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DETERMINING THE FRATTINI SUBGROUP FROM THE CHARACTER TABLE

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1. Introduction. Brauer [**1**, p. 141] has discussed the question of which subgroups of a group can be determined from its character table. He mentions, referring to p-groups, that the Frattini subgroup can be determined. We show that for an arbitrary finite solvable group, the Frattini subgroup can be determined from the character table. Then we exhibit an infinite set of pairs of non-solvable groups such that both members of a given pair have the same character table but Frattini subgroups of different orders. All groups to be considered are finite.

2. The solvable case. For a group G and a prime p, recall that $\mathbf{O}_p(G)$ is the largest normal p-subgroup in G and that the Frattini subgroup, $\Phi(G)$, is the intersection of all maximal subgroups of G.

THEOREM. *Let G be a solvable group and $\mathcal{O}$ be the set of all $N \trianglelefteq G$ such that for some prime p, $\mathbf{O}_p(G/N)$ is the unique minimal normal subgroup of G/N. Then $\cap\, \mathcal{O} = \Phi(G)$. In particular, the Frattini subgroup can be determined from the character table.*

Proof. Let $\mathcal{M}$ be the set of maximal subgroups of G and $x \in \cap \mathcal{M} = \Phi(G)$. We choose $N \in \mathcal{O}$ and show $x \in N$. If N is a maximal subgroup, we are done. Otherwise, choose O and X with $N < O < X \leqq G$ and with O/N and X/O chief factors of G. Since G is solvable, the definition of $\mathcal{O}$ implies there are distinct primes p, q such that $O/N = \mathbf{O}_p(G/N)$ and X/O is a q-group. Let Q/N be a Sylow q-subgroup of X/N so that $QO = X$. Then by the Frattini argument, the normalizer $\mathbf{N}(Q)$ satisfies $G = \mathbf{N}(Q)X = \mathbf{N}(Q)O$. We claim $\mathbf{N}(Q) \in \mathcal{M}$. Indeed if $\mathbf{N}(Q) \leqq M \leqq G$ then $M \cap O$ is normalized by $MO = \mathbf{N}(Q)O = G$. Now O/N is a chief factor and if $M \cap O = O$ then $M = MO = G$. Alternately, if $M \cap O = N$ then

$$M = M \cap \mathbf{N}(Q)O = \mathbf{N}(Q)(M \cap O) = \mathbf{N}(Q)N = \mathbf{N}(Q).$$

Further, Q is not normal in G, so $\mathbf{N}(Q) \in \mathcal{M}$. Since O/N is the unique minimal normal subgroup of G/N, the conjugates of $\mathbf{N}(Q)$ intersect in N. Thus $x \in \cap \mathcal{M}$ implies $x \in N$. But N is an arbitrary member of $\mathcal{O}$ so $\Phi(G) = \cap \mathcal{M} \subseteq \cap\, \mathcal{O}$.

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To show $\cap \mathcal{O} \subseteq \Phi(G)$, pick $M \in \mathcal{M}$. We claim $N = \cap \{M^g | g \in G\}$ is in $\mathcal{O}$. Let O/N be a minimal normal subgroup of G/N. Now N being the maximal normal subgroup of G in M implies the following three statements. First, $O \nsubseteq M$ so by the maximality of M, $MO = G$. Second, since $M \cap O$ is normalized by $MO = G$, $M \cap O = N$. Now suppose O_1 and O_2 are subgroups containing N such that O_i/N is minimal normal in G/N for $i = 1, 2$. Then O_1/N centralizes O_1O_2/N so that $M \cap O_1O_2$ is normalized by $O_1M = G$. We see finally that $M \cap O_1O_2 = N$. It follows that

$$O_1 = O_1N = O_1(M \cap O_1O_2) = G \cap O_1O_2 = O_1O_2.$$

Thus $O_1 = O_2$. Let O/N be this unique minimal normal subgroup of G/N and suppose it has p-power order. We let $X/N = \mathbf{O}_p(G/N)$ and show $X = O$. Since O/N is the unique minimal normal subgroup in G/N, $O/N \subseteq \mathbf{Z}(X/N)$. Thus $MO = G$ normalizes $M \cap X$. As N is the maximal normal subgroup of G in M, we have $M \cap X = N$. It follows that

$$X = MO \cap X = (M \cap X)O = NO = O.$$

Thus $N \in \mathcal{O}$ and we have $\cap \mathcal{O} \subseteq N \subseteq M$. Since M is arbitrary,

$$\cap \mathcal{O} \subseteq \cap \mathcal{M} = \Phi(G).$$

Finally, since the lattice of normal subgroups and their orders can be determined from the character table, the set $\mathcal{O}$ can be found. Thus the classes in $\cap \mathcal{O} = \Phi(G)$ can be determined.

3. Non-solvable counter-examples. Character tables which provide counter-examples to the extension of the above theorem to the non-solvable case include those of groups of the form $G_n = PSL(2, \mathbf{Z}_{p^2})$ for $p \geqq 5$ a prime. The "n" in the notation "G_n" refers to the fact that G_n is a non-split extension of $PSL(2, p)$ by an elementary abelian group of order p^3. Let N_n denote this normal subgroup of G_n. Then the group G_s, defined as the semi-direct product of G_n/N_n and N_n using the natural action, has the same character table as G_n but a Frattini subgroup of a different order.

We now show that G_n and G_s have the same character tables. We must construct a bijection between the characters of G_n and G_s and a bijection between the classes of the two groups so that corresponding characters have the same value on corresponding classes. This will be done separately, but in a coherent manner, for certain blocks (submatrices) of the character table matrix by identifying these blocks with blocks of the character tables of isomorphic sections of G_n and G_s. Before constructing the bijections, some group theoretic and character theoretic information will be needed.

4. Group theoretic information. First we consider the action of G_n/N_n on N_n. We define N_n more explicitly as the kernel of the natural homomorphism from $G_n = PSL(2, \mathbf{Z}_{p^2})$ onto $PSL(2, p)$ obtained by applying to the matrix

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entries the natural homomorphism from $\mathbf{Z}_{p^2}$ to $\mathbf{Z}_p$. Then for $n \in N_n$ we can choose a unique coset representative of the form $I + F(n)$ where I is the 2×2 identity matrix and $F(n) \in \mathscr{L}$, the set of 2×2 matrices with entries from $p\mathbf{Z}_{p^2}$ and trace zero. The last condition holds since $\det(I + F(n)) = 1$ if and only if $\operatorname{tr}(F(n)) = 0$. The map $F : (N_n, \cdot) \to (\mathscr{L}, +)$ is easily seen to be a G_n/N_n isomorphism and so N_n is isomorphic to $(\mathbf{Z}_p)^3$ as claimed. It is convenient to change the ring over which the matrix entries for members of $\mathscr{L}$ are taken. Let $\mathscr{L}'$ be the set of 2×2 matrices with entries in $\mathbf{Z}_p$ and trace zero. We map $\mathscr{L}$ onto $\mathscr{L}'$ by mapping matrix entries of the form $ip(p^2\mathbf{Z}) \in p\mathbf{Z}_{p^2}$ to $i(p\mathbf{Z}) \in \mathbf{Z}_p$. Then the actions by conjugation of G_n/N_n on $(\mathscr{L}, +)$ and of $PSL(2, p)$ on $(\mathscr{L}', +)$ are the same. We now use standard theorems to determine the orbit sizes.

Table 1 gives the necessary information about the orbits of $GL(2, p)$ on $\mathscr{L}'$. The representative of each orbit is the rational canonical form. Here $R = \{\alpha^2 | \alpha \in \mathbf{Z}_p - \{0\}\}$ and $NR = \mathbf{Z}_p - (R \cup \{0\})$.

Table 1.

Representative x	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \beta \\ 1 & 0 \end{bmatrix}$
$\lvert \mathbf{C}_{GL(2,p)}(x) \rvert$	$(p^2 - 1)(p^2 - p)$	$p(p - 1)$	$(p - 1)^2$	$p^2 - 1$
Parameters			$\alpha \in R$	$\beta \in NR$

Now consideration of the centralizer, $\mathbf{C}_{SL(2,p)}(x)$, for x given in Table 1 determines the number of orbits of each size of $SL(2, p)$ on $\mathscr{L}'$. These are given in Table 2.

Table 2.

Orbit size	1	$(p^2 - 1)/2$	$p^2 + p$	$p^2 - p$
Number of orbits of this size	1	2	$(p - 1)/2$	$(p - 1)/2$

The relation between the actions of $PSL(2, p)$ on $\mathscr{L}'$ and G_n/N_n on N_n given above implies that Table 2 also gives the orbits of G_n/N_n acting on N_n. Finally, one calculates that the stabilizer in $PSL(2, p)$ and hence in G_n/N_n of a point from a non-trivial orbit is cyclic of order p, $(p - 1)/2$ or $(p + 1)/2$.

Let N_s denote the normal subgroup $1 \times N_n$ of G_s, the split extension. In what follows G and N will denote both the pair G_n and N_n and the pair G_s and N_s in statements which hold for both pairs and further analogous notation will be introduced. For example, since by definition G_s/N_s acts on N_s just as G_n/N_n acts on N_n, we have that the orbits of G/N on N have sizes 1, $(p^2 - 1)/2$, $p^2 + p$ and $p^2 - p$.

We may now show that G/N acts irreducibly on N. Indeed, suppose K is a proper invariant subgroup of N. Then as K is a union of orbits, $|K| = p^2$. Thus $|N/K| = p$ so that N/K is a trivial G/N-module. But then each orbit is in a fixed coset of K which is impossible for orbits of size $p^2 + p$.

5. Character theoretic information. Proofs of the standard results used in this section may be found in Feit [**2**, I] or Huppert [**4**, V]. It is necessary to calculate the orbit sizes of $\mathrm{Irr}(N)$ under the action of G/N. As noted in the last section, G/N acts on N just as $PSL(2, p)$ acts on $\mathscr{L}'$. Define a bilinear map from $\mathscr{L}' \times \mathscr{L}'$ to $\mathbf{Z}_p$ for $M_1, M_2 \in \mathscr{L}'$ by

$$(M_1, M_2) \rightarrow \mathrm{tr}(M_1M_2).$$

For $M \in PSL(2, p)$ we have $(M_1{}^M, M_2{}^M) = (M_1, M_2)$. Thus the mapping ρ from $\mathscr{L}'$ to the dual space of $\mathscr{L}'$ defined by $\rho(M_1)(M_2) = (M_1, M_2)$ is a $PSL(2, p)$-map. Since $\mathscr{L}'$ is an irreducible module and ρ is non-zero, we have a module isomorphism. Thus N and $\mathrm{Irr}(N)$ are isomorphic G/N modules. In particular, the orbits of G/N on $\mathrm{Irr}(N)$ have the sizes given in Table 2.

We now discuss the irreducible characters of G with kernels not containing N. If $\chi \in \mathrm{Irr}(G)$ is such a character, then χ_N, the restriction of χ to N, has a non-principal constituent λ. By Frobenius Reciprocity χ is a constituent of λ^G, the character of G induced from λ. Thus it suffices to consider the constituents of λ^G for all $\lambda \in \mathrm{Irr}(N) - \{1_N\}$. Let $S \leqq G$ be the inertial group of λ in G. By the permutation isomorphism between the actions of G on N and G on $\mathrm{Irr}(N)$ and the result from Section 4 concerning stabilizers of non-identity elements of N, we see that S/N is cyclic of order p, $(p - 1)/2$ or $(p + 1)/2$. Thus λ extends to $\tilde{\lambda} \in \mathrm{Irr}(S)$ [**2**, I, 9.12]. In what follows if $K \trianglelefteq H$ we will identify $\Phi \in \mathrm{Irr}(H)$ with kernel containing K with the corresponding character of $\mathrm{Irr}(H/K)$. With this convention, Gallagher [**3**, Theorem 2] has shown that

$$\lambda^S = \sum_{\mu \in Irr(S/N)} \mu\tilde{\lambda}$$

where the $\mu\tilde{\lambda}$ are all irreducible and distinct. Since the degree of $(\mu\tilde{\lambda})^G$ is $|G : S|$, exactly the size of the orbit of λ, Clifford's theorem implies $(\mu\tilde{\lambda})^G$ is irreducible. It follows that all irreducible characters of G with kernel not containing N are obtained in the above manner and so have degrees $(p^2 - 1)/2$, $p^2 + p$, or $p^2 - p$.

6. Construction of bijections. We partition the classes of G. Let $\mathscr{K}_1$ denote the set of $p + 2$ conjugacy classes of G which are contained in N. Let $\mathscr{K}_2$, $\mathscr{K}_3$ and $\mathscr{K}_4$ denote respectively those remaining classes whose representatives modulo N have order dividing $(p - 1)/2$, $(p + 1)/2$ and p. This is a partition since every non-identity element of $PSL(2, p)$ has order dividing exactly one of these numbers [**4**, II, 8.5]. As explained earlier $\mathscr{K}_{i,n}$ and $\mathscr{K}_{i,s}$ will denote the corresponding classes of G_n and G_s respectively.

We define the bijection between the classes of G_n and G_s and the bijection between the characters of G_n and G_s in stages. By the definition of G_s as a semi-direct product, $\sigma : N_n \rightarrow N_s = 1 \times N_n$ defined by $\sigma(n) = (1, n)$ is an isomorphism which intertwines with the isomorphism $\tau : G_n/N_n \rightarrow G_s/N_s$ given by $\tau : (xN_n) = (xN_n, 1)N_s$. This induces a bijection between $\mathscr{K}_{1,n}$ and $\mathscr{K}_{1,s}$. Fixing an isomorphism from $(\mathbf{Z}_p, +)$ into $(\mathbf{C}, \cdot)$, the complex numbers, gives an isomorphism from the dual space of N to $\mathrm{Irr}(N)$. Thus the isomorphism $\sigma : N_n \rightarrow N_s$ induces an "adjoint mapping", $\sigma^* : \mathrm{Irr}(N_s) \rightarrow \mathrm{Irr}(N_n)$, defined by $\sigma^*(\theta_s)(n) = \theta_s(\sigma(n))$ for $\theta_s \in \mathrm{Irr}(N_s)$ and $n \in N_n$.

We now construct a bijection between the characters of degree $p^2 + p$ of G_n and G_s. Let $\lambda_{i,s}\, i = 1, \ldots, (p-1)/2$ be representatives of the $(p-1)/2$ orbits in $\mathrm{Irr}(N_s)$ of size $p^2 + p$, chosen so as to have the same inertial group, S_s. This is possible since all subgroups of G/N of order $(p-1)/2$ are conjugate [**4**, II, 8.5]. Let the inertial group of $\sigma^*(\lambda_{i,s}) \in \mathrm{Irr}(N_n)\; i = 1, \ldots, (p-1)/2$ be S_n. Letting S denote either S_n or S_s, we have that $N_{G/N}(S/N)$ is dihedral of order $p - 1$ [**4**, II, 8.3]. By the Schur-Zassenhaus theorem $N_{G_n}(S_n)$ splits over N_n, say with complement C_n. Then $\tilde{\sigma} : \mathbf{N}(S_n) \rightarrow \mathbf{N}(S_s)$ defined for $cn \in C_nN_n = \mathbf{N}(S_n)$ by $\tilde{\sigma}(cn) = (cN_n, n)$ is an isomorphism which extends σ. Let $\tilde{\lambda}_{i,s} \in \mathrm{Irr}(S_s)$ be an extension of $\lambda_{i,s}$ for $i = 1, \ldots, (p-1)/2$. Also let μ_s be a generator of the cyclic group $\mathrm{Irr}(S_s/N_s)$. Just as σ induced σ^*, the isomorphism $\tilde{\sigma}$ induces naturally $\tilde{\sigma}^* : \mathrm{Irr}(S_s) \rightarrow \mathrm{Irr}(S_n)$. Then the mapping

$$(\tilde{\sigma}^*(\mu_s{}^j\tilde{\lambda}_{i,s}))^{G_n} \leftrightarrow (\mu_s{}^j\tilde{\lambda}_{i,s})^{G_s} \quad i, j = 1, \ldots, (p-1)/2$$

is a bijection between the irreducible characters of G_n and G_s of degree $p^2 + p$.

We use σ to define the bijection between $\mathscr{K}_{2,n}$ and $\mathscr{K}_{2,s}$. Notice that every class in $\mathscr{K}_2$ intersects $S - N$, since every element of order dividing $(p-1)/2$ in G/N lies in one of the unique class of cyclic subgroups of order $(p-1)/2$ [**4**, II, 8.5]. Further $\mathbf{N}(S)$ controls fusion in $S - N$ if an odd prime divides $(p-1)/2$ since $\mathbf{N}(S)/N$ and S/N are the normalizer and centralizer of a Sylow subgroup of G/N [**4**, IV, 2.5] and a similar argument works if $(p-1)/2$ is a power of 2. Thus the intersection of a class in $\mathscr{K}_2$ with $S - N$ is a class of $N(S)$. Combining the two bijections whose existence is implied by the last statement with the bijection between the classes of $\mathbf{N}(S_n)$ and $\mathbf{N}(S_s)$ induced by $\tilde{\sigma}$ gives the necessary bijection between $\mathscr{K}_{2,n}$ and $\mathscr{K}_{2,s}$. Notice (for later use) that if the natural mappings

$$\eta_s : \mathbf{N}(S_s) \rightarrow \mathbf{N}(S_s)/N_s \quad \text{and} \quad \eta_n : \mathbf{N}(S_n) \rightarrow \mathbf{N}(S_n)/N_n$$

are defined, then the definition of $\tilde{\sigma}$ implies $\eta_s\tilde{\sigma} = \tau\eta_n$.

We may now show that corresponding characters of degree $p^2 + p$ are equal on corresponding classes regardless of how the bijection from $\mathscr{K}_{i,n}$ to $\mathscr{K}_{i,s}$ is defined for $i = 3, 4$. Indeed, irreducible characters of degree $p^2 + p$ of G_n and G_s are induced from S_n and S_s respectively where $|S/N| = (p-1)/2$ and so must vanish on members of classes from $\mathscr{K}_{i,n}$ and $\mathscr{K}_{i,s}$ for $i = 3, 4$.

Let

$$\chi_s = (\mu_s{}^j\tilde{\lambda}_{i,s})^{G_s} \text{ and } \chi_n = (\tilde{\sigma}^*(\mu_s{}^j\tilde{\lambda}_{i,s}))^{G_n} \text{ for } i, j \in \{1, \ldots, (p-1)/2\}.$$

Then $(\chi_s)_{N_s}$ is the sum of characters of $\mathrm{Irr}(N_s)$ in the orbit of $\lambda_{i,s}$ and $(\chi_n)_{N_n}$ is the sum of characters of $\mathrm{Irr}(N_n)$ in the orbit of $\sigma^*(\lambda_{i,s})$. But for $n \in N_n$, the definition of σ^* gives that

$$(\sigma^*(\lambda_{i,s}))^g(n) = \lambda_{i,s}{}^{\tau(g)}(\sigma(n)) \quad \text{for } g \in G_n/N_n.$$

Finally for classes in $\mathscr{K}_{2,n}$ let $x_n \in S_n - N_n$ so that $\tilde{\sigma}(x_n) \in S_s - N_s$ and represents the corresponding class in $\mathscr{K}_{2,s}$. Then since $\mathbf{N}(S)$ controls fusion in $S - N$, the formula for induced characters gives

$$\chi_n(x_n) = (\tilde{\sigma}^*(\mu_s{}^j\tilde{\lambda}_{i,s}))^{\mathbf{N}(S_n)}(x_n)$$

and similarly

$$\chi_s(\tilde{\sigma}(x_n)) = (\mu_s{}^j\tilde{\lambda}_{i,s})^{\mathbf{N}(S_s)}(\tilde{\sigma}(x_n)).$$

By the definition of $\tilde{\sigma}^*$, these are equal and so we have shown corresponding characters of degree $p^2 + p$ agree on corresponding classes.

The same arguments may be applied to the characters of degree $p^2 - p$ to define a bijection between $\mathscr{K}_{3,n}$ and $\mathscr{K}_{3,s}$ and a bijection between the irreducible characters of G_n and G_s of degree $p^2 - p$ with properties similar to those of the bijections already defined. No additional difficulties arise in this case.

We now consider the correspondences between the characters of degree $(p^2-1)/2$ and between $\mathscr{K}_{4,n}$ and $\mathscr{K}_{4,s}$. The situation here is less straightforward since the normalizers of the appropriate inertial groups in G_n and G_s are not isomorphic. However, we can give an isomorphism between certain sections of G_n and G_s which will meet our needs.

We introduce some notation. Let $\lambda_{1,n} \in \mathrm{Irr}(N_n)$ be chosen from an orbit of size $(p^2-1)/2$. Not all members of $\{\lambda_{1,n}{}^i | i = 1, \ldots, p-1\}$ lie in the same orbit, for otherwise the stabilizer of $\{\lambda_{1,n}{}^i | i = 1, \ldots, p-1\}$ modulo N_n would have index $(p+1)/2$ in G_n/N_n, a simple group containing an element of order p. Let $\lambda_{2,n} \in \mathrm{Irr}(N_n)$ be a power of $\lambda_{1,n}$ in the other orbit of size $(p^2-1)/2$. Let S_n and K_n be respectively the common inertial group and kernel of $\lambda_{1,n}$ and $\lambda_{2,n}$. Notice that since $|S_n : K_n| = p^2$, the commutator subgroup satisfies $S_n' \subseteq K_n$. Let $\lambda_{i,s} \in \mathrm{Irr}(N_s)$ be defined by $\sigma^*(\lambda_{i,s}) = \lambda_{i,n}$ for $i = 1, 2$. Then the inertial group and kernel, S_s and K_s of $\lambda_{1,s}$ and $\lambda_{2,s}$ satisfy $\tau(S_n/N_n) = S_s/N_s$ and $\sigma(K_n) = K_s$.

Let S denote either S_n or S_s. For $x \in S - N$ it is easy to check that the Jordan normal form for x acting on the vector space N has one block. It follows that

$$1 < \mathbf{Z}(S) < S' = K < N.$$

Since

$$x^p \in N \cap \mathbf{C}(x) = \mathbf{Z}(S)$$

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we see that S/K is elementary abelian of order p^2. Now $\mathbf{N}(S)/S$ is cyclic of order $(p-1)/2$ and so S/K is the direct sum of two linear $\mathbf{N}(S)/S$-modules, one being N/K. Let L/K be the other for $K < L < S$. As $\mathbf{N}(S)/S$ modules we have that

$$L/K = L/(N \cap L) \simeq NL/N = S/N.$$

Thus the action of $\mathbf{N}(S)/S$ on L/K is determined in $\mathbf{N}(S)/N$.

We may now define an isomorphism

$$\tilde{\sigma} : \mathbf{N}(S_n)/K_n \rightarrow \mathbf{N}(S_s)/K_s.$$

The symbol $\tilde{\sigma}$ is redefined for the remainder of the section to emphasize the analogy between this and the former construction. Let $t_n \in \mathbf{N}(S_n) - S_n$ with $|t_n| = (p-1)/2$ and define $t_s = (t_nN_n, 1) \in \mathbf{N}(S_s) - S_s$. Then $\tau(t_nN_n) = t_sN_s$. Let $n_n \in N_n - K_n$ and $n_s = \sigma(n_n)$. Finally let $l_n \in L_n - K_n$ and $l_s \in \tau(l_nN_n) \cap L_s$. The last comment in the preceding paragraph shows that t_n acts on $\langle l_nK_n \rangle$ just as t_s acts on $\langle l_sK_s \rangle$. Since σ intertwines with τ we see that t_n acts on $\langle n_nK_n \rangle$ just as t_s acts on $\langle n_sK_s \rangle$. Therefore if we define $\tilde{\sigma}(t_nK_n) = t_sK_s$, $\tilde{\sigma}(l_nK_n) = l_sK_s$ and $\tilde{\sigma}(n_nK_n) = n_sK_s$, then $\tilde{\sigma}$ can be extended uniquely to an isomorphism from $\mathbf{N}(S_n)/K_n$ to $\mathbf{N}(S_s)/K_s$. As in the previous case $\tilde{\sigma}$ induces a bijection

$$\tilde{\sigma}^* : \mathrm{Irr}(\tilde{\sigma}(T_n)/K_n) \rightarrow \mathrm{Irr}(T_n/K_n)$$

for each $K_n \leqq T_n \leqq \mathbf{N}(S_n)$.

Let $\mu_s \in \mathrm{Irr}(S_s/N_s)$ be such that

$$\{\mu_s{}^j | j \in \mathbf{Z}\} = \mathrm{Irr}(S_s/N_s).$$

Let $\tilde{\lambda}_{i,s} \in \mathrm{Irr}(S_s)$ be an extension of $\lambda_{i,s}$ for $i = 1, 2$. Then the correspondence between characters of degree $(p^2-1)/2$ is given by

$$(\tilde{\sigma}^*(\mu_s{}^j\tilde{\lambda}_{i,s}))^{G_n} \leftrightarrow (\mu_s{}^j\tilde{\lambda}_{i,s})^{G_s} \quad i = 1, 2; j = 1, \ldots, p-1.$$

Since all characters of degree $(p^2-1)/2$ are induced from a Sylow p-subgroup, they vanish on $\mathscr{K}_2$ and $\mathscr{K}_3$. Since $\tilde{\sigma}|_{N/K}$ is induced by σ, corresponding characters of degree $(p^2-1)/2$ are equal on corresponding classes in $\mathscr{K}_{1,n}$ and $\mathscr{K}_{1,s}$.

It remains to define the bijection between $\mathscr{K}_{4,n}$ and $\mathscr{K}_{4,s}$. Since $|S : N| = p$, fusion in S/N is controlled in $\mathbf{N}(S)/N$. Thus the conjugacy classes of G in $\mathscr{K}_4$ when intersected with S give the $\mathbf{N}(S)$-conjugacy classes in $S - N$. We claim that the conjugacy classes of $\mathbf{N}(S)$ in $S - N$ are in one to one correspondence with the conjugacy classes of $\mathbf{N}(S)/K$ in $S/K - N/K$. Indeed for $x \in S - N$, a reconsideration of the Jordan normal form for x acting on N shows that $|\mathbf{C}_S(x)| = p^2$. Thus x has p^2 conjugates in S. But two conjugates are always in the same coset of $S' = K$. Finally $|K| = p^2$ implies that xK is the class of x in S. The bijections above, when combined with the map between classes of $\mathbf{N}(S_n)/K_n$ and $\mathbf{N}(S_s)/K_s$ defined by $\tilde{\sigma}$, give the desired bijection. Finally since $\tilde{\sigma}^*$ is an "adjoint mapping" to $\tilde{\sigma}$, we see that corresponding characters of degree $(p^2 - 1)/2$ agree on corresponding classes in $\mathscr{K}_{4,n}$ and $\mathscr{K}_{4,s}$.

We have now completed the definition of the bijection between the classes of G_n and of G_s. It remains to define a bijection between the characters of G_n and G_s which contain N_n and N_s respectively in their kernels, i.e., between $\mathrm{Irr}(G_n/N_n)$ and $\mathrm{Irr}(G_s/N_s)$. But $\tau : G_n/N_n \to G_sN_s$ is an isomorphism. Further the map between classes we have defined has the property that corresponding classes taken modulo N_n and N_s respectively correspond under the map between classes of G_n/N_n and G_s/N_s induced by τ. Thus extending our map between characters to $\mathrm{Irr}(G_n/N_n)$ and $\mathrm{Irr}(G_s/N_s)$ by the "adjoint mapping" τ^* will guarantee that corresponding characters agree on corresponding classes. Thus G_n and G_s have the same character table.

7. Frattini subgroups. It remains only to show that G_n and G_s have Frattini subgroups of different orders. It is easy to see that $\Phi(G/N) = N$ implies $\Phi(G) \subseteq N$. In G_s, $(G_n/N_n) \times 1$ is a maximal subgroup since $N_s = 1 \times N_n$ is an irreducible G_s-module. Thus

$$\Phi(G_s) \subseteq (G_n/N_n \times 1) \cap (1 \times N_n) = 1_{G_s}.$$

In G_n, let M be a maximal subgroup. If $N_n \not\subseteq M$, then $N_nM = G_n$ and so $N_n \cap M \trianglelefteq N_nM = G_n$. Now N_n is an irreducible G_n-module so $N_n \cap M = 1$, in particular there is an element of order p in $G_n - N_n$. Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^p$ is not trivial in $G_n = PSL(2, \mathbf{Z}_{p^2})$ and since $p \geqq 5$, P. Hall's theory of regular p-groups [**4**, III, 10.2, 10.5] shows that all elements of p-power order in $G_n - N_n$ have order p^2. Thus every maximal subgroup of G_n contains N_n and so $\Phi(G_n) = N_n$.

Since the Frattini subgroups have different orders, it is clear that no set of classes of G can be determined from the character table of G such that the union of the set is the Frattini subgroup.

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