w(a, b, c)$ if $a' < a$;
- $w(a', b', c') < w(a, b, c)$ if $a' = a$ and $b' < b$.

Observation 2.6. For $w = x^ay^bz^c$, we denote $\deg_x w = a$, $\deg_y w = b$ and $\deg_z w = c$. It is easy to see that

- $\deg_x w(a', b', c') < \deg_x w(a, b, c)$ if and only if $w(a', b', c') < w(a, b, c)$;
- $\deg_y w(a', b', c') \geq \deg_y w(a, b, c)$ if $w(a', b', c') < w(a, b, c)$;

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- $\deg_z w(a', b', c') \geq \deg_z w(a, b, c)$ if $w(a', b', c') < w(a, b, c)$

if $1 \leq n \leq t-1$.

To construct the minimal free resolution of I^n, we compute generators of $\mathrm{syz}(I^n)$. For $w(a, b, c), w(a', b', c') \in G(I^n)$, we define $w(a', b', c') \lessdot w(a, b, c)$ if $w(a', b', c') < w(a, b, c)$ and there is no monomial $w \in G(I^n)$, such that $w(a', b', c') < w < w(a, b, c)$. Moreover, we put

$$\sigma((a, b, c), (a', b', c')) := \frac{\mathrm{lcm}(w(a, b, c), w(a', b', c'))}{w(a, b, c)} \mathbf{e}_{(a,b,c)} - \frac{\mathrm{lcm}(w(a, b, c), w(a', b', c'))}{w(a', b', c')} \mathbf{e}_{(a',b',c')}.$$

We show that

Claim 1. If $w(a', b', c') \lessdot w(a, b, c)$, then $\{w(a', b', c'), w(a, b, c)\}$ is an edge of $\mathrm{Buch}(I^n)$.

Proof of Claim 1. Note that $w(a', b', c') \lessdot w(a, b, c)$ if and only if either $a' = a, b' = b - 1$ and $c' = c + 1$ or $(a, b, c) = (a, 0, n - a)$ and $(a', b', c') = (a - 1, n - a + 1, 0)$. In the former case, we have $\mathrm{lcm}(w(a, b, c), w(a, b - 1, c + 1)) = x^{ta+b} y^{(t-2)(b-1)+(t-1)(c+1)} z^{n-a}$ from Observation 2.6. It is enough to show that there is no monomial $w \in G(I^n)$ such that $w \mid \mathrm{lcm}(w(a, b, c), w(a, b - 1, c + 1))/xyz = x^{ta+b-1} y^{(t-2)(b-1)+(t-1)(c+1)-1} z^{n-a-1}$.

Assume that there exists such a monomial $w \in G(I^n)$. Then, $\deg_x w \leq ta + b - 1$. Hence, $w \leq w(a, b - 1, c + 1)$ from Observation 2.6. However, $\deg_z w \geq b + c = n - a$ from Observation 2.6 again, this is a contradiction.

Next, we consider the latter case, that is, $(a, b, c) = (a, 0, n - a)$ and $(a', b', c') = (a - 1, n - a + 1, 0)$. As in the former case, it is enough to show that there is no monomial $w \in G(I^n)$, such that $w \mid \mathrm{lcm}(w(a, 0, n - a), w(a - 1, n - a + 1, 0))/xyz = x^{ta-1} y^{(t-2)(n-a+1)-1} z^{n-a}$. Assume that there exists such a monomial $w \in G(I^n)$. Then, $\deg_x w \leq ta - 1$ and $w \leq w(a - 1, n - a + 1, 0)$ from Observation 2.6. But, we have $\deg_z w \geq n - a + 1$ from Observation 2.6 again, this is a contradiction.

Therefore, we have the desired conclusion. □

Here, we put $\Sigma := \{\sigma((a, b, c), (a', b', c')) \mid w(a', b', c') \lessdot w(a, b, c)\}$. Next, we will show the following:

Claim 2. Assume that $w(a', b', c') < w(a, b, c)$ and $w(a', b', c') \not\lessdot w(a, b, c)$. Then, $\sigma((a, b, c), (a', b', c'))$ can be expressed as an B-linear combination of the elements of Σ.

Proof of Claim 2 Let $s \geq 3$ and assume that

$$w(a', b', c') = w(a_s, b_s, c_s) \lessdot w(a_{s-1}, b_{s-1}, c_{s-1}) \lessdot \cdots \lessdot w(a_1, b_1, c_1) = w(a, b, c).$$

From Observation 2.6, we can see that

$$\frac{\mathrm{lcm}(w(a_1, b_1, c_1), w(a_s, b_s, c_s))}{\mathrm{lcm}(w(a_i, b_i, c_i), w(a_{i+1}, b_{i+1}, c_{i+1}))}$$

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is a monomial in B for all $1 \le i \le s-1$. Hence, we have

$$\begin{aligned}\sigma((a,b,c),(a',b',c')) &= \sigma((a_1,b_1,c_1),(a_s,b_s,c_s)) \\ &= \sum_{i=1}^{s-1} \frac{\mathrm{lcm}(w(a_1,b_1,c_1),w(a_s,b_s,c_s))}{\mathrm{lcm}(w(a_i,b_i,c_i),w(a_{i+1},b_{i+1},c_{i+1}))}\sigma((a_i,b_i,c_i),(a_{i+1},b_{i+1},c_{i+1})).\end{aligned}$$

Thus, we have the desired conclusion. □

CLAIM 3. The elements of Σ are linearly independent on B.

Proof of Claim 3 Assume that $w(a',b',c') \lessdot w(a,b,c)$. Recall that $w(a',b',c') \lessdot w(a,b,c)$ if and only if

$$\begin{aligned}&\text{Case}(1): \quad a=a', b=b'+1, c=c'-1;\\ &\text{Case}(2): \quad a=a'+1, b=0, c=n-a, b'=n-a', c'=0.\end{aligned}$$

In addition, $\mathrm{lcm}(w(a,b,c),w(a',b',c')) = x^{ta+b}y^{(t-2)b'+(t-1)c'}z^{b'+c'}$ from Observation 2.6. Hence, we have that each element of Σ is the following form:

$$\sigma((a,b,c),(a',b',c')) = \begin{cases} y\mathbf{e}_{(a,b,n-a-b)} - x\mathbf{e}_{(a,b-1,n-a-b+1)} & \text{Case}(1);\\ y^{t-n-2+a}z\mathbf{e}_{(a,0,n-a)} - x^{t-n-1+a}\mathbf{e}_{(a-1,n-a+1,0)} & \text{Case}(2).\end{cases}$$

Now, we assume that

$$\sum_{\sigma((a,b,c),(a',b',c'))\in\Sigma} u_{\sigma((a,b,c),(a',b',c'))}\,\sigma((a,b,c),(a',b',c')) = 0$$

for some $u_{\sigma((a,b,c),(a',b',c'))} \in B$. Then, we have the following equalities:

$$u_{\sigma((n,0,0),(n-1,1,0))}\, y^{t-2}z = 0;$$

$$-u_{\sigma((a,b,n-a-b),(a,b-1,n-a-b+1))}\, x + u_{\sigma((a,b-1,n-a-b+1),(a,b-2,n-a-b+2))}\, y = 0 \qquad (0 \le a \le n-1,\ 2 \le b);$$

$$-u_{\sigma((a,1,n-a-1),(a,0,n-a))}\, x + u_{\sigma((a,0,n-a),(a-1,n-a+1,0))}\, y^{t-n-2+a} = 0 \ \ (1 \le a \le n-1);$$

$$-u_{\sigma((a,0,n-a),(a-1,n-a+1,0))}\, x^{t-n-1+a} + u_{\sigma((a-1,n-a+1,0),(a-1,n-a,1))}\, y = 0 \ \ (1 \le a \le n);$$

$$u_{\sigma((0,1,n-1),(0,0,n))}\, x = 0.$$

Hence, $u_{\sigma((n,0,0),(n-1,1,0))} = 0$. Therefore, we have that $u_{\sigma((a,b,c),(a',b',c'))} = 0$ for all $\sigma((a,b,c),(a',b',c')) \in \Sigma$. □

Let us return the proof of Proposition 2.5. By Claim 1, 2 and [**5**, Proposition 3.5], Σ is the set of generators of $\mathrm{syz}(I^n)$. Moreover, by Claim 3, the elements of Σ are

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linearly independent on B. Hence,

$$0 \to \bigoplus_j B(-j)^{\beta_{1,j}} \to B(-nt)^{\beta_{0,nt}} \to I^n \to 0$$

is the minimal free resolution of I^n. Therefore, we have $\operatorname{pd} I^n = 1$. □

Now, we can prove Theorem 2.1.

Proof of Theorem 2.1 First, for any integers $i, k \geq 1$, we define the monomial ideal $I_{k,i} := (x_i^{k+1}, x_i y_i^{k-1} z_i, y_i^k z_i)$ in $B_i = K[x_i, y_i, z_i]$. Then, by Proposition 2.5, we obtain

$$\operatorname{depth} B_i/I_{k,i}^t = \begin{cases} 1, & \text{if } t \leq k, \\ 0 & \text{if } t > k. \end{cases}$$

Set $n = a - b$ and $s_i := |f^{-1}(a - i + 1)|$ for each $1 \leq i \leq n$. We show that $I = \sum_{i=1}^n I_{s_i,i}$ in $S = K[x_1, y_1, z_1, \ldots, x_n, y_n, z_n, w_1, \ldots, w_b]$ is the required monomial ideal. By Lemma 2.3 and 2.4, we immediately show the assertion follows. □

Example 2.7. Nonincreasing functions $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ with $f(1) = f(2) = 2$ and $f(3) = f(4) = \cdots = 0$ and $g : \mathbb{Z}_{\geq 0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ with $g(1) = g(2) = 2$, $g(3) = 1$ and $g(4) = g(5) = \cdots = 0$ do not satisfy the assumption of Theorem 2.1. However, there exist monomial ideals I, J of $S = K[x_1, \ldots, x_6]$, such that $\operatorname{depth} S/I^k = f(k)$ and $\operatorname{depth} S/J^k = g(k)$ for $k \geq 1$.

Indeed, $I = (x_1^3, x_1x_2x_3, x_2^3x_3)(x_4^3, x_4x_5x_6, x_5^2x_6) + (x_1^4, x_1^3x_2, x_1x_2^3, x_2^4, x_1^2x_2^2x_3)$ and $J = (x_1^4, x_1x_2^2x_3, x_2^3x_3)(x_4^4, x_4x_5^2x_6, x_5^3x_6) + (x_1^5, x_1^4x_2, x_1x_2^4, x_2^5, x_1^3x_2^2x_3)$ are the desired monomial ideals.

Remark 2.8. In the recent preprint [**2**, Theorem 6.7], Hà-Nguyen-Trung-Trung has settled the conjecture of Herzog and Hibi [**4**, p. 549] in full generality.

3. The number of variables and depth stability number. Let $I \neq (0)$ be a monomial ideal in $S = K[x_1, \ldots, x_n]$ and $f(k)$ the depth function of I. We set $\lim_{k\to\infty} f(k) = d$ and $r = \operatorname{dstab}(I)$. When $n = 1$, we know that $d = 0$ and $r = 1$. Moreover, when $n = 2$, we have $0 \leq d \leq 1$ and $r = 1$.

In this section, for $n \geq 3$, we discuss bounds of the limit depth and depth stability number of a monomial ideal. In fact, we show the following theorem.

Theorem 3.1. *Assume $n \geq 3$. Let $I \neq (0)$ be a monomial ideal in $S = K[x_1, \ldots, x_n]$ and $f(k)$ the depth function of I. We set $\lim_{k\to\infty} f(k) = d$ and $r = \operatorname{dstab}(I)$. Then, one of the followings is satisfied:*

- *$0 \leq d \leq n - 2$ and $r \geq 1$.*
- *$d = n - 1$ and $r = 1$.*

Conversely, for any d and r satisfied one of the above, there exists a monomial ideal J in S such that $\lim_{k\to\infty} g(k) = d$ and $r = \operatorname{dstab}(J)$, where $g(k)$ is the depth function of J.

Proof. In general, for any monomial ideal $I \neq (0)$ in S, we have $0 \leq \operatorname{depth} S/I \leq n - 1$. We assume that $d = n - 1$. Since $\dim S/I^r \leq n - 1$, S/I^r is Cohen–Macaulay. Hence, for any minimal prime ideal P of I^r, we have height $P = 1$. In particular, P is a principle ideal since S is UFD. Hence, I^r is a principle ideal. This says that I is also a principle ideal. Thus, for any $k \geq 1$, S/I^k is a hypersurface. Therefore, we have $r = 1$.

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Next, we show the latter part. Assume that $0 \leq d \leq n-3$ and $r \geq 2$. Let $J_1 = (x_1^r, x_1x_2^{r-2}x_3, x_2^{r-1}x_3) \subset A := K[x_1, x_2, x_3]$. By Proposition 2.5, we have

$$\operatorname{depth} A/J_1^k = \begin{cases} 0, & \text{if } k \geq r, \\ 1, & \text{if } k \leq r-1. \end{cases}$$

Let $J = J_1 + (x_4, \ldots, x_{n-d}) = (x_1^r, x_1x_2^{r-2}x_3, x_2^{r-1}x_3, x_4, \ldots, x_{n-d})$ be a monomial ideal in S and $g_1(k)$ the depth function of J. Then, we have $\lim_{k\to\infty} g_1(k) = d$ and $\operatorname{dstab}(J) = r$. Moreover, an ideal $J_2 = (x_1, \ldots x_{n-d}) \subset S$ satisfies that $\operatorname{depth}(S/J_2^k) = d$ for all $k \geq 1$, that is, $\lim_{k\to\infty} \operatorname{depth}(S/J_2^k) = d$ and $\operatorname{dstab}(J_2) = 1$.

Next, we assume that $d = n-2$ and $r \geq 1$. By [**4**, Proof of Theorem 4.1], we can see that a monomial ideal $J_3 = (x_1^{r+2}, x_1^{r+1}x_2, x_1x_2^{r+1}, x_2^{r+2}, x_1^r x_2^2 x_3) \subset A$ satisfies that $\operatorname{dstab}(J_3) = r$ and

$$\operatorname{depth} A/J_3^k = \begin{cases} 1, & \text{if } k \geq r, \\ 0, & \text{if } k \leq r-1. \end{cases}$$

Let $J' = J_3$ be the monomial ideal in S and let $g_2(k)$ be the depth function of J'. Then, we have $\lim_{k\to\infty} g_2(k) = d$ and $\operatorname{dstab}(J') = r$.

When $d = n-1$ and $r = 1$, we immediately obtain a monomial ideal satisfying the condition by the former part of this proof. □

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