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ON c-NORMALITY OF FINITE GROUPS

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Abstract

A subgroup H of a finite group G is said to be c-normal in G if there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G = \mathrm{Core}_G(H)$. We are interested in studying the influence of the c-normality of certain subgroups of prime power order on the structure of finite groups.

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1. Introduction

All groups in this paper will be finite. We say, following Wang [11], that a subgroup H of G is c-normal in G if there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$, where $H_G = \mathrm{Core}_G(H) = \bigcap_{g\in G} H^g$ is the maximal normal subgroup of G which is contained in H.

Two subgroups H and K of G are said to permute if $HK = KH$. We say, following Kegel [9], that a subgroup of G is S-quasinormal in G if it permutes with every Sylow subgroup of G.

Let p be a prime and let P be a p-subgroup of G, we write

$$\Omega(P) = \begin{cases} \Omega_1(P) & \text{if } p > 2; \\ \Omega_2(P) & \text{if } p = 2, \end{cases}$$

where $\Omega_i(P)$ is the subgroup of P generated by its elements of order dividing p^i.

Let $\mathfrak{F}$ be a class of groups. We call $\mathfrak{F}$ a formation if $\mathfrak{F}$ contains all homomorphic images of a group in $\mathfrak{F}$, and if G/M and G/N are in $\mathfrak{F}$, then $G/(M \cap N)$ is in $\mathfrak{F}$

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for normal subgroups M, N of G. Each group G has a smallest normal subgroup N such that G/N is in $\mathfrak{F}$. This uniquely determined normal subgroup of G is called the *$\mathfrak{F}$-residual subgroup of* G and will be denoted by $G^{\mathfrak{F}}$. A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. Throughout this paper $\mathfrak{U}$ will denote the class of supersolvable groups. Clearly, $\mathfrak{U}$ is a formation. Since a group G is supersolvable if and only if $G/\Phi(G)$ is supersolvable [6, VI, page 713], it follows that $\mathfrak{U}$ is saturated.

With every prime p we associate some formation $\mathfrak{F}(p)$ ($\mathfrak{F}(p)$ could possibly be empty). We say that $\mathfrak{F}$ is the local formation, locally defined by $\{\mathfrak{F}(p)\}$ provided $G \in \mathfrak{F}$ if and only if for every prime p dividing $|G|$ and every p-chief factor H/K of G, $\mathrm{Aut}_G(H/K) \in \mathfrak{F}(p)$ ($\mathrm{Aut}_G(H/K)$ denotes the group of automorphisms induced by G on H/K and it is isomorphic to $G/C_G(H/K)$). It is known (see [5, IV, 4.6]) that a formation is saturated if and only if it is local.

We assume throughout that $\mathfrak{F}$ is a formation, locally defined by the system $\{\mathfrak{F}(p)\}$ of full and integrated formations $\mathfrak{F}(p)$ (that is, $S_p\mathfrak{F}(p) = \mathfrak{F}(p) \subseteq \mathfrak{F}$ for all primes p, where S_p is the formation of all finite p-groups). It is well known (see [5, IV, 3.7]) that for any saturated formation $\mathfrak{F}$, there is a unique integrated and full system which locally defines $\mathfrak{F}$.

A solvable normal subgroup N of a group G is an $\mathfrak{F}$-hypercentral subgroup of G (see Huppert [7]) provided N possesses a chain of subgroups $1 = N_0 \lhd N_1 \lhd \cdots \lhd N_r = N$ satisfying (i) every factor N_{i+1}/N_i is a chief factor of G, and (ii) if N_{i+1}/N_i has order a power of the prime p_i, then $G/C_G(N_{i+1}/N_i) \in \mathfrak{F}(p_i)$. The product of all $\mathfrak{F}$-hypercentral subgroups of G is again an $\mathfrak{F}$-hypercentral subgroup of G, denoted by $Z_{\mathfrak{F}}(G)$ and called the *$\mathfrak{F}$-hypercentre of a group G*.

Ito in [8], proved that a group G of odd order is nilpotent provided that every subgroup of G of prime order lies in the center of G. Wang [11], proved that if all subgroups of G of prime order or order 4 are c-normal in G, then G is supersolvable. Deyu and Xiuyun [4], proved the following: (i) If K is a normal subgroup of a solvable group G of odd order such that G/K is supersolvable and all subgroups of $\mathrm{Fit}(K)$ of prime order are c-normal in G, then G is supersolvable. (ii) If K is a normal subgroup of a solvable group G such that G/K is supersolvable and all maximal subgroups of all Sylow subgroups of $\mathrm{Fit}(K)$ are c-normal in G, then G is supersolvable.

The aim of this paper is to improve and extend the above mentioned results in [4]. The results of our paper are obtained by independent proofs to those in [4].

Our notation is standard and taken mainly from [5].

2. Preliminary results

LEMMA 2.1. *Let $H \leq K \leq G$.*

(i) *If H is c-normal in G, then H is c-normal in K.*

(ii) *If H is a normal subgroup of G, then K is c-normal in G if and only if K/H is c-normal in G/H.*

PROOF. See [11, Lemma 2.1, page 956]. □

LEMMA 2.2. *Let P be a normal p-subgroup of G and let Q be a q-subgroup of G such that $p \neq q$. If Q is c-normal in G then QP/P is c-normal in G/P.*

PROOF. See [13, Lemma 2.4]. □

LEMMA 2.3. *Let p be the smallest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. If all subgroups of P of order p or order 4 are S-quasinormal and, in particular normal, in G, then G is p-nilpotent.*

PROOF. See [10, Theorem 3.2, page 290]. □

LEMMA 2.4. *Let K be a normal subgroup of G such that $G/K \in \mathfrak{F}$, where $\mathfrak{F}$ is a saturated formation. If $\Omega(P) \leq Z_{\mathfrak{F}}(G)$, where P is a Sylow p-subgroup of K, then $G/O_{p'}(K) \in \mathfrak{F}$.*

PROOF. See [3, Theorem, page 2]. □

LEMMA 2.5. *If G is a solvable group and all subgroups of* $\mathrm{Fit}(G)$ *of prime order or order 4 are S-quasinormal and, in particular normal, in G, then G is supersolvable.*

PROOF. See [2, Corollary 2, page 402]. □

LEMMA 2.6. *If $\mathfrak{F}$ is a saturated formation and N is an $\mathfrak{F}$-hypercentral subgroup of G, then $G/C_G(N) \in \mathfrak{F}$.*

PROOF. This is an easy consequence of a result due to Huppert (see [5, IV, 6.10]). □

LEMMA 2.7. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a solvable group with a normal subgroup K such that $G/K \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of* $\mathrm{Fit}(K)$ *are S-quasinormal and, in particular normal, in G, then $G \in \mathfrak{F}$.*

PROOF. See [1, Theorem 1.4, page 3650]. □

LEMMA 2.8. *Let P be a normal p-subgroup of G. If $P \cap \Phi(G) = 1$, then P is a direct product of abelian minimal normal subgroups of G.*

PROOF. See [5, Theorem 10.6, page 36]. □

3. Main results

We begin with the following lemma:

LEMMA 3.1. *Let p be the smallest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. If all subgroups of P of order p or order 4 are c-normal in G, then G is p-nilpotent.*

PROOF. We prove the result by induction on $|G|$. If all subgroups of P of order p or order 4 are normal in G, then G is p-nilpotent by Lemma 2.3. Thus, we may assume that there exists a subgroup H of P of order p or order 4 such that H is not normal in G. By hypothesis, H is c-normal in G. Then there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$, and since H is not normal in G, it follows that $N < G$. Clearly, $P \cap N$ is a Sylow p-subgroup of N. By Lemma 2.1 (i), all subgroups of $P \cap N$ of order p or order 4 are c-normal in N. Then, by induction on $|G|$, N is p-nilpotent and so also does G. □

REMARK. The formation $\mathfrak{U}$ of all supersolvable groups is locally defined by the integrated and full system $\{\mathfrak{U}(p)\}$, where for each prime p, $\mathfrak{U}(p)$ is the class of all strictly p-closed groups (see [12, Theorem 1.9 and Corollary 1.5]). (Let p be a prime. A group G is said to be strictly p-closed whenever P, a Sylow p-subgroup of G, is normal in G with G/P abelian of exponent dividing $p - 1$.)

We can now prove:

THEOREM 3.2. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and let G be a group. Then the following two statements are equivalent:*

(i) $G \in \mathfrak{F}$.

(ii) *There exists a normal subgroup K in G such that $G/K \in \mathfrak{F}$ and all subgroups of K of prime order or order 4 are c-normal in G.*

PROOF. (i) implies (ii): If $G \in \mathfrak{F}$, then (ii) is true with $K = 1$.

(ii) implies (i): Suppose the result is false and let G be a counterexample of minimal order. By Lemma 2.1 (i) and Lemma 3.1, K possesses an ordered Sylow tower and so K has a normal Sylow p-subgroup P, where p is the largest prime dividing $|K|$. Clearly, P is a normal p-subgroup of G and so $(G/P)/(K/P) \cong G/K \in \mathfrak{F}$. By Lemma 2.2, all subgroups of K/P of prime order or order 4 are c-normal in G/P. Then, by the minimality of G, $G/P \in \mathfrak{F}$. Hence, $1 \neq G^{\mathfrak{F}} \leq P$. If all subgroups of $G^{\mathfrak{F}}$ of order p or order 4 are normal in G, then $\Omega(G^{\mathfrak{F}}) \leq Z_{\mathfrak{U}}(G)$ (see the above Remark). Since $\mathfrak{U}$ and $\mathfrak{F}$ are saturated formations with $\mathfrak{U} \subseteq \mathfrak{F}$, it follows that $Z_{\mathfrak{U}}(G) \leq Z_{\mathfrak{F}}(G)$ (see [5, IV, 3.11]). Hence $\Omega(G^{\mathfrak{F}}) \leq Z_{\mathfrak{F}}(G)$. Applying Lemma 2.4, $G \in \mathfrak{F}$; a contradiction. Thus, there exists a subgroup H of $G^{\mathfrak{F}}$ of order p or order 4 such that H is not normal in G. By hypothesis, H is c-normal in G. Then there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$, and since H is not normal in G, it follows that $N < G$. Clearly, $G^{\mathfrak{F}} \nleq N$. Since G/N is a p-group, it follows that $G/N \in \mathfrak{U} \subseteq \mathfrak{F}$. Hence, $G^{\mathfrak{F}} \leq N$; a final contradiction. □

Below we list some immediate corollaries of Theorem 3.2.

COROLLARY 3.3 (Wang [11, Theorem 4.2, page 964]). *If all subgroups of G of prime order or order 4 are c-normal in G, then G is supersolvable.*

COROLLARY 3.4. *If all subgroups of a group G of prime order are c-normal in G, then G is supersolvable if and only if G is p-nilpotent, where p is the smallest prime dividing $|G|$.*

COROLLARY 3.5. *If G is a solvable group and all subgroups of* Fit(G) *of prime order or order 4 are c-normal in G, then G is supersolvable.*

PROOF. We prove the result by induction on $|G|$. If all subgroups of $\mathrm{Fit}(G)$ of prime order or order 4 are normal in G, then G is supersolvable by Lemma 2.5. Thus, we may assume that there exists a subgroup H of $\mathrm{Fit}(G)$ of prime order or order 4 such that H is not normal in G. By hypothesis, H is c-normal in G. Then there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$, and since H is not normal in G, it follows that $N < G$. Clearly, $G = \mathrm{Fit}(G)N$ and $\mathrm{Fit}(N) < \mathrm{Fit}(G)$. By Lemma 2.1 (i), all subgroups of $\mathrm{Fit}(N)$ of prime order or order 4 are c-normal in N. Then, by induction on $|G|$, N is supersolvable. Since $G/\mathrm{Fit}(G) \cong N/(N \cap \mathrm{Fit}(G))$ is supersolvable, it follows by Theorem 3.2, that G is supersolvable. □

The following example shows that the converse of Corollary 3.3, is not true.

EXAMPLE. Let C_n be a cyclic group of order n. Consider the wreath product $G = C_9\, rwr\, C_2$. Then $|G| = |C_2||C_9|^2$ and so G is supersolvable. It is easy to check that $\Phi(G)$ contains a subgroup H of order 3 that fails to be normal in G and hence H is not c-normal in G. The same example shows that the converse of Corollary 3.5, is not true.

We are now ready to prove:

THEOREM 3.6. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and let G be a group. Then the following two statements are equivalent*:

(i) $G \in \mathfrak{F}$.

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(ii) *There exists a normal solvable subgroup K in G such that $G/K \in \mathfrak{F}$ and all subgroups of* $\mathrm{Fit}(K)$ *of prime order or order* 4 *are c-normal in G.*

PROOF. (i) implies (ii): If $G \in \mathfrak{F}$, then (ii) is true with $K = 1$.

(ii) implies (i): Suppose the result is false and let G be a counterexample of minimal order. By Lemma 2.1 (i) and Corollary 3.5, K is supersolvable. Then by [12, Theorem 1.8, page 6], K possesses an ordered Sylow tower and so K has a normal Sylow p-subgroup P, where p is the largest prime dividing $|K|$. Clearly, P is a normal p-subgroup of G. If all subgroups of P of order p or order 4 are normal in G, then $\Omega(P) \leq Z_{\mathfrak{U}}(G)$. Since $\mathfrak{U}$ and $\mathfrak{F}$ are saturated formations with $\mathfrak{U} \subseteq \mathfrak{F}$, it follows that $Z_{\mathfrak{U}}(G) \leq Z_{\mathfrak{F}}(G)$ (see [5, IV, 3.11]). Hence $\Omega(P) \leq Z_{\mathfrak{F}}(G)$. By Lemma 2.6, $G/C_G(\Omega(P)) \in \mathfrak{F}$ and since $G/K \in \mathfrak{F}$, it follows that $G/C_K(\Omega(P)) \in \mathfrak{F}$. Let V be a Sylow p-subgroup of $C_K(\Omega(P))$. Clearly, $\Omega(V) \leq \Omega(P) \leq Z_{\mathfrak{F}}(G)$. Then by Lemma 2.4, $G/O_{p'}(C_K(\Omega(P))) \in \mathfrak{F}$ and since $O_{p'}(C_K(\Omega(P))) \leq O_{p'}(K)$, it follows that $G/O_{p'}(K) \in \mathfrak{F}$. Then

$$(G/P)/(O_{p'}(K)P/P) \cong G/O_{p'}(K)P \cong (G/O_{p'}(K))/(O_{p'}(K)P/O_{p'}(K)) \in \mathfrak{F}$$

Put $\mathrm{Fit}(O_{p'}(K)P/P) = L/P$. Clearly, $L = P(L \cap O_{p'}(K))$ and so $L/P \cong L \cap O_{p'}(K)$ is nilpotent. Since P and $L \cap O_{p'}(K)$ are normal nilpotent subgroups of K, it follows that $L = P(L \cap O_{p'}(K))$ is a normal nilpotent subgroup of K. Then $L \leq \mathrm{Fit}(K)$ and so $\mathrm{Fit}(O_{p'}(K)P/P) = \mathrm{Fit}(K)/P$. Hence, by Lemma 2.2, all subgroups of $\mathrm{Fit}(O_{p'}(K)P/P)$ of prime order or order 4 are c-normal in G/P. By the minimality of G, $G/P \in \mathfrak{F}$. Then by Theorem 3.2, $G \in \mathfrak{F}$; a contradiction. Thus, there exists a subgroup H of P of order p or order 4 such that H is not normal in G. By hypothesis, H is c-normal in G. Then there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$ and since H is not normal in G, it follows that $N < G$. Clearly, $G = PN = KN$ and so $G/K \cong N/(N \cap K) \in \mathfrak{F}$. Since $N \cap K$ is a normal subgroup of K, it follows that $\mathrm{Fit}(N \cap K) \leq \mathrm{Fit}(K)$. Hence, by Lemma 2.1 (i), all subgroups of $\mathrm{Fit}(N \cap K)$ of prime order or order 4 are c-normal in N. By the minimality of G, $N \in \mathfrak{F}$. Since $G/P \cong N/(N \cap P) \in \mathfrak{F}$, it follows by Theorem 3.2, that $G \in \mathfrak{F}$; a final contradiction. □

Finally we prove the following result:

THEOREM 3.7. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and let G be a solvable group. Then the following two statements are equivalent*:

(i) $G \in \mathfrak{F}$.

(ii) *There exists a normal subgroup K in G such that $G/K \in \mathfrak{F}$ and all maximal subgroups of all Sylow subgroups of* $\mathrm{Fit}(K)$ *are c-normal in G.*

PROOF. (i) implies (ii): If $G \in \mathfrak{F}$, then (ii) is true with $K = 1$.

(ii) implies (i): Suppose the result is false and let G be a counterexample of minimal order. We separate the proof into two cases:

Case 1. $K \cap \Phi(G) \neq 1$. Then there exists a prime p such that p divides $|K \cap \Phi(G)|$. Let P be a Sylow p-subgroup of $K \cap \Phi(G)$. Clearly, P is a normal p-subgroup of G and so $(G/P)/(K/P) \cong G/K \in \mathfrak{F}$. By [6, Satz 3.5, page 270], $\mathrm{Fit}(K/P) = \mathrm{Fit}(K)/P$. Then by Lemma 2.1 (ii) and Lemma 2.2, all maximal subgroups of all Sylow subgroups of $\mathrm{Fit}(K/P)$ are c-normal in G/P. By the minimality of G, $G/P \in \mathfrak{F}$. Since $P \leq \Phi(G)$ and $\mathfrak{F}$ is a saturated formation, it follows that $G \in \mathfrak{F}$; a contradiction.

Case 2. $K \cap \Phi(G) = 1$. If all maximal subgroups of all Sylow subgroups of $\mathrm{Fit}(K)$ are normal in G, then $G \in \mathfrak{F}$ by Lemma 2.7; a contradiction. Thus, there exists a maximal subgroup P_1 of a Sylow p-subgroup P of $Fit(K)$, for some prime p, such that P_1 is not normal in G. By hypothesis, P_1 is c-normal in G. Then there exists a normal subgroup H of G such that $G = P_1H$ with $P_1 \cap H \leq (P_1)_G$, and since P_1 is not normal in G, it follows that $H < G$. Let M be a maximal subgroup of G such that $H \leq M < G$. Then M is a normal subgroup of G as G/H is a p-group and so $G = P_1M = PM$. Since $P \cap \Phi(G) = K \cap \Phi(G) = 1$, it follows by Lemma 2.8, that $P = R_1 \times R_2 \times \cdots \times R_n$, where R_i is a minimal normal subgroup of G, for every $1 \leq i \leq n$. Then $R_i \nleq M$, for some i. Hence, $G = R_iM$ and $R_i \cap M = 1$. Clearly, $(G/R_i)/(K/R_i) \cong G/K \in \mathfrak{F}$. Put $\mathrm{Fit}(K/R_i) = L/R_i$. Since $R_i \leq L \leq R_iM = G$, it follows that $L = R_i(L \cap M)$ and so $L/R_i \cong L \cap M$ is nilpotent. Since R_i and $L \cap M$ are normal nilpotent subgroups of G, it follows that $L = R_i(L \cap M)$ is a normal nilpotent subgroup of G. Then $L = \mathrm{Fit}(K)$ and so $\mathrm{Fit}(K/R_i) = \mathrm{Fit}(K)/R_i$. Hence, by Lemma 2.1 (ii) and Lemma 2.2, all maximal subgroups of all Sylow subgroups of $\mathrm{Fit}(K/R_i)$ are c-normal in G/R_i. By the minimality of G, $G/R_i \in \mathfrak{F}$. Since $G/M \cong R_i \in \mathfrak{U} \subseteq \mathfrak{F}$, it follows that $G \cong G/(R_i \cap M) \in \mathfrak{F}$; a final contradiction. □

REMARKS. (i) Our results are not true for saturated formations which do not contain $\mathfrak{U}$. For example, if $\mathfrak{F}$ is the saturated formation of all nilpotent groups, then the symmetric group of degree three is a counterexample.

(ii) Our results are not true for non-saturated formations. Let $\mathfrak{F}$ be the formation composed of all groups G such that $G^{\mathfrak{U}}$, the supersolvable residual, is elementary abelian. Clearly, $\mathfrak{U} \subseteq \mathfrak{F}$ but $\mathfrak{F}$ is not saturated. Put $G = SL(2,3)$ and $K = Z(G)$. Then G/K is isomorphic to the alternating group of degree four and so $G/K \in \mathfrak{F}$, but G does not belong to $\mathfrak{F}$.

(iii) Theorem 3.2 is not true in general if we replace the condition 'prime order or order 4' by 'prime order', as the following example shows. The class $\mathfrak{F} = \mathfrak{N}\mathfrak{U}$ of groups whose derived subgroup is nilpotent is a saturated formation containing the class $\mathfrak{U}$ of supersolvable groups (see [6, VI, 9.1 (b)]). Consider the group $G = GL(2,3)$. This group has a normal subgroup K isomorphic to to the quaternion group of order 8 such that G/K is isomorphic to the symmetric group of degree 3. Therefore we have that $G/K \cong \mathfrak{I}$. Notice that the unique subgroup of K with prime order is $Z(K)$ and this is not only a c-normal subgroup of G. But the derived group $G' = SL(2, 3)$ is not nilpotent, and then $G \notin \mathfrak{I}$. Since K is a nilpotent group, the same example shows Theorem 3.6 is not true in general if we require that all subgroups of $\mathrm{Fit}(K)$ of prime order are c-normal in G.

(iv) Theorems 3.6 and 3.7 are not true if we omit the condition of solvability. Put $G = H \times K$, where $H \in \mathfrak{U}$ and $K = SL(2, 5)$. Then $|\mathrm{Fit}(K)| = 2$ and $G/K \cong H \in \mathfrak{U}$, but G does not belong to $\mathfrak{U}$.

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