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A note on centralizers of involutions involving simple groups

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If a finite group G has a 'central' involution t whose centralizer in G is $\langle t \rangle \times H$ then, under certain conditions on H, G cannot be simple.

The purpose of this note is to observe the following generalization of the theorem of [1]. Let $i(X)$ denote the number of conjugacy classes of involutions of a group X.

THEOREM. *Let G be a finite group possessing a 'central' involution t such that $C_G(t) = \langle t \rangle \times H$ where H is a non-abelian simple group such that*

(i) the centre of a Sylow 2-subgroup S of H is cyclic, and

(ii) the involution τ of $Z(S)$ is a square in H.

Then

(a) if $i(H) = 1$ then $G = O(G).C_G(t)$;

(b) if $i(H) = 2$ then G has a subgroup of index 2 .

Proof. Let χ denote the permutation character of the representation of G on the left cosets of $C_G(t)$. Suppose $i(H) = 1$. Since t is a non-square in G , *(ii)* implies that $t \not\sim \tau$ in G . If $G \neq O(G).C_G(t)$ then $t \sim \tau t$ from [2]. By inducing the identity character of $C_G(t)$ to G one sees that $\chi(t) \equiv 0 \pmod 2$. Therefore

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$$[G : C_G(t)] = \chi(1) \equiv 0 \pmod 2 ,$$

contradicting the fact that t is 'central' in G .

Suppose $i(H) = 2$ and let $\tau_1 \in H$ be an involution not conjugate to τ in H . Again $t \not\sim \tau$ in G . If $t \sim t\tau$ in G then, since $Z(S)$ has a unique involution, one again obtains $\chi(t) \equiv 0 \pmod 2$ and $\chi(1) \equiv 0 \pmod 2$, a contradiction. Therefore $t \not\sim t\tau$ in G . Suppose G has no subgroup of index 2 . Then by a well-known transfer lemma (see, for example, [3], p. 265) S must contain a representative of each conjugacy class of involutions of G . Therefore $i(G) = 2$. Since t is conjugate in G to neither of τ, $t\tau$, one must have $\tau \sim t\tau$ in G . Since $Z(S)$ has a unique involution one sees that $\chi(\tau) \equiv 0 \pmod 2$. Therefore $[G : C_G(t)] = \chi(1) \equiv 0 \pmod 2$, contradicting the fact that t is 'central' in G . The theorem is proved.

REMARK. Most of the sporadic simple groups, including the Mathieu groups, satisfy the conditions imposed on H .

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