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THE WALLACE PROBLEM: A COUNTEREXAMPLE FROM $\mathrm{MA}_{\mathrm{COUNTABLE}}$ AND p-COMPACTNESS

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ABSTRACT. We construct, under $\mathrm{MA}_{\mathrm{countable}}$, a countably compact topological subsemigroup of $T^{\mathfrak{c}}$ which is not a group, hence a counterexample for the Wallace problem. We also show that there is no p-compact counterexample for the Wallace problem, answering a question of D. Grant. Finally, we show that—in some sense—our counterexample for the Wallace problem constructed under $\mathrm{MA}_{\mathrm{countable}}$ cannot be done in ZFC.

Introduction. We call the Wallace problem the question 3L.1 in [1]: *Is every countably compact topological semigroup with two-sided cancelation a topological group*? It was asked by Wallace in 1953 at the annual meeting of the American Mathematical Society in Baltimore, Maryland.

Robbie and Svetlichny showed recently under CH [10] that there is a counterexample for the Wallace problem using Tkachenko's group [11]. We have shown in [12] that there is a countably compact free abelian group without non-trivial convergent sequences under MA (σ-centered), modifying van Douwen's example in [2]. An immediate corollary of the existence of such a group is the existence of a counterexample for the Wallace problem. We show here that there is a counterexample to Wallace's problem under $\mathrm{MA}_{\mathrm{countable}}$ directly, without finding such a group as above. We remind the reader that there is no free abelian group whose ω-th power is countably compact [12], hence one cannot have this example as a subsemigroup of a free abelian group, since our example contains an ω-bounded subgroup.

We note that what makes the solution presented here simpler is the fact that we were able to give a solution to Wallace's problem without having to deal with the product not being countably compact (in [12], our simplest example would contain two counterexamples for the Wallace problem whose product is not countably compact, by using van Douwen's argument of finding two countably compact groups whose product is not countably compact from a countably compact group without non-trivial convergent sequences).

We can get from $\mathrm{MA}_{\mathrm{countable}}$ a counterexample for the Wallace problem whose square is not countably compact. The construction is longer and more complicated and it is part of our Ph.D thesis.

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In the first section, we give a self-contained proof of the existence of the counterexample for the Wallace problem, applying a strong form of the Baire Category Theorem that is equivalent to $\mathrm{MA}_{\mathrm{countable}}$.

In the second section, we show how to modify the construction to obtain other non-isomorphic counterexamples for the Wallace problem which are subsemigroups of T^{c}.

In the third section we show that there are no regular p-compact counterexamples for the Wallace problem, answering a question from [4]. We apply it to show that certain type of counterexamples for the Wallace problem are independent of ZFC.

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1. A counterexample for the Wallace problem from $\mathrm{MA}_{\mathrm{countable}}$. We will first give a general idea of the construction, then we will give the details. We recall the following definitions:

DEFINITION 1.1. A topological semigroup is a semigroup whose addition is continuous.

DEFINITION 1.2. A space is ω-bounded if the closure of any countable subset is compact.

We remind the reader that $\mathrm{MA}_{\mathrm{countable}}$ is Martin's Axiom restricted to countable posets. This axiom is equivalent to a strong form of the Baire Category Theorem:

(#) The circle T is not the union of fewer than continuum many closed nowhere dense sets.

In this work we will only use (#), so no familiarity with partial orders and generic sets will be required.

For someone who is more comfortable with the Continuum Hypothesis but unfamiliar with Martin's Axiom, note that under the Continuum Hypothesis, (#) is just the usual Baire Category Theorem. We recall that MA implies MA (σ-centered) and MA(σ-centered) implies $\mathrm{MA}_{\mathrm{countable}}$ and that the reverse of either implication is not true. A proof of the consistency of MA can be found in Kunen's book [6]. More on MA and its variations can be found in [16].

1.1. *General framework.* We remind the reader that Robbie and Svetlichny's counterexample was a semigroup that was not a group. The same is true for our counterexample we constructed under MA (σ-centered). In fact, if one is looking for a Tychonoff counterexample for the Wallace problem, that must be the case, since a Tychonoff countably compact group which has continuous addition is a topological group (Reznichenko shows in [9] that it is also true for pseudocompact instead of countably compact).

The semigroup S we construct here will be the (algebraic) direct sum of an ω-bounded group G and a semigroup generated by an element x. As in the previous examples of the

Wallace problem, the semigroup will be a subsemigroup of $T^{\mathfrak{c}}$, where T is the circle with the usual group topology.

More precisely, our semigroup S will be the subsemigroup $\{nx + g : n \in \omega, g \in G\}$.

Clearly S is a both-sided cancellative topological semigroup, since S is a subsemigroup of a topological group.

To make S not a group, we will make sure that x does not have an inverse in S. For that, it is enough that

(∗) for all $n \in \omega$, if $nx \in G$ then $n = 0$.

In fact, suppose there exist $n \in \omega$ and $g \in G$ such that $(nx + g) + x = 0$. Then $(n + 1)x = -g \in G$. Therefore, by (∗), $n + 1 = 0$, contradicting that $n \in \omega$.

Countable compactness of S will be achieved using Hart-van Mill's idea from [5], that is, we will make sure that

(∗∗) every infinite subset of $\{nx : n \in \omega\}$ has an accumulation point in G.

As we will see in Lemma 1.1, this will be enough since G is ω-bounded.

We note that unlike in [5], our ω-bounded group is fixed from the beginning and might be definable in ZFC.

Instead of using an ω-independent family on $\mathfrak{c}$ as in [5], we will use a proper σ-ideal on $\mathfrak{c}$ containing all subsets of size less than $\mathfrak{c}$. We also would like to mention that the ω-bounded group in [5] has size $\mathfrak{c}$, but ours will have size at least $\sup\{2^\alpha : \alpha < \mathfrak{c}\}$.

Before giving more details about the construction, we will need some basic definitions:

DEFINITION 1.3. An ultrafilter p on ω is a non-empty family of subsets of ω satisfying the following:

(1) $\emptyset \notin p$.
(2) The intersection of finitely many elements of p is an element of p.
(3) Every subset of ω containing an element of p is an element of p.
(4) For each $A \subseteq \omega$, either $A \in \omega$ or $\omega \setminus A \in \omega$.

A free ultrafilter p on ω is an ultrafilter which does not contain finite subsets of ω.

Let p be a free ultrafilter on ω and X a topological Hausdorff space.

DEFINITION 1.4. An element $x \in X$ is a p-limit of a sequence $\{x_n : n \in \omega\}$ if for every open neighbourhood U of x, $\{n \in \omega : x_n \in U\} \in p$.

Note that by Hausdorffness, the p-limit of a sequence is unique. We will denote this unique point (if it exists) as $p\text{-}\lim\{x_n : n \in \omega\}$. We remind the reader that the p-limits have nice properties of usual limits of a converging sequence, for instance, for a topological semigroup, the sum of p-limits of two sequences equals the p-limit of the sum of those sequences.

We will show now that (∗∗) is sufficient to have S countably compact:

LEMMA 1.1. *Let S be the semigroup generated by x and G, where G is ω-bounded. If* (∗∗) *is satisfied then S is countably compact.*

PROOF OF LEMMA 1.1. The proof of this lemma is as in [5], but we will give it here for completeness sake.

Let $\{y_k : k \in \omega\}$ be an infinite subset of S. Then for each $k \in \omega$ we can find $n_k \in \omega$ and $g_k \in G$ such that $y_k = n_k x + g_k$. We have two cases to consider:

(i) There is an infinite subset A of ω such that $n_k \neq n_l$ if $k, l \in A$ and $k \neq l$.

From $(**)$, $\{n_k x : k \in A\}$ has an accumulation point y in G. Let p be a free ultrafilter over A such that y is a p-limit of $\{n_k x : k \in A\}$. Since G is ω-bounded, there exists $g \in G$ such that g is p-limit of $\{g_k : k \in A\}$. Clearly $y + g \in G \subseteq S$ is a accumulation point of $\{y_n : n \in \omega\}$ and we are done in this case.

(ii) NOT CASE (i). In this case, we can find an infinite subset B of ω and $n' \in \omega$ such that for each $k \in B$ we have $n_k = n'$. Since G is countably compact, there exist $g \in G$ such that g is an accumulation point of $\{g_k : k \in B\}$. Clearly, $n'x + g \in S$ is an accumulation point of $\{y_n : n \in \omega\}$ hence we are also done in this case. ■

From our discussion above, in order to get a counterexample for the Wallace problem, it suffices to find $x \in T^{\mathfrak{c}}$ and G an ω-bounded subgroup of $T^{\mathfrak{c}}$ satisfying properties $(*)$ and $(**)$.

1.2. *More details.* We start by telling exactly what G will look like.

DEFINITION 1.5. Given $y \in T^{\mathfrak{c}}$, $\operatorname{supp} y = \{\alpha < \mathfrak{c} : y(\alpha) \neq 0\}$. Given a σ-ideal I, on $\mathfrak{c}$, the group generated by I is the group $G_I = \{y \in T^{\mathfrak{c}} : \operatorname{supp} y \in I\}$. When it is clear which I we are using, we will denote it simply by G.

We remind that a σ-ideal I on $\mathfrak{c}$ is a family of subsets of $\mathfrak{c}$, satisfying the following:

(1) $\emptyset \in I$,

(2) every subset of an element of I is an element of I and

(3) the union of countable many elements of I is an element of I.

We have now defined all we need to state our goal in this section:

EXAMPLE 1.1 ($MA_{\text{countable}}$). *There exists $x \in T^{\mathfrak{c}}$ such that for each proper σ-ideal I containing all bounded subsets of $\mathfrak{c}$, the semigroup generated by x and G_I is a counterexample for the Wallace problem.*

We will shortly see the reason to restrict ourselves to proper σ-ideals containing all bounded subsets of $\mathfrak{c}$.

We will denote by 0 the neutral element of T or T^{ξ} for $\xi \leq \mathfrak{c}$ and this will be clear by the context.

How will we get $(*)$ and $(**)$?

We already know what the ω-bounded group G will be, but we need yet to say how we are going to get x. At stage $\gamma + 1$, $\gamma < \mathfrak{c}$, we will define $x(\gamma)$.

For $(*)$ and $(**)$ to be satisfied, we will need a couple of inductive hypothesis that must be satisfied during the construction of x.

To show $(*)$, it suffices to show that for each $n \in \omega$, and for every $y \in G$, there is a β in $\mathfrak{c}$ such that $nx(\beta) \neq y(\beta)$. In fact we will have $\beta \in \mathfrak{c}$ such that $y(\beta) = 0 \neq nx(\beta)$. This can be done by making $\operatorname{supp} nx = \mathfrak{c}$, as $\operatorname{supp} y$ is a proper subset of $\mathfrak{c}$.

In order to explain the inductive hypothesis needed to get $(**)$, we have to fix first an enumeration $\{E_\alpha : \alpha < \mathfrak{c}\} = [\omega]^\omega$ of all infinite subsets of ω.

This will be used to code every infinite subset of $\{nx : n \in \omega\}$, that is, for every subset A of $\{nx : n \in \omega\}$ there exists $\alpha < \mathfrak{c}$ such that $A = \{nx : n \in E_\alpha\}$.

To show $(**)$, we need to guarantee an accumulation point for each infinite subset of $\{nx : n \in \omega\}$. Roughly speaking, during the inductive process we will promise to the countable subsets of $\{nx : n \in \omega\}$ an accumulation point.

As we said earlier, we only will know completely x and consequently $\{nx : n \in \omega\}$ in the end of the construction, but we know more about the sequence as we know more about x because of the coding we mentioned earlier. More precisely, at stage β, we will know $x{\restriction}\beta$, therefore we can code $\{nx{\restriction}\beta : n \in E_\beta\}$. We will assign a function $g_\beta \in T^{\mathfrak{c}}$ such that $\operatorname{supp} g_\beta$ is bounded and $g_\beta{\restriction}\beta$ is accumulation point of $\{nx{\restriction}\beta : n \in E_\beta\}$. From then on, we will make sure that for each $\alpha > \beta$, the point $g_\beta{\restriction}\alpha$ is accumulation point of $\{nx{\restriction}\alpha : n \in E_\beta\}$. In particular, at the end of the construction, $g_\beta \in G$ will be an accumulation point of $\{nx : n \in E_\beta\}$.

As every subset of $\{nx : n \in \omega\}$ is coded by some E_β, every subset of $\{nx : n \in \omega\}$ will have an accumulation point in G and $(**)$ will be satisfied.

Note that the x we find works for any group generated by a proper σ-ideal containing all bounded subsets of $\mathfrak{c}$, as $(*)$ and $(**)$ will be simultaneously satisfied for x and any of the groups just described.

Summarizing what we have done so far:

LEMMA 1.2. *Suppose that there exist* $\{x_\alpha : \alpha \leq \mathfrak{c}\}$ *and* $\{g_\beta : \beta < \mathfrak{c}\} \subseteq \{g \in T^{\mathfrak{c}} :$ $\operatorname{supp} g$ *is bounded in* $\mathfrak{c}\}$ *such that the following are satisfied:*

(0) for each $\beta \leq \mathfrak{c}$ *we have* $x_\beta \in T^\beta$.

(1) if $\beta < \alpha \leq \mathfrak{c}$ *then for each* $n \in \mathbb{N}$ *we have* $nx_\alpha(\beta) \neq 0$ *and* $x_\beta \subseteq x_\alpha$.

(2) if $\beta < \alpha \leq \mathfrak{c}$ *then* $g_\beta{\restriction}\alpha$ *is an accumulation point of* $\{nx_\alpha : n \in E_\beta\}$.

Then $(*)$ *and* $(**)$ *are satisfied for* $x = x_{\mathfrak{c}}$ *and any* G_I, *where* I *is a proper* σ*-ideal containing all bounded subsets of* $\mathfrak{c}$. *Therefore the semigroup generated by* x *and each* G_I *is a counterexample for the Wallace problem.*

Therefore, to get Example 1.1, it suffices to show that we can produce $\{x_\alpha : \alpha \leq \mathfrak{c}\}$ and $\{g_\beta : \beta < \mathfrak{c}\}$. For some σ-ideals, such families do not exist in ZFC. In fact, in Section 3, we will see that there is a model in which for the σ-ideal I of all bounded subsets of $\mathfrak{c}$, there is no x such that the semigroup generated by x and G_I is a counterexample for the Wallace problem.

1.3. *The induction.* We will show now that using (#), the conditions in Lemma 1.2 can be satisfied.

We will be done with the construction of Example 1.1 by proving the following:

LEMMA 1.3 ($\mathrm{MA}_{\mathrm{countable}}$). *There exist* $\{x_\alpha : \alpha \leq \mathfrak{c}\}$ *and* $\{g_\beta : \beta < \mathfrak{c}\}$ *satisfying conditions (0)–(2) from Lemma 1.2.*

PROOF OF LEMMA 1.3. These families are constructed by induction.

How g_γ is chosen: At stage $\gamma+1$, we will fix y_γ any accumulation point of $\{nx_\gamma : n \in E_\gamma\}$ and define $g_\gamma = y_\gamma \cup 0 \restriction [\gamma, \mathfrak{c})$.

How to define x_α's.

We will do it in cases, we note that we will use $\mathrm{MA}_{\mathrm{countable}}$ (in fact (#)) only in the successor stage:

CASE 1. $\alpha = 1$. Let $x_1 \in T^1$ be any function such that for each $n \in \mathbb{N}$ we have $nx_1(0) \neq 0$.

CASE 2. α limit. Let $x_\alpha = \bigcup_{\beta<\alpha} x_\beta$.

Clearly the inductive hypothesis are satisfied in both cases so we will now show how to deal with the successor case.

CASE 3. $\alpha = \gamma + 1$. As mentioned before we define $g_\gamma \in T^{\mathfrak{c}}$ such that $g_\gamma \restriction \gamma$ is an accumulation point of $\{nx_\gamma : n \in E_\gamma\}$.

For every $\beta \leq \gamma$, $F \in [\gamma]^{<\omega}$ and $k \in \omega$ define

$$S(\beta, F, k) = \left\{ n \in E_\beta : \forall \eta \in F |nx_\alpha(\eta) - g_\beta(\eta)| < \frac{1}{k+1} \right\}.$$

Note that $S(\beta, F, k)$ is infinite. Now, we have to choose $x_\alpha(\gamma)$ which fulfill the promises.

Note that it suffices that for all β, F and k as above the set $\{n \in S(\beta, F, k) : |nx_\alpha(\gamma) - g_\beta(\gamma)| < \frac{1}{k+1}\}$ is infinite and that $nx_\alpha(\gamma) \neq 0$ for all $n \in \mathbb{N}$.

In order to apply (#) we define now fewer than $\mathfrak{c}$ many dense open sets:

(a) For each $n \in \mathbb{N}$ let $O_n = \{a \in T : na \neq 0\}$.

(b) For each $\beta \leq \gamma$, $F \in [\gamma]^{<\omega}$ and $k, m \in \omega$, let

$$O(\beta, F, k, m) = \left\{ a \in T : \exists n \in S(\beta, F, k) \left[n \geq m \text{ and } |na - g_\beta(\gamma)| < \frac{1}{k+1} \right] \right\}$$

CLAIM 1. *The sets defined above are dense open in T.*

PROOF OF THE CLAIM. (a) Each O_n has finite complement and hence is open and dense in T.

(b) Fix $O(\beta, F, k, m)$.

We will show first that this set is open. Let $b \in O(\beta, F, k, m)$. By definition, there exist $n \in S(\beta, F, k)$ such that $n \geq m$ and $|nb - g_\beta(\gamma)| < \frac{1}{k+1}$. By continuity, clearly there exist an open set W containing b such that for all $a \in W$ we have $|na - g_\beta(\gamma)| < \frac{1}{k+1}$. Therefore $W \subseteq O(\beta, F, k, m)$ and we are done.

We will show now that $O(\beta, F, k, m)$ is dense in T. Let V be any open subset of T. Let $n \in S(\beta, F, k)$ such that $n \geq m$ and $nV = T$. Let $a \in V$ such that $|na - g_\beta(\gamma)| < \frac{1}{k+1}$. Then $a \in O(\beta, F, k, m)$ as well, therefore, $V \cap O(\beta, F, k, m) \neq \emptyset$ so $O(\beta, F, k, m)$ is dense. ∎

By (#), $\bigcap_{\beta,F,k,m} O(\beta, F, k, m) \cap \bigcap_{n\in\mathbb{N}} O_n \neq \emptyset$.

Let a be any element of $\bigcap_{\beta,F,k,m} O(\beta, F, k, m) \cap \bigcap_{n\in\mathbb{N}} O_n$. To end the proof of the Lemma 1.3, it suffices to show the following

CLAIM 2. $x_\alpha = x_\gamma \cup \{\langle \gamma, a \rangle\}$ *satisfies the inductive hypothesis.*

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PROOF OF THE CLAIM. Clearly (0) is satisfied by x_α. Also it is easy to see that (1) is satisfied, since $x_\alpha(\gamma) = a \in O_n = T \setminus \{b \in T : nb = 0\}$. Let us show now that (2) is satisfied for x_α and we will be done. As noted before, to show that $g_\beta \restriction \alpha$ is an accumulation point of $\{nx_\alpha : n \in E_\beta\}$, it suffices to show that for every $F \in [\alpha]^{<\omega}$ and $k \in \omega$ we have that the set $\{l \in S(\beta, F, k) : |lx_\alpha(\gamma) - g_\beta(\gamma)| < \frac{1}{k+1}\} = \{l \in E_\beta : (\forall \eta \in F \cup \{\gamma\}) |lx_\alpha(\eta) - g_\beta(\eta)| < \frac{1}{k+1}\}$ is infinite.

Let $m \in \mathbb{N}$. By construction, $x_\alpha(\gamma) \in O(\beta, F \setminus \{\gamma\}, k, m)$, that is, there exists $n \in S(\beta, F \setminus \{\gamma\}, k)$ such that $n \geq m$ and $|nx_\alpha(\gamma) - g_\beta(\gamma)| < \frac{1}{k+1}$. Therefore $\{l \in S(\beta, F, k) : |lx_\alpha(\gamma) - g_\beta(\gamma)| < \frac{1}{k+1}\}$ is unbounded in ω and we are done. ∎

NOTE. We could have made $\{nx : n \in \omega\}$ dense in $T^{\mathfrak{c}}$ by adding some new dense open sets at successor stages. ∎

REMARKS. It is a short proof (in ZFC) to show that

(+) every countably compact torsion free group such that every countable subset of it has closure of size at least $\mathfrak{c}$ contains a semigroup that is a counterexample for the Wallace problem.

Our example has convergent sequences as well as elements of finite order hence it could not be obtained by Robbie-Svetlichny's approach, since they relied on (+) to obtain their counterexample under CH.

We remind that every countably compact free abelian group without convergent sequences satisfies (+).

In [12] we showed that every countably compact free abelian group without non-trivial convergent sequences contains at least $|\alpha| + \aleph_0$ non-isomorphic counterexamples for the Wallace problem, where $\mathfrak{c} = \aleph_\alpha$.

We also showed that under MA (σ-centered), there exist a semigroup and $\sup\{\kappa < \mathfrak{c} : \kappa \text{ cardinal}\}$ non-homeomorphic topologies that make it a separable counterexample for the Wallace problem whose square is not countably compact (also, under MA (σ-centered), there are $\sup\{\kappa < \mathfrak{c} : \kappa \text{ cardinal}\}$ non-homeomorphic topologies that make the free abelian group generated by $\mathfrak{c}$ many elements a countably compact separable topological group whose square is not countably compact).

2. Some non-isomorphic counterexamples inside $T^{\mathfrak{c}}$.

We will show now how to get some other non-isomorphic counterexamples inside $T^{\mathfrak{c}}$ by choosing some special σ-ideals.

DEFINITION 2.1. A topological space X is κ-bounded if every subset of X of size less or equal to κ has compact closure.

NOTATION. The cofinality of κ will be denoted by $\mathrm{cf}(\kappa)$.

Let z be the unique non-zero element of T such that $z + z = 0$. Given a subset A of $\mathfrak{c}$, we will denote by χ_A the function from $\mathfrak{c}$ into $\{0, z\}$ such that $\chi_A(a) = z$ if and only if $a \in A$. We will call it here the characteristic function of A.

EXAMPLE 2.1 ($MA_{countable}$). *Suppose $\kappa < cf(\mathfrak{c})$ is an infinite cardinal. Then there exist $x \in T^{\mathfrak{c}}$ and $G_\kappa \subseteq T^{\mathfrak{c}}$ that is κ-bounded but not κ^+-bounded such that the semigroup generated by x and G_κ is a counterexample for the Wallace problem.*

In order to get a κ-bounded group, we will choose a proper κ-ideal. For this, we will use the fact that $\kappa < cf(\mathfrak{c})$.

THE CHANGES. As mentioned above we choose some special ideals. Fix $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ such that for all $\alpha < \mathfrak{c}$ A_α is a subset of $\mathfrak{c}$ of size $\mathfrak{c}$ and if $\alpha \neq \beta$ then $A_\alpha \cap A_\beta = \emptyset$.

Let I_κ be the κ-ideal generated by $\mathcal{A} \cup \{\alpha : \alpha < \mathfrak{c}\}$. (Since $\kappa < cf(\mathfrak{c})$, I_κ is proper). Let G_κ be the group generated by I_κ.

Clearly G_κ is κ-bounded. To check that G_κ is not κ^+-bounded, first note that $A_\alpha \in I_\kappa$ and $\bigcup_{\alpha<\kappa^+} A_\alpha \notin I_\kappa$.

Then $\{\chi_{\bigcup_{\alpha\in F} A_\alpha} : F \in [\kappa^+]^{<\omega}\} \in [G_\kappa]^{\kappa^+}$, $\chi_{\bigcup_{\alpha<\kappa^+} A_\alpha} \in \overline{\{\chi_{\bigcup_{\alpha\in F}} A_\alpha : F \in [\kappa^+]^{<\omega}\}}$, but $\chi_{\bigcup_{\alpha<\kappa^+} A_\alpha} \notin G_\kappa$. Hence G_κ is not κ^+-bounded. Since I_κ satisfies the conditions from Example 1.1, we are done. ∎

REMARK. I would like to thank the referee for pointing out that an enumeration we were using to obtain a condition weaker than $(*)$ was not necessary and for suggesting the sets O_n to obtain $(*)$. From that we could use more general σ-ideals as well (our σ-ideals used to be the ones generated by $\mathcal{A} \cup \{\alpha : \alpha < \mathfrak{c}\}$). Previously, because of the enumeration, the example above was not a particular case of the example in Section 1.

We note that we could use O_n for T but not for 2, since $\{0\}$ is open in 2. In [13], we constructed a subgroup of $2^{\mathfrak{c}}$ whose square is countably compact but whose cube is not. In that case, it seems we need our original condition related to $(*)$ using the enumeration.

COROLLARY 2.1 ($MA_{countable}$). *$T^{\mathfrak{c}}$ contains at least $|\gamma|$ many non-isomorphic counterexamples for the Wallace problem, where $cf(\mathfrak{c}) = \aleph_\gamma$.*

PROOF OF THE COROLLARY. From Example 2.1, for all $\alpha < \gamma$ there exist x and $G_{\aleph_\alpha}$ such that the semigroup S_α generated is a counterexample for the Wallace problem. We claim that for $\alpha < \beta < \gamma$, the semigroups S_α and S_β are not isomorphic. For this, it suffices to notice that the set of elements of S_α that have inverse is $G_{\aleph_\alpha}$ and that $G_{\aleph_\alpha}$ is not homeomorphic to $G_{\aleph_\beta}$. ∎

REMARKS. All the semigroups we have constructed here so far have size at least $\sup\{2^\lambda : \lambda < \mathfrak{c}\}$, as it contains all elements of $T^{\mathfrak{c}}$ whose support is bounded. However, using closing-off arguments one can get a counterexample for the Wallace problem of size $\mathfrak{c}$.

3. **Further results and limitations.** As we discussed in the introduction, the first counterexample for the Wallace problem was only obtained in 1994. Some authors have proved before that a counterexample for the Wallace problem cannot have certain properties. In 1957, Ellis [3] showed that a locally compact group whose addition is assumed only to be separately continuous is a topological group, in particular a

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counterexample for the Wallace problem cannot be algebraically a group and locally compact.

We recall that a locally compact topological semigroup is not necessarily a group, in fact, $[0, 1)$ with the usual multiplication is an example. In 1972, Mukherjea and Tserpes [7] showed that there is no first countable counterexample for the Wallace problem.

Pfister [8] showed that every (locally) countably compact regular paratopological group (addition is continuous, but the inversion is not necessarily continuous) is a topological group, in particular, a counterexample for the Wallace problem cannot be algebraically a group.

Grant [4] gave some other properties that a counterexample for the Wallace problem cannot have, among them, sequential compactness. He also mentioned that it was known that there is no ω-bounded counterexample for the Wallace problem.

3.1. *p-compactness.* We recall the following:

DEFINITION 3.1. A p-compact space is a Hausdorff space such that every sequence has a p-limit.

We remind that every ω-bounded space is p-compact for every free ultrafilter p on ω. More on p-limits and p-compact spaces can be found in [15].

The fact that there is no sequentially compact or ω-bounded counterexample for the Wallace problem motivated the following question asked in [4]:

QUESTION 3. *Is every p-compact* ($T_{3\frac{1}{2}}$), *cancellative topological semigroup a topological group?*

We have shown the following:

THEOREM 3.1. *Let p be a free ultrafilter. Then every p-compact both-sided cancellative semigroup is a group.*

From Theorem 3.1 and Pfister's result mentioned above, we answer Question 3. We note that one can modify the proof of Theorem 3.1 below to show that every sequentially compact both-sided cancellative semigroup is a group.

PROOF OF THEOREM 3.1 . Suppose there exist an S that is p-compact both-sided cancellative semigroup but is not a group. Without loss of generality we can assume that S has the neutral element 0. In fact, if S does not have the neutral element then let $0 \notin S$ and define $S \cup \{0\}$, where $\{0\}$ is an open subset and we consider the topological sum of S and $\{0\}$.

Note that we still would have a p-compact both-sided cancellative semigroup that is not a group. Although the semigroup is not assumed to be abelian, we will use + to denote the binary operation. Since the addition is continuous, it is easy to see that given $\{a_n : n \in \omega\}$ and $\{b_n : n \in \omega\}$, $p\text{-}\lim\{a_n + b_n : n \in \omega\} = p\text{-}\lim\{a_n : n \in \omega\} + p\text{-}\lim\{b_n : n \in \omega\}$. S is not a group, therefore, there exist $x \in S$ that does not have inverse, that is, for all $y \in S$ we have $x + y \neq 0$ or $y + x \neq 0$.

Define $f_n \in S^\omega$ as follows:

$$f_n(i) = \begin{cases} (n-i)x & \text{if } n > i \\ 0 & \text{otherwise} \end{cases}.$$

For all $i \in \omega$, let $f(i) = p\text{-}\lim\{f_n(i) : n \in \omega\}$. Note that $f_n(i) = (n-i)x$ for all but finitely many n's. Therefore, $f(i) + ix = p\text{-}\lim\{f_n(i) : n \in \omega\} + ix = p\text{-}\lim\{f_n(i) + ix : n \in \omega\} = p\text{-}\lim\{nx : n \in \omega\} = f(0)$. Hence $(*)$ $\forall i \in \omega f(i) + ix = f(0)$.

From $(*)$, $f(0) = p\text{-}\lim\{f(2i) + (2i)x : i \in \omega\} = p\text{-}\lim\{f(2i) : i \in \omega\} + p\text{-}\lim\{2(ix) : i \in \omega\} = p\text{-}\lim\{f(2i) : i \in \omega\} + p\text{-}\lim\{ix : i \in \omega\} + p\text{-}\lim\{ix : i \in \omega\} = p\text{-}\lim\{f(2i) : i \in \omega\} + 2f(0)$. By the cancelation property, $p\text{-}\lim\{f(2i) : i \in \omega\} + f(0) = 0$.

However, from $(*)$ applied to 1, we have $f(1) + x = f(0)$, therefore, $\big(p\text{-}\lim\{f(2i) : i \in \omega\} + f(1)\big) + x = p\text{-}\lim\{f(2i) : i \in \omega\} + \big(f(1) + x\big) = p\text{-}\lim\{f(2i) : i \in \omega\} + f(0) = 0$. Since S is cancellative, this means that x has inverse, contradiction. ∎

3.2. *Products.* As we mentioned before the proof of Theorem 3.1, no counterexample for the Wallace problem is p-compact.

We recall that if the $2^{\mathfrak{c}}$-th power of a space is countably compact, then the space is p-compact for some free ultrafilter p. Therefore, no counterexample for the Wallace problem can have the $2^{\mathfrak{c}}$-th power countably compact.

From this, it is natural to ask: *which is the minimal power we have to consider in order not to have a counterexample for the Wallace problem?*

As we mentioned earlier we can obtain a counterexample for the Wallace problem under $\mathrm{MA}_{\mathrm{countable}}$ whose square is not countably compact, but it is not known yet whether one can obtain a counterexample for the Wallace problem whose square is countably compact. We have shown in [12] that the ω-th power of a subsemigroup of a free abelian group without non-trivial convergent sequences is not countably compact.

We have also shown that the same is true for semigroups without identity that are subsemigroups of a free abelian group. Therefore, the previously known counterexamples for the Wallace problem, namely the one we have obtained under MA (σ-centered) in [12] and Robbie and Svetlichny's CH example in [10] have the ω-th power not countably compact. Related to this, we also showed in [12] that the ω-th power of every free abelian group is not countably compact.

I believe it is still unknown whether one can find a free abelian group whose square is countably compact. Related to these type of construction, in [13] we have obtained a group under $\mathrm{MA}_{\mathrm{countable}}$ whose square is countably compact but whose cube is not. However that proof relies on the fact that we are working on $2^{\mathfrak{c}}$ so the same arguments would not be enough to get a counterexample for the Wallace problem whose square is countably compact.

PROPOSITION 3.1. *Let H be a topological abelian group and 0 its neutral element. Let $E \ni 0$ be a subsemigroup of H of size at most $\mathfrak{c}$ and G be an ω-bounded subgroup of H. Then $(E+G)^{\mathfrak{c}}$ countably compact implies that $E+G$ is not a counterexample for the Wallace problem.*

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PROOF OF THE PROPOSITION. Suppose that $(E+G)^{\mathfrak{c}}$ is countably compact. It suffices to show that $E+G$ is a group, since it is a subgroup of a topological group. Since H is abelian, $E+G$ is a subsemigroup. Clearly $E+G$ is cancellative, since is a subsemigroup of a group. E^{ω} has size at most $\mathfrak{c}$, since E has size at most $\mathfrak{c}$. For each $n \in \omega$ define $F_n \in (E+G)^{E^{\omega}}$ as follows:

For each $f \in E^{\omega}$ let $F_n(f) = f(n)$. By the countable compactness of $(E+G)^{E^{\omega}}$, there exist $F \in (E+G)^{E^{\omega}}$ that is an accumulation point of $\{F_n : n \in \omega\}$. Let p be a free ultrafilter such that $F = p\text{-}\lim\{F_n : n \in \omega\}$.

CLAIM 3. *$E+G$ is p-compact.*

PROOF OF THE CLAIM. Let $\{x_n : n \in \omega\}$ be a sequence in $E+G$. Fix $a_n \in E$ and $b_n \in G$ such that $a_n + b_n = x_n$. Let $f \in E^{\omega}$ such that $f(n) = a_n$, for all $n \in \omega$.

Then $F(f)$ is the $p\text{-}\lim\{F_n(f) : n \in \omega\} = p$-limit of $\{a_n : n \in \omega\}$. Since G is ω-bounded, there exist $b \in G$ such that b is the p-limit of $\{b_n : n \in \omega\}$. Hence $F(f) + b$ is the the p-limit of $\{x_n : n \in \omega\}$. Since the sequence of x_n's was chosen arbitrarily, we are done. ∎

Now, $E+G$ satisfies the condition from Theorem 3.1, hence $E+G$ is a group and we are done. ∎

An immediate corollary of this:

COROLLARY 3.1. *The Example 1.1 has the $\mathfrak{c}$-th power not countably compact.*

3.3. *Limitations.* We recall that our counterexample for the Wallace problem presented in this paper is, algebraically, a direct sum of $\{ny : n \in \omega\}$ for some $y \in T^{\mathfrak{c}}$ and an ω-bounded group which could be $\{x \in T^{\mathfrak{c}} : \operatorname{supp} x \text{ is bounded}\}$.

In this sense, such a counterexample cannot be achieved in ZFC. Before we give more details, we recall that Kunen's Axiom (KA) is the following statement:

There exist a free ultrafilter p on ω that is generated by a basis of size $\aleph_1$.

Note that CH implies KA trivially. We will use the fact that KA $+\mathfrak{c} = \omega_2$ is consistent (see Exercise VIIIA.10 in [6]).

DEFINITION 3.2. A space is initially ω_1-compact if every open cover of size at most $\aleph_1$ has a finite subcover.

We recall that a space is initially ω_1-compact if and only if every infinite subset X of size at most $\aleph_1$ has a complete accumulation point x, that is for each open neighbourhood U of x, $|X| = |U \cap X|$.

PROPOSITION 3.2 (KA). *Every initially ω_1-compact both-sided cancellative semigroup is a group.*

PROOF OF THE PROPOSITION. If p is any free ultrafilter generated by $\aleph_1$ many elements, then every initially ω_1-compact space is p-compact. We are done by applying Theorem 3.1. ∎

We are ready now to show that the type of counterexample for the Wallace problem we obtained needs some set-theoretic assumption.

THEOREM 3.2 (KA). *Let H be a topological abelian group. Let E be a countable subset of H and G be an ω_1-bounded subgroup of H. Let $S = S_H(E, G)$ be the smallest semigroup containing $E \cup G$. Then S is not a counterexample for the Wallace problem.*

PROOF OF THE THEOREM. Suppose by contradiction that S is a counterexample for the Wallace problem. In particular, S is countably compact. Since E is countable and G is ω_1-bounded, it is easy to show that S is initially ω_1-compact. Then by Proposition 3.2, S is a group, hence a topological group, since it is a subgroup of a topological group, contradiction. ∎

LEMMA 3.3 ($\mathrm{MA}_{\mathrm{countable}} + \mathrm{cf}(\mathfrak{c}) > \aleph_1$). *There exists an initially ω_1-compact counterexample for the Wallace problem.*

PROOF OF THE LEMMA. In Example 2.1, we have seen that under $\mathrm{MA}_{\mathrm{countable}} + \mathrm{cf}(\mathfrak{c}) > \aleph_1$, there exist x and an ω_1-bounded group H (for instance the group generated by the ideal of bounded subsets of $\mathfrak{c}$) such that the semigroup generated was a counterexample for the Wallace problem.

It is not difficult to show that this semigroup is initially ω_1-compact. ∎

THEOREM 3.4. *The existence of an initially ω_1-compact Tychonoff counterexample for the Wallace problem is independent of $\mathfrak{c} = \aleph_2$.*

We could have considered any cardinal arithmetic for $\mathfrak{c}$, as long as KA and $\mathrm{MA}_{\mathrm{countable}}$ are consistent with it and $\mathrm{cf}(\mathfrak{c}) > \aleph_1$.

PROOF OF THE THEOREM. KA and $\mathrm{MA}_{\mathrm{countable}}$ are consistent with $\mathfrak{c} = \aleph_2$. Lemma 3.3 gives a model where such a counterexample exists.

Proposition 3.2 gives us a model where every initially ω_1-compact cancellative semigroup is a group. From results mentioned earlier, such Tychonoff groups are topological groups, hence they are not counterexamples for Wallace's problem. ∎

Related to Theorem 3.4, we have shown in [12] that the existence of a initially ω_1-compact free abelian group is independent of $\mathfrak{c} = \aleph_2$.

COROLLARY 3.2. *Let $G = \{y \in T^{\mathfrak{c}} : \mathrm{supp}\, y$ is bounded in $\mathfrak{c}\}$.*
Then the following is independent of $\mathfrak{c} = \aleph_2$: There exist $x \in T^{\mathfrak{c}}$ such that the semigroup generated by x and G is a counterexample for the Wallace problem.

FINAL REMARK. D. Grant has also asked in [4] whether a counterexample for the Wallace problem could have the square countably compact or pseudocompact. It is easy to see that our example is pseudocompact for any of its powers, since it contains a dense ω-bounded subgroup.

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