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ISOMORPHISM CLASSES OF GRAPH BUNDLES

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ABSTRACT. Recently, M. Hofmeister [4] counted all nonisomorphic double coverings of a graph by using its $\mathbf{Z}_2$ cohomology groups, and J. Kwak and J. Lee [5] did the same work for some finite-fold coverings. In this paper, we give an algebraic characterization of isomorphic graph bundles, from which we get a formula to count all nonisomorphic graph-bundles. Some applications to wheels are also discussed.

1. Graph Bundles. Let G be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$, and let $|X|$ denote the cardinality of a set X. The number $\beta(G) = |E(G)| - |V(G)| + 1$ is equal to the number of independent cycles in G and it is referred to as the *Betti number* of G. We denote the set of vertices adjacent to $v \in V(G)$ by $N(v)$ and call it the *neighborhood* of a vertex v. A graph means a finite simple graph throughout this paper.

A graph $\tilde{G}$ is called a *covering* of G with the projection $p : \tilde{G} \longrightarrow G$ if there is a surjection $p : V(\tilde{G}) \longrightarrow V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \longrightarrow N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We say that $\tilde{G}$ is an *n-fold covering* of G if the covering projection p is n-to-one.

Every edge of a graph G gives rise to a pair of oppositely directed edges. We denote the set of directed edges of G by $D(G)$. By e^{-1} we mean the reverse edge to an edge e. Each directed edge e has an initial vertex i_e and a terminal vertex t_e. Following [3], a *permutation voltage assignment* ϕ on a graph G is a map $\phi : D(G) \longrightarrow S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$, where S_n is the symmetric group on n elements $\{1, \ldots, n\}$. The *permutation derived graph* G^ϕ is defined as follows: $V(G^\phi) = V(G) \times \{1, \ldots, n\}$, and for each edge $e \in D(G)$ and $j \in \{1, \ldots, n\}$ let there be an edge (e, j) in $D(G^\phi)$ with $i_{(e,j)} = (i_e, j)$ and $t_{(e,j)} = (t_e, \phi(e)j)$. The natural projection $p_\phi : G^\phi \longrightarrow G$ is a covering. An *ordinary voltage assignment* ϕ on G, with values in a finite group Γ, is a map $\phi : D(G) \longrightarrow \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The *ordinary derived graph* $G \times_\phi \Gamma$ has the vertex set $V(G) \times \Gamma$ and the edge set $E(G) \times \Gamma$. An edge (e, g) has $i_{(e,g)} = (i_e, g)$ and $t_{(e,g)} = (t_e, \phi(e)g)$. The natural projection $p_\phi : G \times_\phi \Gamma \longrightarrow G$ commutes with the left multiplication action of the $\phi(e)$ and the right action of Γ on the fibres $p_\phi^{-1}(v)$, $v \in V(G)$, which is free and transitive, so that p_ϕ is Γ-regular. It is well-known [3] that every covering (resp. regular covering) graph $\tilde{G}$ of a given graph G can be described by a permutation (resp. ordinary) voltage assignment ϕ such that the edges of an arbitrary fixed spanning tree T of G are assigned identity voltages.

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We now consider a triple of the form (X, p, G), where X is a graph and p is a cell preserving projection of X onto G, written $p : X \longrightarrow G$. The graph G is called the base and X is the total graph of the triple (X, p, G). Here we allow p to be degenerate. In other words, p maps vertices to vertices, but an image of an edge can be either an edge or a vertex. We say that an edge e is *degenerate* if $p(e)$ is a vertex, and *non-degenerate* otherwise. The projection p thus induces a (fundamental) factorization $X = \tilde{G} \cup R$ of X into $\tilde{G}$ and R, where if D is the set of degenerate edges then $R = (V(X), D)$ contains all degenerate edges and $\tilde{G} = (V(X), E(X) - D)$ contains non-degenerate ones. For each vertex $v \in V(G)$ we define a fibre of v to be the graph $R_v = p^{-1}(v)$. Obviously, $R = \cup_{v \in V(G)} R_v$.

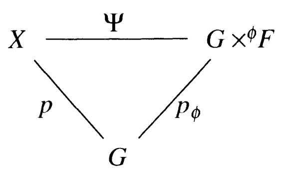
Let F be a graph. A triple $p : X \longrightarrow G$ will be called a *graph bundle* with fibre F (or, briefly, an *F-bundle*) [**6**] if the following three conditions are satisfied:

(a) Each fibre R_v is isomorphic to F.

(b) $\tilde{p} = p|_{\tilde{G}} : \tilde{G} \longrightarrow G$ is a $|V(F)|$-fold covering projection; this implies that for an arbitrary edge $e \in D(G)$ the set of $p^{-1}(e)$ of the lifted edges induces a bijection $\phi_e : V(R_{i_e}) \longrightarrow V(R_{t_e})$.

(c) Each mapping ϕ_e determines a graph isomorphism $\phi_e : R_{i_e} \longrightarrow R_{t_e}$.

To construct an F-bundle over an arbitrary graph G one can proceed as follows. Take a permutation voltage assignment ϕ on $D(G)$ into $S_{|V(F)|}$ with values in the automorphism group Aut(F) of the graph F. Define a graph X so that $V(X) = V(G^\phi)$ and $X = G^\phi \cup R$, where $R = (G - E(G)) \times F$ is the cartesian product. We denote the resulting graph X by $G \times^\phi F$. Then X is clearly an F-bundle over G. Conversely, every F-bundle over G admits such a description. More precisely, if (X, p, G) is an arbitrary F-bundle then G admits a permutation voltage assignment ϕ and there is an isomorphism $\Psi : X \longrightarrow G \times^\phi F$ such that the diagram

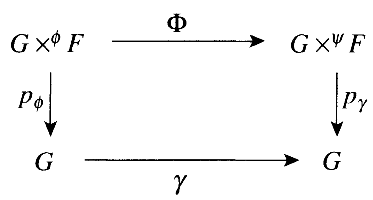


commutes.

Clearly, a graph bundle is just an n-fold covering graph if its fibre F is the complement $\bar{K}_n$ of the complete graph K_n of n vertices. Intuitively speaking, a graph bundle is the 1-skeleton of a fibre bundle where both the base and the fibre are graphs.

2. A characterization of isomorphic F-bundles. Let G be a graph and let Γ be a group of (graph-) automorphisms of G.

DEFINITION 1. *Two F-bundles $G \times^\phi F$ and $G \times^\psi F$ are isomorphic with respect* *to* Γ *if there exists an isomorphism* $\Phi : G\times^{\phi} F \longrightarrow G\times^{\psi} F$ *and* $\gamma \in \Gamma$ *such that the diagram*



commutes. We write $G\times^{\phi} F \simeq_{\Gamma} G\times^{\psi} F$. *The corresponding isomorphism classes are called* F*-bundles over* G *with respect to* Γ.

Example. It is well-known that the torus and the Klein bottle are the only topological bundles over the 1-sphere with the 1-sphere as fibre. Let's triangulate the 1-sphere as the complete graph K_3. Then their total spaces receive the structure of 2-dimensional complexes and their 1-skeletons are graph bundles with base K_3 and fibre K_3. But Figure 1 gives total graphs of at least three nonisomorphic graph bundles with base K_3 and fibre K_3.

It will be shown later that any graph bundle with base K_3 and fibre K_3 is isomorphic to one of three bundles in Figure 1. Graph bundles of Types I and III are 1-skeletons of the torus, and a graph bundle of Type II is a 1-skeleton of the Klein bottle.

An isomorphism class of F-bundles over G can be characterized through the corresponding equivalence class of functions $\phi : D(G) \longrightarrow \mathrm{Aut}(F)$ such that $\phi(e^{-1}) = \phi(e)^{-1}$.

Let $C^0(G;\mathrm{Aut}(F))$ denote the set of functions $f : V(G) \longrightarrow \mathrm{Aut}(F)$ and let $C^1(G;\mathrm{Aut}(F))$ denote the set of functions $\phi : D(G) \longrightarrow \mathrm{Aut}(F)$ such that $\phi(e^{-1}) = \phi(e)^{-1}$. Note that the set $C^1(G;\mathrm{Aut}(F))$ can fail to be a group with pointwise multiplication.

We define Γ-actions on the set $C^0(G;\mathrm{Aut}(F))$ and on the set $C^1(G;\mathrm{Aut}(F))$ as follows:

$$\gamma(f)(v) = f(\gamma^{-1}(v))$$

and

$$\gamma(\phi)(e) = \phi(\gamma^{-1}(i_e)\gamma^{-1}(t_e))$$

for any $\gamma \in \Gamma, f \in C^0(G;\mathrm{Aut}(F))$, and $\phi \in C^1(G;\mathrm{Aut}(F))$.

THEOREM 1. *Two* F*-bundles* $G\times^{\phi} F$ *and* $G\times^{\psi} F$ *are isomorphic with respect to* Γ, $\Gamma \leqq \mathrm{Aut}(G)$, *if and only if there exist* $\gamma \in \Gamma$ *and* $f \in C^0(G;\mathrm{Aut}(F))$ *such that* $\gamma^{-1}\psi(e) = f(t_e)\phi(e)f(i_e)^{-1}$ *for all* $e \in D(G)$.

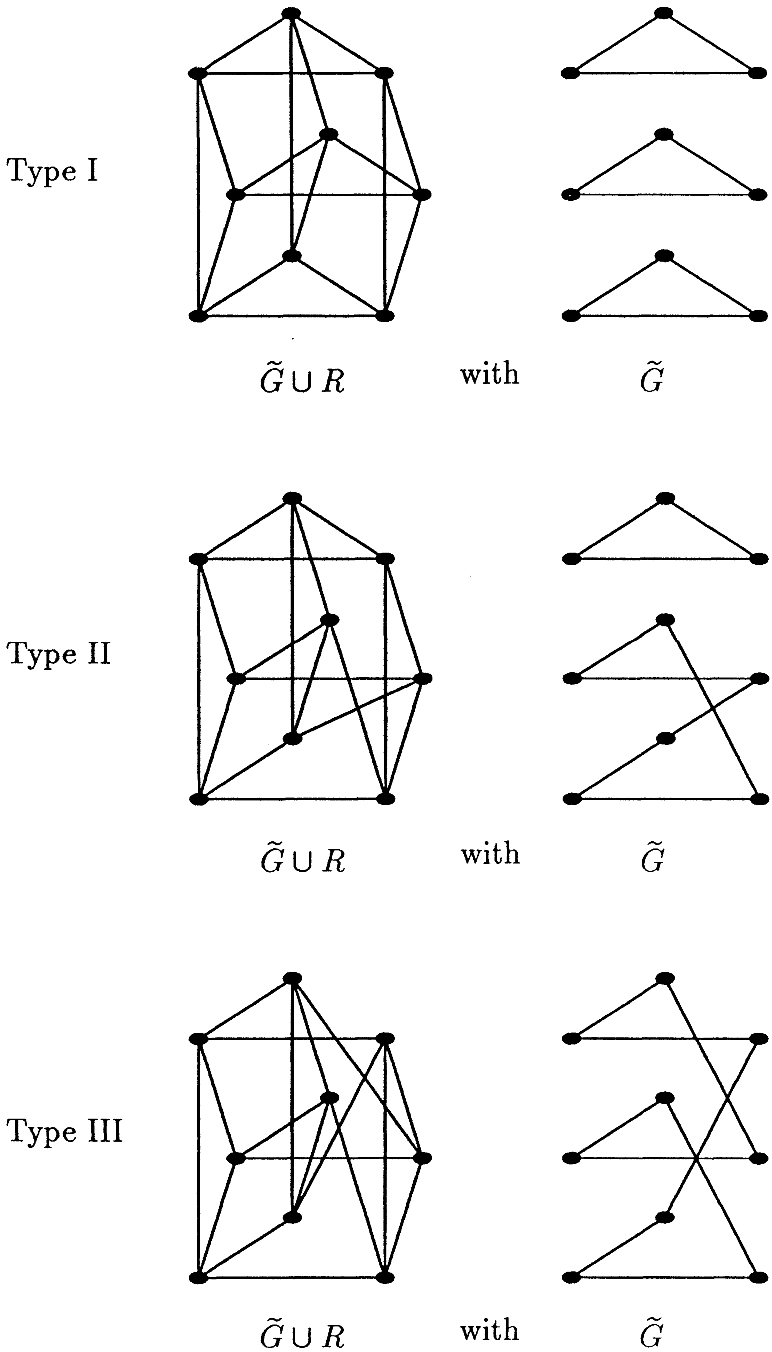


Figure 1. Three nonisomorphic graph bundles

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Proof. Assume that $G \times^{\phi} F \simeq_{\Gamma} G \times^{\psi} F$ with an isomorphism $\Phi : G \times^{\phi} F \longrightarrow G \times^{\psi} F$. Then $\Phi|_{p_{\phi}^{-1}(v)} : p_{\phi}^{-1}(v) \longrightarrow p_{\psi}^{-1}(\gamma(v))$ is an isomorphism for all $v \in V(G)$ and for some $\gamma \in \Gamma$. Now, we define $f : V(G) \longrightarrow \mathrm{Aut}(F)$ by $f(v) = \Phi|_{p_{\phi}^{-1}(v)}$ for all $v \in V(G)$. If (i_e, h) is joined to (t_e, k) in $G \times^{\phi} F$, then $\phi(e)(h) = k$ and $(\gamma(i_e), f(i_e)(h))$ is joined to $(\gamma(t_e), f(t_e)(k))$ in $G \times^{\psi} F$. Thus

$$\gamma^{-1}\psi(e) = \psi(\gamma(i_e)\gamma(t_e)) = f(t_e)\phi(e)f(i_e)^{-1}$$

for all $e \in D(G)$. Conversely, define $\Phi : G \times^{\phi} F \longrightarrow G \times^{\psi} F$ by $\Phi(v, h) = (\gamma(v), f(v)(h))$ for any (v, h) in $V(G \times^{\phi} F)$. If (i_e, h) is joined to (t_e, k) in $G \times^{\phi} F$, then $\phi(e)(h) = k$ and $\Phi(i_e, h) = (\gamma(i_e), f(i_e)(h))$ is joined to $\Phi(t_e, k) = (\gamma(t_e), f(t_e)(k))$. Thus Φ is the desired isomorphism to complete the proof. □

Let T be a fixed spanning tree in G with root v_0. Define a map $\Im^{\#} : C^1(G; \mathrm{Aut}(F)) \longrightarrow C^0(G; \mathrm{Aut}(F))$ as follows: for any $v \in V(G)$ there exists a unique path $e_1 e_2 \cdots e_m$ in the tree T from v_0 to v and we define

$$\Im^{\#}(\phi)(v) = (\phi(e_m) \cdots \phi(e_1))^{-1} = \phi(e_1)^{-1} \cdots \phi(e_m)^{-1}.$$

We write

$$C_T^1(G; \mathrm{Aut}(F)) = \{\phi \in C^1(G; \mathrm{Aut}(F)) : \phi(e) = \text{identity for each } e \in D(T)\},$$

and define $\Im^* : C^1(G; \mathrm{Aut}(F)) \longrightarrow C_T^1(G; \mathrm{Aut}(F))$ by

$$\Im^*(\phi)(e) = \Im^{\#}(\phi)(t_e)\phi(e)\Im^{\#}(\phi)(i_e)^{-1}$$

for any $\phi \in C^1(G; \mathrm{Aut}(F))$ and any $e \in D(G)$. Then, $\Im^*$ is clearly well-defined and the identity on $C_T^1(G; \mathrm{Aut}(F))$. Hence, we have

COROLLARY 1. *Any F-bundle $G \times^{\phi} F$ over G, $\phi \in C^1(G; \mathrm{Aut}(F))$, is isomorphic to an F-bundle $G \times^{\psi} F$ with respect to the identity automorphism of G for some $\psi \in C_T^1(G; \mathrm{Aut}(F))$.*

3. Some counting formulas. Let T be a fixed spanning tree of G and let $\mathrm{Aut}(G, T)$ denote the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms f of G fixing T, i.e., $f(T) = T$. Then for any subgroup Γ of $\mathrm{Aut}(G, T)$, the subset $C_T^1(G; \mathrm{Aut}(F))$ of $C^1(G; \mathrm{Aut}(F))$ is invariant under the Γ-action. Denote the number of nonisomorphic F-bundles over G with respect to a subgroup Γ of $\mathrm{Aut}(G)$ by $\mathrm{Iso}_{\Gamma}(G; F)$. From now on, we only consider a group Γ of automorphisms of G which fix a given spanning tree T of G and voltage assignments ϕ which are in $C_T^1(G; \mathrm{Aut}(F))$. Note that $|C_T^1(G; \mathrm{Aut}(F))| = |\mathrm{Aut}(F)|^{\beta(G)}$, and it will be used later. Let T^* denote the cotree of T in the graph G.

THEOREM 2. *$G\times^{\phi}F \simeq_{\Gamma} G\times^{\psi}F$ if and only if there exist $\gamma \in \Gamma$ and $g \in \mathrm{Aut}(F)$ such that $\gamma^{-1}\psi(e) = g\phi(e)g^{-1}$ for all $e \in D(T^*) = D(G) - D(T)$.*

Proof. Since both $\gamma^{-1}\psi$ and ϕ are identity on the spanning tree T, the map f satisfying $\gamma^{-1}\psi(e) = f(t_e)\phi(e)f(i_e)^{-1}$ must be constant. The proof is now clear by Theorem 1. □

LEMMA 1. *For any $\gamma \in \Gamma$, any $\phi \in C_T^1(G;\mathrm{Aut}(F))$, and any $g \in \mathrm{Aut}(F)$, we have $g(\gamma\phi)g^{-1} = \gamma(g\phi g^{-1})$.*

Proof. For any edge e in $D(G)$, $(g(\gamma\phi)g^{-1})(e) = g(\gamma\phi(e))g^{-1} = g(\phi(\gamma^{-1}(i_e)\gamma^{-1}(t_e)))g^{-1} = \gamma(g\phi g^{-1})(e)$. □

With the conjugate action of $\mathrm{Aut}(F)$ on $C_T^1(G;\mathrm{Aut}(F))$, we define an action of the product group $\Gamma \times \mathrm{Aut}(F)$ on $C_T^1(G;\mathrm{Aut}(F))$ by $(\gamma, g)(\phi) = \gamma(g\phi g^{-1})$ for $(\gamma, g) \in \Gamma \times \mathrm{Aut}(F)$ and $\phi \in C_T^1(G;\mathrm{Aut}(F))$. It is well-defined by Lemma 1. Now, Burnside's Lemma and Theorem 2 give

THEOREM 3. *For any subgroup Γ of $\mathrm{Aut}(G,T)$*

$$\mathrm{Iso}_{\Gamma}(G;F) = \frac{1}{|\Gamma|\,|\mathrm{Aut}(F)|} \sum_{(\gamma,g)\in\Gamma\times\mathrm{Aut}(F)} |\mathrm{Fix}_{(\gamma,g)}|,$$

where $\mathrm{Fix}_{(\gamma,g)} = \{\phi \in C_T^1(G;\mathrm{Aut}(F)) : (\gamma, g)\phi = \phi\}$.

It is easy to show that if (γ_1, g_1) and (γ_2, g_2) are conjugate in $\Gamma \times \mathrm{Aut}(F)$, then $|\mathrm{Fix}_{(\gamma_1,g_1)}| = |\mathrm{Fix}_{(\gamma_2,g_2)}|$. Thus, we can rewrite

THEOREM 4. *For any subgroup Γ of $\mathrm{Aut}(G,T)$*

$$\mathrm{Iso}_{\Gamma}(G;F) = \frac{1}{|\Gamma|\,|\mathrm{Aut}(F)|} \sum_{(\gamma,g)} |C(\gamma,g)||\mathrm{Fix}_{(\gamma,g)}|,$$

where (γ, g) runs over all representatives of the conjugacy classes of $\Gamma\times\mathrm{Aut}(F)$, and $C(\gamma, g)$ denotes the conjugacy class of (γ, g) in $\Gamma \times \mathrm{Aut}(F)$.

Every group is isomorphic to the automorphism group of some graph and many graphs can have the same automorphism group. For example, the automorphism group of a graph F is isomorphic to that of its complement $\bar{F}$, and with four small-order exceptions, the automorphism group of a connected graph is isomorphic to that of its line graph (see **[8]**).

COROLLARY 2. *If $\mathrm{Aut}(F_1)$ is isomorphic to $\mathrm{Aut}(F_2)$, then $\mathrm{Iso}_{\Gamma}(G;F_1) = \mathrm{Iso}_{\Gamma}(G;F_2)$.*

COROLLARY 3. *If Γ is trivial, then*

$$\mathrm{Iso}_{\{1\}}(G;F) = \frac{1}{|\mathrm{Aut}(F)|} \sum_{g} |C(g)||\mathrm{Fix}_g|,$$

where g runs over all representatives of conjugacy classes of Aut(F), *and* $C(g)$ *is the conjugacy class of g in* Aut(F).

If G is tree or Aut(F) is trivial, then $C_T^1(G; \mathrm{Aut}(F))$ is trivial and $\mathrm{Iso}_\Gamma(G; F) = 1$ for any Γ. Hence, we have

COROLLARY 4. *(a) Any two bundles over a tree G with the same fibre are isomorphic with respect to any subgroup Γ of* Aut(G). *(b) Any two bundles over a graph G with a fibre having the trivial automorphism group are isomorphic with respect to any subgroup Γ of* Aut(G).

Consider the case that Aut(F) is abelian, so that the action of Aut(F) on $C_T^1(G; \mathrm{Aut}(F))$ is trivial. Then the isomorphism classes of F-bundles $G \times^\phi F$ over G for $\phi \in C_T^1(G; \mathrm{Aut}(F))$ depend only on the Γ-action. Hence, Burnside's Lemma gives

THEOREM 5. *If* Aut(F) *is abelian, then*

$$\mathrm{Iso}_\Gamma(G; F) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |\mathrm{Fix}_\gamma|$$

for any subgroup Γ of Aut(G, T). *In particular, if Γ is trivial*

$$\mathrm{Iso}_{\{1\}}(G; F) = |\mathrm{Aut}(F)|^{\beta(G)}.$$

Next, we aim to find a formula to find $|\mathrm{Fix}_{(\gamma,g)}|$ for a given (γ, g) in $\Gamma \times \mathrm{Aut}(F)$.

LEMMA 2. *Let ϕ be an element in* $\mathrm{Fix}_{(\gamma,g)}$. *Then the voltage $\phi(\gamma^n e)$ is completely determined by the voltage $\phi(e)$ for all n and $e \in D(G)$.*

Proof. For any $\phi \in \mathrm{Fix}_{(\gamma,g)}$ and for any $e \in D(G)$,

$$\begin{aligned} \phi(e) &= g(\phi(\gamma^{-1}(i_e)\ \gamma^{-1}(t_e)))g^{-1} \\ &= g^2(\phi(\gamma^{-2}(i_e)\ \gamma^{-2}(t_e)))g^{-2} \\ &= g^3(\phi(\gamma^{-3}(i_e)\ \gamma^{-3}(t_e)))g^{-3} \\ &= \cdots. \end{aligned}$$

Hence, for all n

$$\phi(\gamma^n e) = g^n \phi(e) g^{-n}. \qquad \square$$

Since all voltages are assumed to be the identity on the tree T, we need to consider only the voltages of edges which are in the cotree T^* of T.

For an element γ in Γ, we define an equivalence relation $\sim_\gamma$ on $D(T^*) = D(G) - D(T)$ as follows: $e_1 \sim_\gamma e_2$ if and only if $e_1 = \gamma^\ell e_2$ for some ℓ. Note that if ϕ is an element of $\mathrm{Fix}_{(\gamma,g)}$, then the voltages ϕ in an equivalence class

$[e]$ containing e are completely determined by the voltage of $\phi(e)$, by Lemma 2. An equivalence class $[e]$ of e is called of *class 1* if e and e^{-1} are contained in the same class, and of *class 2* otherwise. For any edge $e \in D(T^*)$, we define a number $\eta(\gamma, e)$ to be the smallest natural number ℓ such that $e^{-1} = \gamma^{\ell} e$ if $[e]$ is of class 1, and the smallest natural number ℓ such that $e = \gamma^{\ell} e$ if $[e]$ is of class 2. This number is well-defined because γ has finite order in Γ.

Now, for an element ϕ in $\mathrm{Fix}_{(\gamma,g)}$ the voltage $\phi(e)$ of e must satisfy $g^{\eta(\gamma,e)}\phi(e)g^{-\eta(\gamma,e)} = \phi(e)^{-1}$ if $[e]$ is of class 1, and $g^{\eta(\gamma,e)}\phi(e)g^{-\eta(\gamma,e)} = \phi(e)$ if $[e]$ is of class 2. Denote that

$$I(g^n) = \{h \in \mathrm{Aut}(F) : g^n h g^{-n} = h^{-1}\}$$

and

$$Z(g^n) = \{h \in \mathrm{Aut}(F) : g^n h g^{-n} = h\}$$

as a subset of $\mathrm{Aut}(F)$. Now, for $\phi \in \mathrm{Fix}_{(\gamma,g)}$ the voltage $\phi(e)$ of e must be contained in $I(g^{\eta(\gamma,e)})$ if $[e]$ is of class 1, and contained in $Z(g^{\eta(\gamma,e)})$ if $[e]$ is of class 2. Note that if $[e]$ is of class 2, so is $[e^{-1}]$, and the voltages of edges in $[e^{-1}]$ are also completely determined by the voltage of $\phi(e)$. Hence, we get the following formula to compute $|\mathrm{Fix}_{(\gamma,g)}|$:

THEOREM 6.

$$|\mathrm{Fix}_{(\gamma,g)}| = \left(\prod_{[e]\in Class\,1} |I(g^{\eta(\gamma,e)})|\right)\left(\prod_{[e]\in Class\,2} |Z(g^{\eta(\gamma,e)})|\right)^{\frac{1}{2}},$$

where the product over the empty index set is defined to be 1.

Let Γ be trivial. Then, every edge in $D(T^*)$ is of class 2, and for any $g \in \mathrm{Aut}(F)$, ϕ is contained in Fix_g if and only if $\phi(e) \in Z(g)$ for every positively oriented edge e in $D(T^*)$. Hence, we get

COROLLARY 5. *If Γ is trivial, then*

$$|\mathrm{Fix}_g| = |Z(g)|^{\beta(G)}$$

for any $g \in \mathrm{Aut}(F)$.

We recall that a bundle having $F = \bar{K}_n$ as fibre over G is an n-fold covering of G and that each permutation in $\mathrm{Aut}(\bar{K}_n) = S_n$ can be resolved into a product of disjoint cycles in a unique manner up to the order of the cycle factors. And, each conjugacy class $C(g)$ of S_n is determined by the cycle type $(\ell_1, \cdots, \ell_n)$ of g, where ℓ_k is the number of cycles of length k in the factorization of an element g in S_n into disjoint cycles, so that $\ell_1 + 2\ell_2 + \cdots + n\ell_n = n$. Then $|Z(g)| = \ell_1!\,2^{\ell_2}\ell_2! \cdots n^{\ell_n}\ell_n!$ if g is of the cycle type $\ell = (\ell_1, \cdots, \ell_n)$.

THEOREM 7. *The number of isomorphism classes of n-fold coverings of G with respect to the trivial automorphism group, is*

$$\mathrm{Iso}_{\{1\}}(G;\bar{K}_n) = \sum_{\ell_1+2\ell_2+\cdots+n\ell_n=n} (\ell_1!\,2^{\ell_2}\ell_2!\cdots n^{\ell_n}\ell_n!)^{\beta(G)-1}.$$

Proof. Clearly, $\mathrm{Aut}(\bar{K}_n) = S_n$, $|\mathrm{Aut}(\bar{K}_n)| = n!$ and $|C(g)|\,|Z(g)| = n!$ for any $g \in S_n$. The theorem comes from Corollaries 3 and 5. □

For example, the number of isomorphism classes of n−fold coverings of the complete graph K_m with respect to the trivial automorphism group, is

$$\mathrm{Iso}_{\{1\}}(K_m;\bar{K}_n) = \sum_{\ell_1+2\ell_2+\cdots+n\ell_n=n} (\ell_1!\,2^{\ell_2}\ell_2!\cdots n^{\ell_n}\ell_n!)^{\frac{1}{2}m(m-3)}.$$

If $\mathrm{Aut}(F)$ is abelian, then the set $I(g^n)$ is the subgroup of $\mathrm{Aut}(F)$ consisting of all elements of order 2, and $Z(g^n)$ is the total group $\mathrm{Aut}(F)$ for all n. Hence, if we denote $\kappa(F) = |\{g \in \mathrm{Aut}(F) : g^2 = \text{identity}\}|$, $o(F) = |\mathrm{Aut}(F)|$, and the number of equivalence classes in $D(T^*)/\sim_\gamma$ of class j by $\kappa_j(\gamma)$ for $j = 1, 2$, then we have

COROLLARY 6. *If* $\mathrm{Aut}(F)$ *is abelian,*

$$|\mathrm{Fix}_\gamma| = \kappa(F)^{\kappa_1(\gamma)} o(F)^{\frac{1}{2}\kappa_2(\gamma)}.$$

For example, if $\mathrm{Aut}(F) = \mathbf{Z}_{p_1^{m_1}} \times \cdots \times \mathbf{Z}_{p_n^{m_n}}$, then, by Theorem 5 and Corollary 6,

$$\mathrm{Iso}_\Gamma(G;F) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} 2^{\alpha\kappa_1(\gamma)}\left(\prod_{i=1}^{n} p_i^{m_i}\right)^{\frac{1}{2}\kappa_2(\gamma)},$$

where α is the number of p_i which is 2. In particular, if Γ is trivial, then

$$\mathrm{Iso}_{\{1\}}(G;F) = \left(\prod_{i=1}^{n} p_i^{m_i}\right)^{\beta(G)}.$$

4. Applications to wheels. Let K_1 denote the trivial graph with vertex 0 and C_m an m-cycle with consecutively labelled vertices $1, 2, \ldots, m$. Then the join $W_m = K_1 \vee C_m$ of K_1 and C_m is called a *wheel* for $m \geqq 3$. Let T_m be the spanning tree of W_m consisting of all edges incident with the vertex 0. For convenience to apply our result, we only consider $m \geqq 4$.

First we evaluate $\mathrm{Iso}_\Gamma(W_m;F)$, $\Gamma \leqq \mathrm{Aut}(W_m)$ when $\mathrm{Aut}(F)$ is an abelian group. Note that $\mathrm{Aut}(W_m)$ is the dihedral group D_m for $m \geqq 4$. Let D_m denote the dihedral group generated by two permutations ρ and τ, where $\tau(i) = m+1-i$

and $\rho(i) = i + 1$. Note that all arithmetic is done modulo m. Then D_m is the semi-direct product of $\mathbf{Z}_m$ and $\mathbf{Z}_2$, where $\mathbf{Z}_m$ and $\mathbf{Z}_2$ are cyclic groups generated by ρ and τ respectively, and any subgroup Γ of $\mathrm{Aut}(W_m) = D_m$ fixes T_m for all $m \geqq 4$. For each $\rho^k \in \mathbf{Z}_m$, let $o(k)$ denote the order of ρ^k in $\mathbf{Z}_m$ and $\imath(k)$ the index of the subgroup generated by ρ^k in $\mathbf{Z}_m$.

With notation discussed before Theorem 6, we first compute $\kappa_i(\gamma)$ and $\eta(\gamma, e)$ for any $\gamma \in \mathrm{Aut}(W_m)$ and $e \in D(W_m)$. Note that an element of $\mathrm{Aut}(W_m)$ is either of the form $\tau\rho^k$ or of the form ρ^k, i.e., either a reflection or a rotation. Geometrically, we can identify $\mathrm{Aut}(W_m) = D_m$ as the symmetric group of the regular m-gons. Hence, for any nontrivial symmetry $\gamma \in \mathrm{Aut}(W_m)$, $[e]$ is of class 1 and $(\gamma, e) = 1$ if an edge e is fixed by γ, and $[e]$ is of class 2 and $\eta(\gamma, e)$ is the order of γ otherwise. Hence, we get the following lemma.

LEMMA 3. *Let $G = W_m, m \geqq 4$ be a wheel. Then,*

$$(a) \qquad \kappa_1(\tau\rho^k) = \begin{cases} 1 & \text{if } m \text{ is odd and } 0 \leqq k \leqq m-1 \\ 2 & \text{if } m \text{ is even and } 0 \leqq k = \text{even} \leqq m-1 \\ 0 & \text{if } m \text{ is even and } 0 \leqq k = \text{odd} \leqq m-1, \end{cases}$$

and $\eta(\tau\rho^k, e) = 1$ for any k and any $[e]$ in $D(T^*)/\sim_{\tau\rho^k}$ of class 1.

$$(b) \qquad \kappa_2(\tau\rho^k) = \begin{cases} m-1 & \text{if } m \text{ is odd and } 0 \leqq k \leqq m-1 \\ m-2 & \text{if } m \text{ is even and } 0 \leqq k = \text{even} \leqq m-1 \\ m & \text{if } m \text{ is even and } 0 \leqq k = \text{odd} \leqq m-1, \end{cases}$$

and $\eta(\tau\rho^k, e) = 2$ for any k and any $[e]$ in $D(T^*)/\sim_{\tau\rho^k}$ of class 2.

(c) $\kappa_1(\rho^k) = 0$, $\kappa_2(\rho^k) = 2\,\imath(k)$ and $\eta(\rho^k, e) = o(k)$ for any k and any e in $D(T^*)$.

By Theorem 5, Corollary 6 and Lemma 3, we get the following theorem.

THEOREM 8. *Let $\mathrm{Aut}(F)$ be an abelian group.*
(a) If Γ is the total group $\mathrm{Aut}(W_m) = D_m$, then

$$\mathrm{Iso}_{\mathrm{Aut}(W_m)}(W_m; F) = \begin{cases} \dfrac{1}{2m}\left(\displaystyle\sum_{k=0}^{m-1} o(F)^{\imath(k)} + m\kappa(F)o(F)^{\frac{m-1}{2}}\right) & \text{if } m \text{ is odd} \\[2ex] \dfrac{1}{2m}\left(\displaystyle\sum_{k=0}^{m-1} o(F)^{\imath(k)} + \dfrac{m}{2}\,(o(F) + \kappa(F)^2)o(F)^{\frac{m-2}{2}}\right) & \text{if } m \text{ is even.} \end{cases}$$

(b) If $\Gamma \simeq \mathbf{Z}_m$ is the cyclic group generated by ρ, then

$$\mathrm{Iso}_{\mathbf{Z}_m}(W_m; F) = \frac{1}{m}\sum_{k=0}^{m-1} o(F)^{\imath(k)} \quad \text{for all } m.$$

(c) If $\Gamma \simeq \mathbf{Z}_2$ *is the cyclic group generated by* τ*, then*

$$\mathrm{Iso}_{Z_2}(W_m;F) = \begin{cases} \frac{1}{2}\left(o(F)^m + \kappa(F)o(F)^{\frac{m-1}{2}}\right) & \text{if } m \text{ is odd} \\ \frac{1}{2}\left(o(F)^m + \kappa(F)^2 o(F)^{\frac{m-2}{2}}\right) & \text{if } m \text{ is even.} \end{cases}$$

(d) If Γ *is the cyclic group generated by* $\tau\rho$*, then*

$$\mathrm{Iso}_{\Gamma}(W_m;F) = \begin{cases} \frac{1}{2}\left(o(F)^m + \kappa(F)o(F)^{\frac{m-1}{2}}\right) & \text{if } m \text{ is odd} \\ \frac{1}{2}\left(o(F)^m + o(F)^{\frac{m}{2}}\right) & \text{if } m \text{ is even.} \end{cases}$$

If the fibre $F = \bar{K}_n$ has only n vertices, then an F-bundle $G \times^{\phi} F$ over a graph G is an n-fold covering of G. Note that $\mathrm{Aut}(\bar{K}_n)$ is abelian only for $n = 2$.

COROLLARY 7. *(a) The number of isomorphism classes of double covers of* W_m *with respect to* $\mathrm{Aut}(W_m) = D_m$ *is*

$$\mathrm{Iso}_{\mathrm{Aut}(W_m)}(W_m;\bar{K}_2) = \begin{cases} \frac{1}{2m}\left(\sum_{k=0}^{m-1} 2^{\imath(k)} + m2^{\frac{m+1}{2}}\right) & \text{if } m \text{ is odd} \\ \frac{1}{2m}\left(\sum_{k=0}^{m-1} 2^{\imath(k)} + 3m2^{\frac{m-2}{2}}\right) & \text{if } m \text{ is even.} \end{cases}$$

(b) The number of isomorphism classes of double covers of W_m *with respect to* $\mathbf{Z}_m$ *is*

$$\mathrm{Iso}_{\mathbf{Z}_m}(W_m;\bar{K}_2) = \frac{1}{m}\sum_{k=0}^{m-1} 2^{\imath(k)} \quad \text{for all } m.$$

(c) The number of isomorphism classes of double covers of W_m *with respect to* $\mathbf{Z}_2$ *is*

$$\mathrm{Iso}_{\mathbf{Z}_2}(W_m;\bar{K}_2) = \begin{cases} 2^{\frac{m-1}{2}}(2^{\frac{m-1}{2}} + 1) & \text{if } m \text{ is odd} \\ 2^{\frac{m}{2}}(2^{\frac{m-2}{2}} + 1) & \text{if } m \text{ is even.} \end{cases}$$

In particular, if m is prime

(d) $\mathrm{Iso}_{\mathrm{Aut}(W_m)}(W_m;\bar{K}_2) = \dfrac{1}{2m}(2^m + 2m - 2 + m2^{\frac{m+1}{2}})$.

(e) $\mathrm{Iso}_{Z_m}(W_m;\bar{K}_2) = \dfrac{1}{m}(2^m + 2m - 2)$.

(f) $\mathrm{Iso}_{Z_2}(W_m;\bar{K}_2) = 2^{\frac{m-1}{2}}(2^{\frac{m-1}{2}} + 1)$.

Finally, we consider the general case, i.e., Aut(F) is not necessarily abelian. By using Theorems 4, 6 and Lemma 3, we get

THEOREM 9. *Let F be any graph as the fibre of W_m.*
(a) If Γ is the total group Aut$(W_m) = D_m$, *then*

$$\mathrm{Iso}_{\mathrm{Aut}(W_m)}(W_m;F) = \begin{cases} \frac{1}{2m}\frac{1}{o(F)}\sum_g |C(g)|\Big(\sum_{k=0}^{m-1}|Z(g^{o(k)})|^{\imath(k)} \\ \quad + m|I(g)||Z(g^2)|^{\frac{m-1}{2}}\Big) & \text{if } m \text{ is odd} \\ \frac{1}{2m}\frac{1}{o(F)}\sum_g |C(g)|\Big(\sum_{k=0}^{m-1}|Z(g^{o(k)})|^{\imath(k)} \\ \quad + \frac{m}{2}(|I(g)|^2 + |Z(g^2)|)|Z(g^2)|^{\frac{m-2}{2}}\Big) & \text{if } m \text{ is even.} \end{cases}$$

(b) If $\Gamma \simeq \mathbf{Z}_m$ is the cyclic group generated by ρ, then

$$\mathrm{Iso}_{\mathbf{Z}_m}(W_m;F) = \frac{1}{m}\frac{1}{o(F)}\sum_g |C(g)|\left(\sum_{k=0}^{m-1}|Z(g^{o(k)})|^{\imath(k)}\right) \quad \text{for all } m.$$

(c) If $\Gamma \simeq \mathbf{Z}_2$ is the cyclic group generated by τ, then

$$\mathrm{Iso}_{Z_2}(W_m;F) = \begin{cases} \frac{1}{2}\frac{1}{o(F)}\sum_g |C(g)|\big(|Z(g)|^m + |I(g)||Z(g^2)|^{\frac{m-1}{2}}\big) \\ \qquad\qquad \text{if } m \text{ is odd} \\ \frac{1}{2}\frac{1}{o(F)}\sum_g |C(g)|\big(|Z(g)|^m + |I(g)|^2|Z(g^2)|^{\frac{m-2}{2}}\big) \\ \qquad\qquad \text{if } m \text{ is even.} \end{cases}$$

(d) If Γ is the cyclic group generated by $\tau\rho$, then

$$\mathrm{Iso}_{\Gamma}(W_m;F) = \begin{cases} \frac{1}{2}\frac{1}{o(F)}\sum_g |C(g)|\big(|Z(g)|^m + |I(g)||Z(g^2)|^{\frac{m-1}{2}}\big) \\ \qquad\qquad \text{if } m \text{ is odd} \\ \frac{1}{2}\frac{1}{o(F)}\sum_g |C(g)|\big(|Z(g)|^m + |Z(g^2)|^{\frac{m}{2}}\big) \\ \qquad\qquad \text{if } m \text{ is even.} \end{cases}$$

Here, all summations are taken over the representatives g over the conjugacy classes of Aut(F).

In particular, if the fibre F is $\bar{K}_n, n \geqq 3$, then we can count the number of isomorphism classes of n-fold covering of W_m. Note that $|Z(g)||C(g)| = n!$ for all $g \in S_n$, where $Z(g)$ is the centralizer subgroup of g in S_n and $C(g)$ is the conjugacy class of g in S_n.

COROLLARY 8. *(a) The number of isomorphism classes of n-fold coverings of W_m with respect to* $\mathrm{Aut}(W_m) = D_m$ *is*

$$\mathrm{Iso}_{\mathrm{Aut}(W_m)}(W_m;\bar{K}_n) = \begin{cases} \dfrac{1}{2m}\displaystyle\sum_g \frac{1}{|Z(g)|}\left(\sum_{k=0}^{m-1} |Z(g^{o(k)})|^{\imath(k)} \right. \\ \qquad \left. + m|I(g)|\,|Z(g^2)|^{\frac{m-1}{2}}\right) & \text{if } m \text{ is odd} \\ \dfrac{1}{2m}\displaystyle\sum_g \frac{1}{|Z(g)|}\left(\sum_{k=0}^{m-1} |Z(g^{o(k)})|^{\imath(k)} \right. \\ \qquad \left. + \dfrac{m}{2}(|Z(g^2)| + |I(g^2)|\,|Z(g^2)|^{\frac{m-2}{2}}\right) \\ & \text{if } m \text{ is even.} \end{cases}$$

(b) The number of isomorphism classes of n–fold coverings of W_m with respect to $\mathbf{Z}_m$ is

$$\mathrm{Iso}_{\mathbf{Z}_m}(W_m;\bar{K}_n) = \frac{1}{m}\sum_g \sum_{k=0}^{m-1} \frac{1}{|Z(g)|}\,|Z(g^{o(k)})|^{\imath(k)}.$$

(c) The number of isomorphism classes of n–fold coverings of W_m with respect to $\mathbf{Z}_2$ is

$$\mathrm{Iso}_{Z_2}(W_m;\bar{K}_n) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_g \frac{1}{|Z(g)|}\left((|Z(g)|^m + |I(g)|\,|Z(g^2)|^{\frac{m-1}{2}}\right) \\ & \text{if } m \text{ is odd} \\ \dfrac{1}{2}\displaystyle\sum_g \frac{1}{|Z(g)|}\left((|Z(g)|^m + |I(g)|^2|Z(g^2)|^{\frac{m-2}{2}}\right) \\ & \text{if } m \text{ is even.} \end{cases}$$

In particular, if m is prime, then

$$(d) \quad \mathrm{Iso}_{\mathrm{Aut}(W_m)}(W_m;\bar{K}_n) = \frac{1}{2m}\sum_g \frac{1}{|Z(g)|}\left((|Z(g)|^m + (m-1)|Z(g^m)|\right.$$
$$\left. + m|I(g)|\,|Z(g^2)|^{\frac{m-1}{2}})\right)$$

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(e) $\mathrm{Iso}_{Z_m}(W_m;\bar{K}_n) = \frac{1}{m}\sum_g \frac{1}{|Z(g)|}\left((|Z(g)|^m + (m-1)|Z(g^m)|)\right).$

Here, all summations are taken over the representatives g of the conjugacy classes of S_n.

5. Counting of regular p-fold covering graphs. Let p be a prime number, and let T be a fixed spanning tree in a graph G. For the graph $\bar{K}_p$ of p vertices, $\mathrm{Aut}(\bar{K}_p)$ is the symmetric group S_p. Let $\mathbf{Z}_p$ denote the subgroup of S_p generated by the p-cycle $\rho = (0\,1\,2\cdots p-1)$ in S_p. Then, it is well-known [3] that every regular p-fold covering of G can be considered as an $\bar{K}_p$-bundle $G\times^{\phi}\bar{K}_p$ with ϕ in $C^1_T(G;\mathbf{Z}_p)$, where $C^1_T(G;\mathbf{Z}_p)$ denotes the set of functions $\phi : D(G) \longrightarrow \mathbf{Z}_p$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ and ϕ is the identity on $D(T)$. Let $\mathrm{Iso}^R_{\{1\}}(G;p)$ denote the number of isomorphism classes of regular p-fold coverings of G with respect to the identity automorphism of G.

Let any two coverings $G\times^{\phi}\bar{K}_p$ and $G\times^{\psi}\bar{K}_p$, $\phi,\psi \in C^1_T(G;\mathbf{Z}_p)$, be isomorphic with respect to the identity automorphism, then there exists an element $\alpha \in \mathrm{Aut}(\bar{K}_p) = S_p$ such that $\psi(e) = \alpha\phi(e)\alpha^{-1}$ for all $e \in D(G)-D(T)$, by Theorem 2, and such α must be contained in the normalizer $N(\mathbf{Z}_p)$ of $\mathbf{Z}_p$ in $\mathrm{Aut}(\bar{K}_p) = S_p$. But the normalizer $N(\mathbf{Z}_p)$ of $\mathbf{Z}_p$ in S_p is the set $N(\mathbf{Z}_p) = \{\alpha \in S_p : \alpha\rho\alpha^{-1} = \rho^i$ for some $i = 1,\ldots,p-1\}$. The $\mathrm{Aut}(\bar{K}_p)$-action on $C^1_T(G;\mathrm{Aut}(\bar{K}_p))$ induces an $N(\mathbf{Z}_p)$-action on $C^1_T(G;\mathbf{Z}_p)$, on which $\mathbf{Z}_p$ acts trivially. Hence, it induces an $N(\mathbf{Z}_p)/\mathbf{Z}_p$-action on $C^1_T(G;\mathbf{Z}_p)$, and the quotient group $N(\mathbf{Z}_p)/\mathbf{Z}_p$ is clearly isomorphic to the cyclic group of order $p-1$. Let us write $A_p = N(\mathbf{Z}_p)/\mathbf{Z}_p = \{g_1,\ldots,g_{p-1}\}$ with $g_ig_j = g_{ij(\mathrm{mod}\,p)}$.

THEOREM 10 ([**5**]) .

$$\mathrm{Iso}^R_{\{1\}}(G;p) = \frac{1}{(p-1)}\left(p^{\beta(G)} + p - 2\right).$$

Proof. Clearly

$$|\mathrm{Fix}_{g_i}| = \begin{cases} p^{\beta(G)} & \text{if } i = 1 \\ 1 & \text{otherwise,}\end{cases}$$

and Burnside's Lemma gives our theorem. □

COROLLARY 9 ([**4**]). *The number of double covers of G is*

$$\mathrm{Iso}^R_{\{1\}}(G;2) = 2^{\beta(G)}.$$

Let Γ be any subgroup of $\mathrm{Aut}(G)$ which fixes a spanning tree T of G. Let $\mathrm{Iso}^R_{\Gamma}(G;p)$ denote the number of isomorphism classes of regular p-fold coverings of G with respect to Γ. If we apply Theorem 3 to this situation, we have the following theorem.

THEOREM 11.

$$\mathrm{Iso}^R_\Gamma(G;p) = \frac{1}{(p-1)|\Gamma|} \sum_{(g_i,\gamma)\in A_p\times\Gamma} |\mathrm{Fix}_{(g_i,\gamma)}|.$$

COROLLARY 10. *The number of double covers of G with respect to $\Gamma \leqq \mathrm{Aut}(G;T)$ is*

$$\mathrm{Iso}^R_\Gamma(G;2) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} 2^{\kappa_1(\gamma)+\frac{1}{2}\kappa_2(\gamma)}.$$

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