Π-PRINCIPAL HEREDITARY ORDERS

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Introduction. Let S denote the integral closure of a complete discrete rank one valuation ring R in a finite Galois extension of the quotient field of R, G the Galois group of the quotient field extension, and f an element of $Z^2(G, U(S))$ where $U(S)$ denotes the multiplicative group of units of S. A crossed product $\Delta(f, S, G)$ whose radical is generated as a left ideal by the prime element Π of S is an hereditary order according to the Corollary to Thm. 2. 2 of [2], and we call such a crossed product a Π-principal hereditary order. In previous papers the author has studied Π-principal hereditary orders $\Delta(f, S, G)$ for tamely and wildly ramified extensions S of R (see [10] and [11]). The purpose of this paper is to study Π-principal hereditary orders $\Delta(f, S, G)$ with no restriction on the extension S of R.

In Section 1 we present necessary and sufficient conditions for a crossed product $\Delta(f, S, G)$ to be Π-principal. Let G_p denote the Galois group of the quotient field of S over the quotient field of the maximal tamely ramified extension of R in S. We associate to the cohomology class $[f]$ a subgroup R_f of the center of G_p called its radical group and prove that the following statements are equivalent

(1) $\Delta(f, S, G)$ is a Π-principal hereditary order

(2) G_p is an Abelian group and $R_f = (1)$

(3) $R_f = (1)$.

Thus we generalize a result obtained in [11] for wildly ramified extensions S of R.

It is natural to ask if each hereditary crossed product is Π-principal. In Section 2 we present an example of an hereditary crossed product which is not Π-principal. However, if the residue class field extension $\bar{S}$ of $\bar{R}$ is separable, then a crossed product $\Delta(f, S, G)$ is hereditary if and only if it is Π-principal. In order to prove this main result we make use of facts

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concerning the cohomology of wildly ramified extensions presented in an appendix.

Finally, in Section 3 we present a criterion for determining the number of maximal two-sided ideals in a Π-principal hereditary order by generalizing a result obtained by the author for crossed products over tamely ramified extensions (see [10]).

The following notation shall be in use throughout the entire paper. The multiplicative group of units of a ring R shall be denoted by $U(R)$; rad R shall denote the radical of R and ctr R its center. If R is a local ring, then $\bar{R}$ shall denote its residue class field. Unless otherwise stated, R shall always denote a complete discrete rank one valuation ring, S the integral closure of R in a finite Galois extension of the quotient field of R, and G the Galois group of the quotient field extension. The prime elements of R and S shall be denote by π and Π respectively, and p shall denote the characteristic of $\bar{R}$.

1. The radical group. The purpose of this section is to present necessary and sufficient conditions for a crossed product $\Delta(f, S, G)$ over an integrally closed extension S of a complete discrete rank one valuation ring R to be a Π-principal hereditary order. According to Thm. 3-4-7 of [9] we may consider the maximal tamely ramified extension T of R in S. Let G_p denote the Galois group of the quotient field extension of $S \supset T$. The criteria for determining whether or not a crossed product $\Delta(f, S, G)$ is Π-principal shall be given in terms of a subgroup R_f of the center of G_p called the radical group of $[f]$ (see Thm. 1. 9).

Observe that the subgroup G_p of G defined above is a p-group. In the case when the residue class field extension $\bar{S}$ of $\bar{R}$ is separable, G_p is the first ramification group G_1 of S over R. It is easy to construct an example to show that when the extension $\bar{S}$ of $\bar{R}$ is inseparable, G_p need not equal G_1. The following relation between the inertia group G_0 of S over R and G_p shall be useful throughout the paper.

PROPOSITION 1. 1. *The inertia group G_0 of S over R is the semi-direct product $G_0 = J \times G_p$ where J is a cyclic group of order relatively prime to the characteristic p of $\bar{R}$. Moreover, G_p is a normal subgroup of G.*

Proof. We first observe that G_p is a normal subgroup of G_0. Consider the chain of rings $R \subset U \subset T \subset S$ where U and T denote the maximal

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unramified and tamely ramified extensions (respectively) of R in S. Let π_t denote a prime element of T and recall that $\pi_t^e = \pi$ for some prime element π of U and positive integer e relatively prime to the characteristic p of $\bar{R}$ (see Prop. 3-4-3 of [9]). The conjugates of π_t relative to U are therefore of the form $\zeta^i \pi_t$ for $1 \leq i \leq e$ where ζ denotes a primitive e^{th} root of unity. Since the quotient field extension of $S \supset R$ is Galois, ζ must be in S. Let $\bar{\zeta}$ denote the image of ζ under the natural map of S onto $\bar{S}$. The extension $\bar{U} \subset \bar{U}(\bar{\zeta})$ is separable since $(e, p) = 1$, so that $\bar{\zeta}$ is in $\bar{U}$ because $\bar{U}$ is the separable closure of $\bar{R}$ in $\bar{S}$. The polynomial $X^e - \bar{1}$ of $\bar{U}[X]$ is separable and has $\bar{\zeta}$ as a root; by Hensel's lemma we may now conclude that ζ is in U. Let τ denote an element of G_p and σ an element of G_0. Since $T = U[\pi_t]$ (see Thm. 3-3-1 of [9]) it suffices to show that $\sigma^{-1}\tau\sigma(\pi_t) = \pi_t$ to prove that G_p is a normal subgroup of G_0. Using the fact that $\sigma(\pi_t) = \zeta^i \pi_t$ for some i together with the fact that ζ is in U it is easy to check that $\sigma^{-1}\tau\sigma(\pi_t) = \pi_t$.

We may now verify that G_0 is a semi-direct product. For the factor group G_0/G_p is a cyclic group of order e relatively prime to the order of the normal subgroup G_p. Thm. 15. 2. 2 of [4] now implies that there exists a cyclic group J of order e such that $G_0 = J \times G_p$.

Finally we shall make use of the fact that the inclusions $G_p \subset G_0$ and $G_0 \subset G$ are normal to prove that G_p is a normal subgroup of G. Consider elements σ of G and τ of G_p, and let n denote the order of τ. Then $\sigma\tau\sigma^{-1}$ is in G_0 so we may write $\sigma\tau\sigma^{-1} = \rho\omega$ for some element ρ of J and ω of G_p. Using the definition of semi-direct product we may now obtain the equalities $1 = (\rho\omega)^n = \rho^n \prod_i \omega^{\rho^{n-i}}$ where $1 \leq i \leq n$, from which it follows that $\rho^n = 1$. The order of ρ is relatively prime to n. Therefore $\rho = 1$ and $\sigma\tau\sigma^{-1}$ is in G_p.

We proceed to define the radical group R_f of $[f]$. Let C denote the center of G_p and consider the crossed product $\Delta(\bar{f}, \bar{S}, C)$ where $\bar{f}$ denotes the image of f under the natural maps $Z^2(G, U(S)) \to Z^2(G, U(\bar{S})) \to Z^2(C, U(\bar{S}))$. The radical group of $[\bar{f}]$ was defined by the author in [11]. For the convenience of the reader we present the definition here. Let $C = E_1 \times \cdots \times E_t$ be a decomposition of C into a direct product of cyclic p-groups. According to Cor. A. 3 of [11] we may assume that $\bar{f}$ is normalized on $C \times C$ in the sense of Abelian p-groups, so that $\bar{f} = f_1 \cdots f_t$ where each element f_i of $Z^2(E_i, U(\bar{S}))$ is normalized in the sense of cyclic groups. For

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$1 \leq i \leq t$ let a_i denote the element of $U(\bar{S})$ which corresponds to f_i under the canonical identification $H^2(E_i, U(\bar{S})) = U(\bar{S})/[U(\bar{S})]^{e_i}$ where e_i denotes the order of E_i, and consider the polynomials $h_i(X) = X^{e_i} - a_i$ of $\bar{S}[X]$. The element $[\bar{f}]$ of $H^2(C, U(\bar{S}))$ determines a chain of fields $L_0 \subseteq \cdots \subseteq L_i \subseteq L_{i+1} \subseteq \cdots \subseteq L_{t-1}$ defined inductively in the following way. Let $L_0 = \bar{S}$, and when L_i has been defined we then define L_{i+1} to be a splitting field for the polynomial $h_{i+1}(X)$ over L_i. We next define $R_{f,i}$ for $1 \leq i \leq t$ to be the maximal subgroup of E_i with the property that $[f_i]$ is in the kernel of the natural map $H^2(E_i, U(\bar{S})) \to H^2(R_{f,i}, U(L_{i-1}))$. The *radical group* $R_{\bar{f}}$ of the element $[\bar{f}]$ of $H^2(C, U(\bar{S}))$ is defined to be the direct product $R_{f,1} \times \cdots \times R_{f,t}$. The significance of the radical group of $[\bar{f}]$ is indicated by the fact that the crossed product $\Delta(\bar{f}, \bar{S}, C)$ is semi-simple if and only if $R_{\bar{f}} = (1)$, (see Prop. 1. 10 of [11]).

Definition. The *radical group* R_f of an element $[f]$ of $H^2(G, U(S))$ is defined to be the radical group of $[\bar{f}]$ where $\bar{f}$ denotes the image of f under the natural map $Z^2(G, U(S)) \to Z^2(C, U(\bar{S}))$ and C is the center of the subgroup G_p of G.

It follows at once from the definition that a crossed product $\Delta(f, S, G)$ is a Π-principal hereditary order if and only if the crossed product $\Delta(\bar{f}, \bar{S}, G)$ is a semi-simple ring. And according to Prop. 3. 1 of [11], $\Delta(\bar{f}, \bar{S}, G)$ is semi-simple if and only if the subring $\Delta(\bar{f}, \bar{S}, G_0)$ is semi-simple. Observe that the inertia group G_0 acts trivially on $\bar{S}$.

The notion of a splitting field of a crossed product shall be useful for studying $\Delta(\bar{f}, \bar{S}, G_0)$. Given a finite group G, fields F and K such that K is a G-ring over F, an extension L of K is called a *splitting field* of $\Delta(f, K, G)$ if $[f]$ is in the kernel of the natural map $H^2(G, U(K)) \to H^2(G, U(L))$ induced by the inclusion of K in L. If in addition L is a purely inseparable extension of K, then L is called a *purely inseparable splitting field* of $\Delta(f, K, G)$.

The next two propositions establish the existence of splitting fields for certain crossed products. In the proof of Prop. 1. 2 we shall make use of the notion of the *central series* of a p-group G_p (see Section 2 of [11]). which is defined to be the (normal) series $G_p = C_n \supset \cdots \supset C_i \supset \cdots \supset C_0 \supset C_{-1} = (1)$ where $C_{-1} = (1)$ and C_{i+1} is the preimage in G_p of the center of G_p/C_i for $0 \leq i \leq n-1$.

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PROPOSITION 1. 2. *Let G_p denote a p-group with trivial action on a field F of characteristic p. Each crossed product $\Delta(f, F, G_p)$ has a purely inseparable splitting field.*

Proof. The proof is by induction of the length $l_c(G_p)$ of the central series of G_p. If $l_c(G_p) = 1$ then G_p is an Abelian p-group, so that $\Delta(f, F, G_p)$ has a purely inseparable splitting field according to Lemma 2. 1 of [11].

For the inductive step we assume that the assertion of the proposition is true for p-groups H for which $l_c(H) \leq n$, and consider a group G_p with $l_c(G_p) = n+1$. Let $G_p = C_n \supset C_{n-1} \supset \cdots \supset C_{-1} = (1)$ be the central series of G_p. It is easy to check that $l_c(C_{n-1}) \leq n$, so that the crossed product $\Delta(f, F, C_{n-1})$ has a purely inseparable splitting field L_{n-1} according to the induction hypothesis. The sequence $H^2(G_p/C_{n-1}, U(L_{n-1})) \to H^2(G_p, U(L_{n-1})) \to H^2(C_{n-1}, U(L_{n-1}))$ (where the maps are inflation and restriction) is exact according to Prop. A. 7 of [11]. For convenience of notation denote the image of f under the natural map $Z^2(G, U(F)) \to Z^2(G, U(L_{n-1}))$ by f also. From the definition of L_{n-1} it follows that $[f]$ is in the kernel of the restriction map $H^2(G_p, U(L_{n-1})) \to H^2(C_{n-1}, U(L_{n-1}))$. The exactness of the above sequence implies that there exists an element $[g]$ of $H^2(G_p/C_{n-1}, U(L_{n-1}))$ such that $\inf([g]) = [f]$. Form the crossed product $\Delta(g, L_{n-1}, G_p/C_{n-1})$. The factor group G_p/C_{n-1} is an Abelian p-group with trivial action on L_{n-1}, so that $\Delta(g, L_{n-1}, G_p/C_{n-1})$ has a purely inseparable splitting field L according to Prop. 2. 1 of [11]. Observe that L is a purely inseparable extension of F.

It remains to show that L is a splitting field of $\Delta(f, F, G_p)$. Consider the following diagram of cohomology groups and homomorphisms.

$$\begin{array}{ccccc} H^2(G_p, U(F)) & \to & H^2(G_p, U(L_{n-1})) & \to & H^2(G_p, U(L)) \\ & & \uparrow{\scriptstyle \inf} & & \uparrow{\scriptstyle \inf} \\ & & H^2(G_p/C_{n-1}, U(L_{n-1})) & \to & H^2(G_p/C_{n-1}, U(L)) \end{array}$$

where the horizontal maps are induced by the inclusions $F \subset L_{n-1} \subset L$. Using the commutativity of this diagram together with the fact that the image of $[g]$ under the map $H^2(G_p/C_{n-1}, U(L_{n-1})) \to H^2(G_p/C_{n-1}, U(L))$ is trivial, one may obtain by diagram chasing the fact that $[f]$ is in the kernel of the map $H^2(G_p, U(F)) \to H^2(G_p, U(L))$, i.e. that L is a purely inseparable splitting field for $\Delta(f, F, G_p)$.

COROLLARY 1. 3. *Let G_p be a p-group with trivial action on a field F of characteristic p. A crossed product $\Delta = \Delta(f, F, G_p)$ has the property that $\Delta/\text{rad}\,\Delta$ is a field. (In fact $\Delta/\text{rad}\,\Delta$ is a purely inseparable extension of F and is contained in every splitting field of Δ).*

Proof. Let L denote a purely inseparable splitting field of Δ whose existence is guaranteed by Prop. 1. 2. Since $[f]$ is in the kernel of the natural map $H^2(G_p, U(F)) \to H^2(G_p, U(L))$ the crossed product $\Delta(f, L, G_p)$ is L-algebra isomorphic to the trivial crossed product $\Delta(1, L, G_p)$. Now $\Delta(1, L, G_p)/\text{rad}\,\Delta(1, L, G_p)$ is isomorphic to L (see p. 435 of [3]) so that $\Delta(f, L, G_p)/\text{rad}\,\Delta(f, L, G_p)$ is isomorphic to L. The natural map $\Delta/\text{rad}\,\Delta \to \Delta(f, L, G_p)/\text{rad}\,\Delta(f, L, G_p)$ is well-defined because $\text{rad}\,\Delta$ is contained in $\text{rad}\,\Delta(f, L, G_p)$ according to Lemma 1. 4 of [11]; and it is an injection because the intersection $[\text{rad}\,\Delta(f, L, G_p)] \cap \Delta$ is contained in $\text{rad}\,\Delta$ (see Lemma 2. 4 of [11]). We may conclude now that $\Delta/\text{rad}\,\Delta$ is a field since a semi-simple subring of a field is a field.

Combining Cor. 1. 3 with Prop. 2. 9 of [11] we obtain at once the following result.

COROLLARY 1. 4. *Let G_p denote a p-group with trivial action on a field F of characteristic p, and f an element of $Z^2(G_p, U(F))$. Then the following statements are equivalent:*

(1) *$\Delta(f, F, G_p)$ is a semi-simple ring*

(2) *$\Delta(f, F, G_p)$ is a field*

(3) *$\Delta(f, F, C)$ is a field where C denote the center of G_p.*

Observe that the equivalence of statements (1) and (2) of Cor. 1. 4 does not depend upon the fact that $\Delta(f, F, G_p)$ has a splitting field which is *purely inseparable.* However we did make use of the existence of a purely inseparable splitting field to prove that (3) implies (1), (see Section 2 of [11]). This stronger implication shall be used to prove the main result of Section 2 of this paper.

COROLLARY 1. 5. *Let S denote an inertial extension of a complete discrete rank one valuation ring R with no tame part, and let G_p denote the Galois group of the quotient field extension. If $[f]$ is an element of $H^2(G_p, U(S))$, then the following statements are equivalent:*

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(1) $\Delta(f, S, G_p)$ *is an hereditary order*

(2) $\Delta(f, S, G_p)$ *is a maximal order.*

Proof. Assume that the crossed product $\Delta(f, S, G_p)$ is hereditary. The fact that $\Delta(f, S, G_p)/\text{rad}\, \Delta(f, S, G_p)$ is a simple ring (Cor. 1. 3) implies that rad $\Delta(f, S, G_p)$ is the unique maximal two-sided ideal of $\Delta(f, S, G_p)$. Therefore $\Delta(f, S, G_p)$ is a maximal order according to the Corollary to Thm. 2. 2 of [2]. To complete the proof we recall that each maximal order is hereditary.

Consider the inertia group G_0 of an extension S of R and the Galois group G_p of the quotient field of S over the quotient field of the maximal tamely ramified extension of R in S. The next proposition concerning the existence of splitting fields shall be useful in proving that $\Delta(\bar{f}, \bar{S}, G_0)$ is semi-simple if and only if $\Delta(\bar{f}, \bar{S}, G_p)$ is semi-simple.

PROPOSITION 1. 6. *Let G_0 denote the inertia group of S over R. The crossed product $\Delta(\bar{f}, \bar{S}, G_0)$ has a splitting field.*

Proof. Prop. 1. 2. implies that the crossed product $\Delta(\bar{f}, \bar{S}, G_p)$ has a splitting field L_p. For convenience of notation denote the image of $\bar{f}$ under the natural map $Z^2(G_0, U(\bar{S})) \to Z^2(G_0, U(L_p))$ by $\bar{f}$ also. Consider the sequence $(1) \to H^2(G_0/G_p, U(L_p)) \to H^2(G_0, U(L_p)) \to H^2(G_p, U(L_p))$ where the maps are inflation and restriction. This sequence is exact according to Prop. 5 p. 126 of [7] because $H^1(G_p, U(L_p)) = (1)$, (see Lemma A. 6 of [11]). The definition of L_p implies that $[\bar{f}]$ is in the kernel of the restriction map $H^2(G_0, U(L_p)) \to H^2(G_p, U(L_p))$. The exactness of the above sequence implies that there exists a 2-cocycle g in $Z^2(G_0/G_p, U(L_p))$ such that $\inf([g]) = [\bar{f}]$, and we may assume that g has been normalized in the sense of cyclic groups. Consider the crossed product $\Delta(g, L_p, G_0/G_p)$. Let a be an element of $U(L_p)$ corresponding to g under the canonical identification $H^2(G_0/G_p, U(L_p)) = U(L_p)/[U(L_p)]^e$ which holds because G_0/G_p is a cyclic group. Let α denote a root of the polynomial $P(X) = X^e - a$ of $L_p[X]$, and define $L = L_p(\alpha)$. It is easy to check that L is a splitting field for the crossed product $\Delta(g, L_p, G_0/G_p)$.

In order to prove that L is in fact a splitting field for the crossed product $\Delta(\bar{f}, \bar{S}, G_0)$ consider the following diagram:

$$
\begin{array}{ccccc}
 & & H^2(G_0, U(\bar{S})) & & \\
 & & \downarrow & & \\
H^2(G_0/G_p, U(L_p)) & \rightarrow & H^2(G_0, U(L_p)) & \rightarrow & H^2(G_p, U(L_p)) \\
\downarrow & & \downarrow & & \\
H^2(G_0/G_p, U(L)) & \rightarrow & H^2(G_0, U(L)) & &
\end{array}
$$

where the horizontal maps are inflation and restriction, and the vertical maps are the obvious ones. The commutativity of this diagram together with the above observations implies that $[\bar{f}]$ is in the kernel of the map $H^2(G_0, U(\bar{S})) \rightarrow H^2(G_0, U(L))$. Therefore L is a splitting field for $\Delta(\bar{f}, \bar{S}, G_0)$ and this completes the proof.

PROPOSITION 1. 7. *The radical of* $\Delta(\bar{f}, \bar{S}, G_0)$ *is generated both as a left and a right ideal by the radical of* $\Delta(\bar{f}, \bar{S}, G_p)$.

Proof. According to Prop. 1. 6 we may consider a splitting field L for the crossed product $\Delta(\bar{f}, \bar{S}, G_0)$. The definition of splitting field implies that $\Delta(\bar{f}, L, G_0)$ is L-algebra isomorphic to the trivial crossed product $\Delta(1, L, G_0)$. We shall make use of this isomorphism to prove first of all that the radical of $\Delta(\bar{f}, L, G_0)$ is generated as a right ideal by rad $\Delta(\bar{f}, L, G_p)$. For the exercise on p. 435 of [3] implies that rad $\Delta(1, L, G_0)$ is generated by rad $\Delta(1, L, G_p)$. Let $\phi : G_0 \rightarrow U(L)$ be the map which makes $\bar{f}$ cohomologous to the trivial 2-cocycle in $Z^2(G_0, U(L))$. Consider the L-algebra isomorphism $\psi : \Delta(\bar{f}, L, G_0) \rightarrow \Delta(1, L, G_0)$ induced by ϕ. The restriction of ψ to $\Delta(\bar{f}, L, G_p)$ establishes an isomorphism of $\Delta(\bar{f}, L, G_p)$ with $\Delta(1, L, G_p)$. From the above observation concerning $\Delta(1, L, G_0)$ we may conclude therefore that rad $\Delta(\bar{f}, L, G_0)$ is generated as a right ideal by rad $\Delta(\bar{f}, L, G_p)$.

Now we may prove that rad $\Delta(\bar{f}, \bar{S}, G_0)$ is generated as a right ideal by rad $\Delta(\bar{f}, \bar{S}, G_p)$. The radical of $\Delta(\bar{f}, \bar{S}, G_p)$ is contained in rad $\Delta(\bar{f}, L, G_p)$, (see Lemma 1. 4 of [11]) and so rad $\Delta(\bar{f}, \bar{S}, G_p)$ is contained in rad $(\bar{f}, \bar{S}, G_0)$ by the above observation. The fact that [rad $\Delta(\bar{f}, L, G_0)] \cap \Delta(\bar{f}, \bar{S}, G_0)$ is contained in rad $\Delta(\bar{f}, \bar{S}, G_0)$ (Lemma 2. 4 of [11]) now implies that the right ideal generated by rad $\Delta(\bar{f}, \bar{S}, G_p)$ is contained in rad $\Delta(\bar{f}, \bar{S}, G_0)$. To obtain the opposite inclusion consider a disjoint right coset decomposition $G_0 = \cup G_p\sigma_i$ of G_0 relative to the subgroup G_p. The fact that rad $\Delta(\bar{f}, \bar{S}, G_0)$ is contained in rad $\Delta(\bar{f}, L, G_0)$ (see Lemma 1. 4 of [11]) implies that an

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element δ of rad $\Delta(\bar{f}, \bar{S}, G_0)$ may be written uniquely in the form $\delta = \sum_i n_i u_{\sigma_i}$ with each n_i in rad $\Delta(\bar{f}, L, G_p)$, since rad $\Delta(\bar{f}, L, G_0)$ is generated as a right ideal by rad $\Delta(\bar{f}, L, G_p)$. Each n_i must be in $\Delta(\bar{f}, \bar{S}, G_p)$ since δ is an element of $\Delta(\bar{f}, \bar{S}, G_0)$. The intersection [rad $\Delta(\bar{f}, L, G_p)$] $\cap$ $\Delta(\bar{f}, \bar{S}, G_p)$ is contained in rad $\Delta(\bar{f}, \bar{S}, G_p)$ by Lemma 2. 4 of [11]. Therefore each n_i is in rad $\Delta(\bar{f}, \bar{S}, G_p)$, and this completes the proof of the fact that rad $\Delta(\bar{f}, \bar{S}, G_0)$ is generated as a right ideal by rad $\Delta(\bar{f}, \bar{S}, G_p)$. A similar computation shows that rad $\Delta(\bar{f}, \bar{S}, G_0)$ is generated as a left ideal by rad $\Delta(\bar{f}, \bar{S}, G_p)$.

The following corollary follows at once from Prop. 1. 7 and shall be useful in Section 2 of this paper (see Prop. 2. 1).

COROLLARY 1. 8. *The radical of* $\Delta(f, S, G_0)$ *is generated both as a left and a right ideal by the radical of* $\Delta(f, S, G_p)$.

Now we may prove the main theorem of this section.

THEOREM 1. 9. *Let* S *denote the integral closure of a complete discrete rank one valuation ring* R *in a finite Galois extension of the quotient field of* R *and let* G *denote the Galois group of the quotient field extension. If* $[f]$ *is an element of* $H^2(G, U(S))$, *then the following statements are equivalent:*

(1) $\Delta(f, S, G)$ *is a* Π-*principal hereditary order*

(2) G_p *is an Abelian group and* $R_f = (1)$

(3) $R_f = (1)$.

Proof. We have already observed that $\Delta(f, S, G)$ is a Π-principal hereditary order if and only if $\Delta(\bar{f}, \bar{S}, G)$ is a semi-simple ring and that this in turn is equivalent to the semi-simplicity of $\Delta(\bar{f}, \bar{S}, G_0)$. Prop. 1. 7 now implies that $\Delta(f, S, G)$ is Π-principal if and only if $\Delta(\bar{f}, \bar{S}, G_p)$ is semi-simple.

According to Cor. 1. 4, $\Delta(\bar{f}, \bar{S}, G_p)$ is semi-simple if and only if it is a field. Using Prop. 1. 10 of [11] we see that $\Delta(\bar{f}, \bar{S}, G_p)$ is a field if and only if G_p is Abelian and $R_f = (1)$. Therefore statements (1) and (2) are equivalent. On the other hand, $\Delta(\bar{f}, \bar{S}, G_p)$ is semi-simple if and only if $\Delta(\bar{f}, \bar{S}, C)$ is a field (Cor. 1. 4) which is equivalent to $R_f = (1)$.

2. **Wild ramification.** The purpose of this section is to prove that a crossed product $\Delta(f, S, G)$ is hereditary if and only if it is Π-principal in the case when the residue class field extension $\bar{S}$ of $\bar{R}$ is separable. And we present an example to show the necessity of the assumption that the

residue class field extension be separable. In [6], Harada has proved that if $\bar{R}$ is perfect, a crossed product $\Delta(f, S, G)$ is hereditary if and only if S is a tamely ramified extension of R. The proof of this fact suggested to the author a way of viewing the more general problem considered here. Each crossed product over a tamely ramified extension is Π-principal; so for the purpose of this section we may as well restrict our attention to crossed products over wildly ramified extensions.

Unless otherwise stated, throughout this section S shall always denote a wildly ramified extension of a complete discrete rank one valuation ring R. The first step is to reduce the problem to a study of the crossed product $\Delta(f, S, G_p)$ where G_p denotes as usual the Galois group of the quotient field of S over the quotient field of the maximal tamely ramified extension of R in S. For Prop. 2.1 we make no restriction on the extension S of R.

PROPOSITION 2.1. *The crossed product $\Delta(f, S, G)$ is hereditary if and only if the subring $\Delta(f, S, G_p)$ is hereditary.*

Proof. According to Harada's criterion (Lemma 3 of [6]) a necessary and sufficient condition for an order Λ to be hereditary is that there exist an element α in Λ and a positive integer t such that $(\text{rad}\,\Lambda)^t = \alpha\Lambda = \Lambda\alpha$. For convenience of notation denote $\Delta(f, S, G)$ by Δ and the subring $\Delta(f, S, G_p)$ by Δ_p; let $N = \text{rad}\,\Delta$ and $N_p = \text{rad}\,\Delta_p$. Prop. 3.1 of [11] together with Cor. 1.8 implies that $N = N_p\Delta = \Delta N_p$.

Let π denote a prime element of R. According to Thm. 6.1 of [5], the assumption that Δ is hereditary implies the existence of a positive integer t such that $N^t = \pi\Delta$ because $\pi\Delta$ is an invertible ideal. We shall show that $N_p^t = \pi\Delta_p$. The equalities $N = N_p\Delta = \Delta N_p$ imply that $\pi\Delta = N^t = N_p^t\Delta$. Let $G = \cup G_p\sigma_i$ be a disjoint right coset decomposition of G relative to the subgroup G_p. Using the fact that $\Delta(f, S, G)$ is a free left $\Delta(f, S, G_p)$-module with free basis $\{u_{\sigma_i}\}$ one may obtain the inclusion $(N_p^t\Delta) \cap \Delta_p \subset N_p^t$, from which it follows that $\pi\Delta_p$ is contained in N_p^t. To obtain the opposite inclusion, observe that N_p^t is contained in $(\pi\Delta) \cap \Delta_p$. Using the fact that $\Delta(f, S, G)$ is a free left S-module with free basis $\{u_\sigma\}$ for σ in G, one may obtain the equality $(\pi\Delta) \cap \Delta_p = \pi\Delta_p$, so that N_p^t is contained in $\pi\Delta_p$. Therefore $N_p^t = \pi\Delta_p = \Delta_p\pi$ since π is in $\text{ctr}\Delta_p$. It now follows from Harada's criterion that Δ_p is an hereditary order.

The proof of the assertion in the other direction follows at once from Harada's criterion together with the equalities $N = N_p\Delta = \Delta N_p$.

We proceed to prove that if $\Delta(f, S, G)$ is hereditary, then it is Π-principal. The proof shall be indirect; so we assume that $\Delta(f, S, G)$ is an hereditary order which is not Π-principal and contradict the assumption that S is a wildly ramified extension of R.

Consider a decomposition $C = E_1 \times \cdots \times E_t$ of the center C of G_p into a direct product of cyclic groups. We next observe that we may assume that the restriction of f to $E_i \times E_i$ is normalized in the sense of cyclic groups. Since cohomologous 2-cocycles determine isomorphic crossed products it suffices to prove the following lemma.

LEMMA 2. 2. *There exists a 2-cocycle g in $Z^2(G, U(S))$ cohomologous to f such that the image of g under the restriction map $Z^2(G, U(S)) \to Z^2(E_i, U(S))$ is normalized in the sense of cyclic groups for each i.*

Proof. Let f_i denote the restriction of f to $E_i \times E_i$. It is well known (see p. 82 of [1]) that there exists a 2-cocycle g_i in $Z^2(E_i, U(S))$ such that f_i is cohomologous to g_i and g_i is normalized in the sense of cyclic groups. For each i let $\phi_i : E_i \to U(S)$ be the map satisfying $g_i(\sigma, \tau) = f_i(\sigma, \tau)\phi_i(\sigma)\phi_i^\sigma(\tau)/\phi_i(\sigma\tau)$ for all elements σ and τ in E_i, and note that $\phi_i(1) = 1$. We next extend the ϕ_i to a map $\phi : G \to U(S)$ by defining $\phi(\sigma) = \phi_i(\sigma)$ if σ is in E_i and $\phi(\sigma) = 1$ if σ is not in any subgroup E_i. It is easy to verify that the 2-cocycle g of $Z^2(G, U(S))$ defined by $g(\sigma, \tau) = f(\sigma, \tau)\phi(\sigma)\phi^\sigma(\tau)/\phi(\sigma\tau)$ has the desired properties.

The assumption that $\Delta(f, S, G)$ is not Π-principal implies that the radical group R_f of $[f]$ is non-trivial according to Thm. 1. 9. Recall (see Section 1) that R_f is by definition a direct product of cyclic groups $R_f = R_{f,1} \times \cdots \times R_{f,t}$ where $R_{f,i}$ is a subgroup of E_i. Since R_f is non-trivial we may consider the subgroup Q_x of order p contained in the first non-trivial component $R_{f,x}$ of R_f. Observe that the choice of x implies that the crossed product $\Delta(\bar{f}, \bar{S}, E_1 \times \cdots \times E_{x-1})$ is a field, and that there exists an element b in $\Delta(\bar{f}, \bar{S}, E_1 \times \cdots \times E_{x-1})$ such that $\bar{f}(\rho, \rho^{-1}) = b^p$ where ρ denotes a generator of Q_x. Write b in the form $b = \sum \bar{a}_\sigma u_\sigma$ with σ in $E_1 \times \cdots \times E_{x-1}$ and $\bar{a}_\sigma$ in $\bar{S}$. Since $\Delta(\bar{f}, \bar{S}, E_1 \times \cdots \times E_{x-1})$ is a commutative ring of characteristic p, it follows that $b^p = \sum (\bar{a}_\sigma)^p (u_\sigma)^p$. Observe that $b^p = \sum (\bar{a}_\sigma)^p (u_\sigma)^p$ with $\operatorname{ord} \sigma = p$ since b^p is in $\bar{S}$. Therefore the element

b of $\Delta(f, S, E_1 \times \cdots \times E_{x-1})$ satisfying $\bar{f}(\rho, \rho^{-1}) = b^p$ may be taken to be of the form $b = \sum \bar{a}_\sigma u_\sigma$ where each element σ has order p. Now let β denote an element of $\Delta(f, S, E_1 \times \cdots \times E_{x-1})$ in the preimage of b. Since $\bar{U} = \bar{S}$ where U denotes the inertia ring of S over R, the element β may be chosen in such a way that $\beta = \sum a_\sigma u_\sigma$ where each a_σ is in U and each element σ of $E_1 \times \cdots \times E_{x-1}$ has order p. The notation introduced in this paragraph shall be in use throughout the rest of this section. The following technical lemma shall be useful in proving that the non-triviality of the radical group of $[f]$ implies that $\Delta(f, S, G)$ is not hereditary when S is a wildly ramified extension of R.

LEMMA 2. 3. *Let ρ denote a generator of Q_x and let $\beta^{\rho^{-i}}$ denote the element of $\Delta(f, S, E_1 \times \cdots \times E_{x-1})$ defined by the equality $\beta u_{\rho^i} = u_{\rho^i} \beta^{\rho^{-i}}$ for $0 \leq i \leq p-1$. Then the element $f(\rho, \rho^{-1}) - \prod_{i=0}^{p-1} \beta^{\rho^{-i}}$ is in $\Pi^2 \Delta(f, S, E_1 \times \cdots \times E_x)$.*

Proof. Recall that by Lemma 2. 2 we may assume that the restriction of f to $E_i \times E_i$ is normalized in the sense of cyclic groups. In order to make use of Props. A. 4 and A. 5 of the appendix, we first observe that we can restrict our attention to a crossed product over an elementary Abelian p-group. For $1 \leq i \leq x$, let Q_i denote the (unique) subgroup of E_i with order p, and observe that $Q_1 \times \cdots \times Q_x$ is an elementary Abelian p-group. Recall that β is of the form $\beta = \sum a_\sigma u_\sigma$ where each a_σ is in the inertia ring U and each element σ of $E_1 \times \cdots \times E_{x-1}$ has order p, so that β is in fact an element of the crossed product $\Delta(f, S, Q_1 \times \cdots \times Q_x)$.

The next step is to show that there exists an element a in the fixed ring S_x of $Q_x = (\rho)$ such that $\beta^p \equiv a \bmod \Pi^2 \Delta(f, S, E_1 \times \cdots \times E_x)$. Consider the crossed product $\Delta(\tilde{f}, S/\Pi^2 S, Q_1 \times \cdots \times Q_x)$ where $\tilde{f}$ denotes the image of f under the natural map $Z^2(Q_1 \times \cdots \times Q_x, U(S)) \to Z^2(Q_1 \times \cdots \times Q_x, U(S/\Pi^2 S))$. According to Prop. A. 4, the crossed product $\Delta(\tilde{f}, S/\Pi^2 S, Q_1 \times \cdots \times Q_x)$ is a commutative ring with characteristic p, so that the image $\tilde{\beta}$ of $\beta = \sum a_\sigma u_\sigma$ in $\Delta(\tilde{f}, S/\Pi^2 S, Q_1 \times \cdots \times Q_x)$ satisfies the equalities $\tilde{\beta}^p = \sum (\tilde{a}_\sigma)^p (u_\sigma)^p = \sum (\tilde{a}_\sigma)^p \tilde{f}(\sigma, \sigma^{-1})$. The element $\sum (\tilde{a}_\sigma)^p \tilde{f}(\sigma, \sigma^{-1})$ of $S/\Pi^2 S$ is in the image of the fixed ring S_Q of $Q_1 \times \cdots \times Q_x$ under the natural map of S onto $S/\Pi^2 S$ (see Prop. A. 5). It suffices therefore to let a denote an element of S_Q in the preimage of $\sum (\tilde{a}_\sigma)^p \tilde{f}(\sigma, \sigma^{-1})$ to guarantee that $\beta^p \equiv a \bmod \Pi^2 \Delta(f, S, E_1 \times \cdots \times E_x)$.

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Now we may complete the proof of the lemma. The congruences $f(\rho,\rho^{-1})-\beta^p\equiv 0 \mod \Pi\Delta(f,S,E_1\times\cdots\times E_x)$ and $f(\rho,\rho^{-1})-\beta^p\equiv f(\rho,\rho^{-1})-a \mod \Pi^2\Delta(f,S,E_1\times\cdots\times E_x)$ imply that $f(\rho,\rho^{-1})-a\equiv 0 \mod \Pi S$ since $f(\rho,\rho^{-1})-a$ is in S. The fact that the extension S of S_x is a wildly ramified inertial extension of degree p implies that $f(\rho,\rho^{-1})-a\equiv 0 \mod \Pi^2 S$ since $f(\rho,\rho^{-1})-a$ is in S_x. On the other hand, the fact that $\Delta(\tilde{f},S/\Pi^2 S,Q_1\times\cdots\times Q_x)$ is a commutative ring implies that $f(\rho,\rho^{-1})-\beta^p \equiv f(\rho,\rho^{-1})-(\beta\beta^{\rho^{-1}}\cdots\beta^{\rho}) \mod \Pi^2\Delta(f,S,E_1\times\cdots\times E_x)$. By combining the above congruences we may now conclude that $f(\rho,\rho^{-1})-\prod_{i=0}^{p-1}\beta^{\rho^{-i}}$ is in $\Pi^2\Delta(f,S,E_1\times\cdots\times E_x)$.

PROPOSITION 2. 4. *Let S be a wildly ramified extension of R, and $[f]$ an element of $H^2(G,U(S))$ such that R_f is non-trivial. Then the crossed product $\Delta(f,S,G)$ is not an hereditary order.*

Proof. The proof is by contradiction. Suppose therefore that $\Delta(f,S,G)$ is hereditary. Then the subring $\Delta_p=\Delta(f,S,G_p)$ is hereditary according to Prop. 2. 1. The fact that $\Delta_p/\mathrm{rad}\,\Delta_p$ is a field (Cor. 1. 3) now implies that Δ_p is a maximal order with the property that all ideals are two-sided and are powers of the radical (see Thm. 3. 11 of [2]).

Throughout the proof of this proposition we shall assume the notation introduced in the statement of Lemma 2. 3. The ideals $\Pi\Delta_p$ and $(u_\rho-\beta)\Delta_p$ are therefore two-sided and either $\Pi\Delta_p$ is contained in $(u_\rho-\beta)\Delta_p$ or the opposite inclusion holds. Since the residue class ring $\Delta_p/\Pi\Delta_p$ is not semi-simple, we may conclude that the ideal $\Pi\Delta_p$ is contained in $(u_\rho-\beta)\Delta_p$. This inclusion of ideals shall be used to contradict the assumption that S is a wildly ramified extension of R.

According to the above we may write $\Pi=(u_\rho-\beta)\delta$ for some element δ of Δ_p. Observe that the elements of E_x may be taken as part of a system of representatives of a disjoint coset decomposition $G_p=\cup(E_1\times\cdots\times E_{x-1})\sigma$ of G_p relative to the subgroup $E_1\times\cdots\times E_{x-1}$. Therefore δ has a (unique) expression in the form $\delta=\sum_\sigma u_\sigma\delta_\sigma$ with the δ_σ in the crossed product $\Delta(f,S,E_1\times\cdots\times E_{x-1})$ and so $\Pi=(u_\rho-\beta)\sum u_\sigma\delta_\sigma$.

Now $(u_\rho-\beta)\sum u_\sigma\delta_\sigma=\sum f(\rho,\sigma)u_{\rho\sigma}\delta_\sigma-\sum u_\sigma\beta^{\sigma^{-1}}\delta_\sigma$ where $\beta^{\sigma^{-1}}$ denotes the element of $\Delta(f,S,E_1\times\cdots\times E_{x-1})$ defined by the equality $\beta u_\sigma=u_\sigma\beta^{\sigma^{-1}}$. Let $\tau=\rho\sigma$. From this change of variable we obtain the equality $\Pi=\sum_\tau u_\tau[f^{\tau^{-1}}(\rho,\rho^{-1}\tau)\delta_{\rho^{-1}\tau}-\beta^{\tau^{-1}}\delta_\tau]$. Using the fact that the elements $\{u_{\rho^i}\}$.

form part of a free basis for $\Delta(f, S, G_p)$ over $\Delta(f, S, E_1 \times \cdots \times E_{x-1})$ together with the fact f is normalized on $E_x \times E_x$ in the sense of cyclic groups we may now obtain the equalities

$$\Pi = f(\rho, \rho^{-1})\delta_{\rho^{-1}} - \beta\delta_1$$
$$0 = \delta_{\rho^{i-1}} - \beta^{\rho^{-i}}\delta_{\rho^i} \qquad \text{for } 0 < i < p,$$

which in turn combine to imply that $\Pi = [f(\rho, \rho^{-1}) - \prod_{i=0}^{p-1}\beta^{\rho^{-i}}]\delta_{\rho^{-1}}$.

Now we may complete the proof of the proposition. For according to Lemma 2. 3 the element $f(\rho, \rho^{-1}) - \prod_{i=0}^{p-1}\beta^{\rho^{-i}}$ is in the submodule $\Pi^2\Delta(f, S, G_p)$. The fact that $\Delta(f, S, G_p)$ is a free left S-module with free basis $\{u_\sigma\}$ for σ in G_p now implies that the equality $\Pi = [f(\rho, \rho^{-1}) - \prod_{i=0}^{p-1}\beta^{\rho^{-i}}]\delta_{\rho^{-1}}$ cannot hold. This contradiction completes the proof of the proposition.

Thus we have established the following main theorem.

THEOREM 2. 5. *Let S denote the integral closure of a complete discrete rank one valuation ring R in a finite Galois extension of the quotient field of R, and G the Galois group of the quotient field extension. If the residue class field extension $\bar{S} \supset \bar{R}$ is separable, then for each element $[f]$ of $H^2(G, U(S))$ the following statements are equivalent:*

(1) *$\Delta(f, S, G)$ is an hereditary order*

(2) *$\Delta(f, S, G)$ is a Π-principal hereditary order.*

Finally, we present an example to show the necessity of the assumption that the residue class field extension be separable.

EXAMPLE 2. 6. Let $R = Z[X]_{(2)}$ be the localization of the ring of polynomials with integral coefficients at the minimal prime ideal generated by 2. Let $K = k(X^{\frac{1}{2}})$ where k denotes the quotient field of R, and let $G = \{1, \sigma\}$ denote the Galois group of K over k. The integral closure of R in K is $S = R[X^{\frac{1}{2}}]$ and the residue class field extension $\bar{S}$ of $\bar{R}$ is purely inseparable of degree two. Let f be the element of $Z^2(G, U(S))$ corresponding to the element $2 - X$ of $U(R)$ under the canonical identification $H^2(G, U(S)) = U(R)/N(U(S))$, and consider the crossed product $\Delta = \Delta(f, S, G)$. An easy computation shows that rad $\Delta = (u_\sigma - X^{\frac{1}{2}})\Delta$ is a free right Δ-module, so that Δ is an hereditary order according to the Corollary to Thm. 2. 2

of [2]. However, Δ is not a Π-principal hereditary order since $\Pi\Delta$ is strictly contained in rad Δ.

3. **The conductor group.** Harada has shown in [5] that the number of maximal two-sided ideals in an hereditary order Λ in a central simple algebra Σ over the quotient field of a discrete rank one valuation ring R is equal to the length of a saturated chain of orders in Σ containing Λ. We are interested therefore in determining the number of maximal two-sided ideals in a Π-principal hereditary order $\Delta(f,S,G)$. In [10] the author proved that the number of maximal two-sided ideals in a crossed product $\Delta(f,S,G)$ over a tamely ramified extension S of R is equal to the order of the conductor group H_f of $\Delta(f,S,G)$ where H_f is defined to be the maximal subgroup of the inertia group of S over R such that $[\bar{f}]$ is in the image of the inflation map $H^2(G/H_f, U(\bar{S})) \to H^2(G, U(\bar{S}))$. In this section we shall generalize the notion of the conductor group to the case of any Π-principal hereditary order $\Delta(f,S,G)$ and then observe that the number of maximal two-sided ideals in $\Delta(f,S,G)$ is equal to the order of its conductor group.

The number of maximal two-sided ideals in a Π-principal hereditary order $\Delta(f,S,G)$ is equal to the number of primitive orthogonal idempotents required to generate the center of the (semi-simple) ring $\Delta(\bar{f},\bar{S},G)$.

PROPOSITION 3.1. *Let S denote the integral closure of a complete discrete rank one valuation ring R in a finite Galois extension of the quotient field of R, and G the Galois group of the quotient field extension. Then the center of $\Delta(\bar{f},\bar{S},G)$ is contained in the center of $\Delta(\bar{f},\bar{S},G_0)$ where G_0 denotes the inertia group of S over R.*

Proof. Consider the separable closure $\bar{U}$ of $\bar{R}$ in $\bar{S}$, and let θ denote an element of $\bar{U}$ for which $\bar{U} = \bar{R}(\theta)$. A non-zero element $\delta = \sum s_\sigma u_\sigma$ (with $s_\sigma \neq 0$) in the center of $\Delta(\bar{f},\bar{S},G)$ has the property that $\delta\theta = \theta\delta$. Now $\delta\theta = \sum s_\sigma \theta^\sigma u_\sigma$ so that $\delta\theta = \theta\delta$ if and only if $\theta^\sigma = \theta$ for each σ. But $\theta^\sigma = \theta$ if and only if σ is in G_0 since G/G_0 is the Galois group of $\bar{U}$ over $\bar{R}$. Therefore δ is in the subring $\Delta(\bar{f},\bar{S},G_0)$.

The next two propositions pertain to the center of $\Delta(\bar{f},\bar{S},G_0)$. Recall (Prop. 1.1) that the inertia group G_0 is the semi-direct product $J \times G_p$ where G_p is a p-group normal in G, and the order e of J is relatively prime to p.

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PROPOSITION 3. 2. *The center of* $G_0 = J \times G_p$ *is of the form* $J_c \times C_c$ *(direct product) where* J_c *is a subgroup of* J *and* C_c *is a subgroup of the center of* G_p. *Furthermore,* J_c *is a normal subgroup of* G.

Proof. Let $\rho\tau$ denote an element of the center $C(G_0)$ of G_0, where ρ is in J and τ is in G_p. To prove the proposition it suffices to show that both ρ and τ are in $C(G_0)$. To prove that ρ is in $C(G_0)$ we first observe that the fact that J is an Abelian group may be used to show that τ commutes (element-wise) with every element of J. Let n denote the order of τ. Then $(\rho\tau)^n = \rho^n$ since τ commutes with ρ, so that ρ^n is in $C(G_0)$. The fact that the order of ρ is relatively prime to n implies that ρ is in $C(G_0)$. We may conclude at once that τ is in $C(G_0)$ since $\rho\tau$ and ρ are in $C(G_0)$.

We next show that J_c is a normal subgroup of G. Let σ denote a generator of the cyclic group J_c, and τ an element of G. Since σ is in G_0 and G_0 is a normal subgroup of G, it follows that $\tau\sigma\tau^{-1}$ is in G_0. Let $\bar{\rho}$ denote the image of an element ρ of G under the natural map of G onto G/G_p. The homomorphic image $\bar{J}$ of J under this map is a normal subgroup of G/G_p since $\bar{J}$ is the inertia subgroup of G/G_p. From this it follows that the subgroup $\bar{J}_c$ of the cyclic group $\bar{J}$ is also a normal subgroup of G/G_p. Therefore $\overline{\tau\sigma} = \bar{\sigma}^i\bar{\tau}$ for some integer i, and so we may write $\tau\sigma = \rho\sigma^i\tau$ for some element ρ of G_p. It remains to show that $\rho = 1$. Let n denote the order of σ and observe that n is relatively prime to p. Then $\tau\sigma\tau^{-1} = \rho\sigma^i$ has order n. The fact that σ is in J_c implies that $1 = (\rho\sigma^i)^n = \rho^n$. Since ρ is in the p-group G_p and $(n, p) = 1$, we conclude at last that $\rho = 1$.

PROPOSITION 3. 3. *The crossed product* $\Delta(\bar{f}, \bar{S}, J_c \times C_c)$ *is contained in the center of* $\Delta(\bar{f}, \bar{S}, G_0)$.

Proof. In order to establish the inclusion $\Delta(\bar{f}, \bar{S}, J_c \times C_c) \subset \mathrm{ctr}\Delta(\bar{f}, \bar{S}, G_0)$ it suffices to show that every element of the form u_α with α in $J_c \times C_c$ commutes with every element of the form u_β with β in G_0. Now $u_\alpha u_\beta = u_\beta u_\alpha$ if and only if $\bar{f}(\alpha, \beta) = \bar{f}(\beta, \alpha)$ since α is in the center of G_0.

It remains to show that $\bar{f}(\alpha, \beta) = \bar{f}(\beta, \alpha)$ for each α in $J_c \times C_c$ and β in G_0. Write α in the form $\alpha = \sigma_1\tau_1$ with σ_1 in J_c and τ_1 in C_c, and write β in the form $\beta = \sigma_2\tau_2$ with σ_2 in J and τ_2 in G_p. We first observe

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that $\bar{f}(\sigma_2\tau_2,\sigma_1)=\bar{f}(\sigma_1,\sigma_2\tau_2)$. For the equalities $\bar{f}(\sigma_2\tau_2,\sigma_1)\bar{f}(\sigma_2,\tau_2)=\bar{f}(\sigma_2,\tau_2\sigma_1)\bar{f}(\tau_2,\sigma_1)$ and $\bar{f}(\sigma_2,\sigma_1\tau_2)\bar{f}(\sigma_1,\tau_2) = \bar{f}(\sigma_2\sigma_1,\tau_2)\bar{f}(\sigma_2,\tau_1)$ together imply $\bar{f}(\sigma_2\tau_2,\sigma_1) = \bar{f}(\sigma_2\sigma_1,\tau_2)\bar{f}(\sigma_2,\sigma_1)\bar{f}(\tau_2,\sigma_1)/\bar{f}(\sigma_2,\tau_2)\bar{f}(\sigma_1,\tau_2)$ since $\tau_2\sigma_1=\sigma_1\tau_2$. Now $\bar{f}(\tau_2,\sigma_1)=\bar{f}(\sigma_1,\tau_2)$ according to Lemma A. 1 of [11] because the order of τ_2 is a p^{th} power. Therefore $\bar{f}(\sigma_2\tau_2,\sigma_1)=\bar{f}(\sigma_2\sigma_1,\tau_2)\bar{f}(\sigma_2,\sigma_1)/\bar{f}(\sigma_2,\tau_2)$. On the other hand, the associativity property of $\bar{f}$ implies that $\bar{f}(\sigma_1,\sigma_2\tau_2)\bar{f}(\sigma_2,\tau_2) = \bar{f}(\sigma_1\sigma_2,\tau_2)\bar{f}(\sigma_1,\sigma_2)$. Since J is a cyclic group it follows that $\bar{f}(\sigma_1,\sigma_2) = \bar{f}(\sigma_2,\sigma_1)$. Therefore $\bar{f}(\sigma_2\tau_2,\sigma_1)=\bar{f}(\sigma_1,\sigma_2\tau_2)$.

Now we may prove that $\bar{f}(\sigma_1\tau_1,\sigma_2\tau_2)=\bar{f}(\sigma_2\tau_2,\sigma_1\tau_1)$. The equalities $\bar{f}(\sigma_1\tau_1,\sigma_2\tau_2)\bar{f}(\sigma_1,\tau_1) = \bar{f}(\sigma_1,\tau_1\sigma_2\tau_2)\bar{f}(\tau_1,\sigma_2\tau_2)$ and $\bar{f}(\sigma_1,\sigma_2\tau_2\tau_1)\bar{f}(\sigma_2\tau_2,\tau_1) = \bar{f}(\sigma_1\sigma_2\tau_2,\tau_1)\bar{f}(\sigma_1,\sigma_2\tau_2)$ imply that $\bar{f}(\sigma_1\tau_1,\sigma_2\tau_2)=\bar{f}(\sigma_1\sigma_2\tau_2,\tau_1)\bar{f}(\sigma_1,\sigma_2\tau_2)\bar{f}(\tau_1,\sigma_2\tau_2)/\bar{f}(\sigma_2\tau_2,\tau_1)\bar{f}(\sigma_1,\tau_1)$ since $\tau_1\sigma_2\tau_2=\sigma_2\tau_2\tau_1$. On the other hand, $\bar{f}(\sigma_2\tau_2,\sigma_1\tau_1)\bar{f}(\sigma_1,\tau_1) = \bar{f}(\sigma_2\tau_2\sigma_1,\tau_1)\bar{f}(\sigma_2\tau_2,\sigma_1)$. Now $\bar{f}(\tau_1,\sigma_2\tau_2)=\bar{f}(\sigma_2\tau_2,\tau_1)$ by Lemma A. 1 of [11], and $\bar{f}(\sigma_1,\sigma_2\tau_2)=\bar{f}(\sigma_2\tau_2,\sigma_1)$ by the above observation. Therefore $\bar{f}(\sigma_1\tau_1,\sigma_2\tau_2) = \bar{f}(\sigma_2\tau_2,\sigma_1\tau_1)$ and this completes the proof.

Observe that for Props. 3. 1, 3. 2 and 3. 3 we did not need to assume that $\Delta(f,S,G)$ is Π-principal.

PROPOSITION 3. 4. *If the crossed product* $\Delta(f,S,G_0)$ *is* Π*-principal, then the center of* $\Delta(\bar{f},\bar{S},G_0)$ *is contained in* $\Delta(\bar{f},\bar{S},J_c\times G_p)$.

Proof. Recall that the assumption that $\Delta(f,S,G_0)$ is Π-principal implies that G_p is Abelian (Thm. 1. 9). Since G_0 is the semi-direct product $J\times G_p$, the elements of J may be taken as representatives of a disjoint coset decomposition of G_0 relative to the (normal) subgroup G_p. An element δ of $\Delta(\bar{f},\bar{S},G_0)$ has therefore a unique expression in the form $\delta=\sum\delta_\sigma u_\sigma$ with each σ in J and each δ_σ in the subring $\Delta(\bar{f},\bar{S},G_p)$ according to Lemma 2. 5 of [11]. If $\delta=\sum\delta_\sigma u_\sigma$ (with $\delta_\sigma\neq 0$) is in ctr $\Delta(\bar{f},\bar{S},G_0)$ then $u_\tau\delta=\delta u_\tau$ for each element τ of G_p. By an easy computation one may obtain the equality $\delta u_\tau=\sum\delta_\sigma[\bar{f}(\sigma,\tau)/\bar{f}(\tau^\sigma,\sigma)]u_{\tau^\sigma}u_\tau$ where τ^σ is the element of G_p defined by $\sigma\tau=\tau^\sigma\sigma$. The fact that $u_\tau\delta=\delta u_\tau$ now implies that $\delta_\sigma=\delta_\sigma[\bar{f}(\sigma,\tau)/\bar{f}(\tau^\sigma,\sigma)]u_{\tau^\sigma}$ for each σ. The assumption that $\Delta(f,S,G_0)$ is Π-principal implies that $\Delta(\bar{f},\bar{S},G_p)$ is a field (see Thm. 1. 9). Therefore $1=[\bar{f}(\sigma,\tau)/\bar{f}(\tau^\sigma,\sigma)]u_{\tau^\sigma}$ which implies that u_{τ^σ} is an element of $\bar{S}$ and so τ^σ must equal 1. We have shown that each σ in the expression $\delta=\sum\delta_\sigma u_\sigma$ for an element δ in the center of $\Delta(\bar{f},\bar{S},G_0)$ commutes with each element of G_p, and this completes the proof.

Combining Props. 3. 3 and 3. 4 we may now determine the idempotents in the center of $\Delta(\bar{f}, \bar{S}, G_0)$ when $\Delta(f, S, G)$ is Π-principal.

PROPOSITION 3. 5. *If $\Delta(f, S, G)$ is a Π-principal hereditary order then the idempotents in the center of $\Delta(\bar{f}, \bar{S}, G_0)$ are precisely the idempotents of the commutative ring $\Delta(\bar{f}, \bar{S}, J_c)$.*

Proof. Prop. 3. 4 implies that the idempotents in the center of $\Delta(\bar{f}, \bar{S}, G_0)$ are present in the commutative ring $\Delta(\bar{f}, \bar{S}, J_c \times G_p)$. Let d denote an idempotent element in $\Delta(\bar{f}, \bar{S}, J_c \times G_p)$ and observe that d has an expression in the form $d = \sum d_\tau u_\tau$ with each τ in G_p and d_τ in $\Delta(\bar{f}, \bar{S}, J_c)$. The assumption that d is an idempotent implies that $d^n = d$ where n denotes the order of G_p. The fact that $\Delta(\bar{f}, \bar{S}, J_c \times G_p)$ is a commutative ring of characteristic p implies that $d^n = \sum (d_\tau)^n (u_\tau)^n$ since n is a p^{th} power; thus d^n is in $\Delta(\bar{f}, \bar{S}, J_c)$ since $(u_\tau)^n$ is in $\bar{S}$ by the choice of n. Therefore d is in $\Delta(\bar{f}, \bar{S}, J_c)$.

On the other hand, Prop. 3. 3 implies that each idempotent of $\Delta(\bar{f}, \bar{S}, J_c)$ is in the center of $\Delta(\bar{f}, \bar{S}, G_0)$.

If $\Delta(f, S, G)$ is Π-principal, then Props. 3. 1 and 3. 5 together imply that the idempotents in the center of $\Delta(\bar{f}, \bar{S}, G)$ are precisely those idempotents of $\Delta(\bar{f}, \bar{S}, J_c)$ which are also in the center of $\Delta(\bar{f}, \bar{S}, G)$. This motivates us to generalize the notion of the conductor group in the following way.

DEFINITION. Let $\Delta(f, S, G)$ be a Π-principal hereditary order, and let J_c denote the subgroup of the inertia group defined in Prop. 3. 2. Then the *conductor group* H_f of $\Delta(f, S, G)$ is defined to be the maximal subgroup of J_c with the property that $[\bar{f}]$ is in the image of the inflation map $H^2(G/H_f, U(\bar{S})) \to H^2(G, U(\bar{S}))$ where $\bar{f}$ denotes the image of f under the natural map $Z^2(G, U(S)) \to Z^2(G, U(\bar{S}))$.

Observe that $J_c = G_0$ when S is a tamely ramified extension of R. Therefore the above definition of conductor group is indeed a generalization of the definition given in [10] for the tamely ramified case.

The arguments used in Section 2 of [10] may now be extended to prove that the number of maximal two-sided ideals in a Π-principal hereditary order is equal to the order of its conductor group.

LEMMA 3. 6. *Let c denote the order of J_c. For each element τ of G we* *have* $\tau(\zeta)=\zeta^{n(\tau)}$ *for each* c^{th} *root of unity* ζ *in* $\bar{S}$ *where* $n(\tau)$ *is the integer defined modulo* c *by the equality* $\tau\sigma\tau^{-1}=\sigma^{n(\tau)}$ *and* σ *denotes a generator of* J_c.

Proof. Consider the maximal tamely ramified extension T of R in S, and recall (Prop. 1. 1) that $\bar{T}$ contains a primitive e^{th} root of unity where e denotes as usual the order of J. The image $\bar{J}$ of J under the natural map of G onto G/G_p is the inertia group of T over R. Denote the image of an element τ of G in G/G_p by $\bar{\tau}$. Then Prop. 2. 1 of [10] implies that $\bar{\tau}(\zeta)=\zeta^{n(\bar{\tau})}$ for each e^{th} root of unity ζ in $\bar{S}$ where $n(\bar{\tau})$ is the integer defined modulo e by the equality $\bar{\tau}\bar{\omega}\bar{\tau}^{-1}=\bar{\omega}^{n(\bar{\tau})}$ where $\bar{\omega}$ denotes a generator of $\bar{J}$. Let σ denote a generator of J_c. The equality $\tau\sigma\tau^{-1}=\sigma^{n(\bar{\tau})}$ holds because J_c is a normal subgroup of G. This is sufficient to prove the lemma.

It is convenient to introduce the following subgroup of J_c in order to determine the number of primitive orthogonal idempotents in ctr $\Delta(\bar{f},\bar{S},G)$.

Definition. Let Γ_f denote the maximal subgroup of J_c with the property that the image of $[\bar{f}]$ under the restriction map $H^2(G, U(\bar{S}))\to H^2(\Gamma_f, U(\bar{S}))$ is trivial.

Observe that the conductor group H_f of $\Delta(f,S,G)$ is a subgroup of Γ_f. An easy computation shows that $\bar{f}$ is cohomologous to a 2-cocycle whose restriction to $\Gamma_f\times\Gamma_f$ is trivial. Thus we shall always assume that $\bar{f}$ is a *properly normalized* 2-cocycle; i.e. that $\bar{f}(\sigma,\tau)=1$ for all σ and τ in Γ_f.

The next two lemmas are essentially the same as Props. 2. 2 and 2. 3 of [10] and so we refer the reader to [10] for their proofs.

Lemma 3. 7. *The number of simple components of* $\Delta(\bar{f},\bar{S},J_c)$ *is equal to the number of simple components of* $\Delta(\bar{f},\bar{S},\Gamma_f)$ *and the primitive orthogonal idempotents are given by* $\eta_i=\frac{1}{m}\sum_{k=1}^{m}(\zeta_i u_T)^k$ *for* $1\leq i\leq m$ *where* m *is the order of* Γ_f *and the* ζ_i *are the* m *distinct* m^{th} *roots of unity.*

Lemma 3. 8. *Let* $\bar{f}$ *be a properly normalized 2-cocycle and* ρ *an element of* Γ_f. *Then the cyclic group generated by* ρ *is contained in* H_f *if and only if* $\bar{f}(\tau,\rho)=\bar{f}(\rho^{n(\tau)},\tau)$ *for each element* τ *in* G.

Combining these three lemmas we may now obtain the following result.

PROPOSITION 3. 9. *The number of simple components of* $\Delta(\bar{f}, \bar{S}, G)$ *is equal to the order of the conductor group* H_f.

Proof. The number of simple components of $\Delta(\bar{f}, \bar{S}, G)$ is equal to the number of primitive orthogonal idempotents required to generate its center. According to Props. 3. 1 and 3. 5 the idempotents in $\operatorname{ctr} \Delta(\bar{f}, \bar{S}, G)$ are precisely those partial sums P of elements η_i such that P is in $\operatorname{ctr} \Delta(\bar{f}, \bar{S}, G)$ where the η_i are defined in Prop. 3. 7. Let $P = \sum_{i=1}^{t} \eta_i$ be any partial sum of elements η_i (with a suitable reordering) and observe that P is in $\operatorname{ctr} \Delta(\bar{f}, \bar{S}, G)$ if and only if $u_\tau P = P u_\tau$ for every τ in G. By an easy computation we obtain that

$$u_\tau P = \sum_{k=1}^{m} \sum_{i=1}^{t} \frac{1}{m} \tau(\zeta_i^k)[\bar{f}(\tau, \gamma^k)/\bar{f}(\gamma^{kn(\tau)}, \tau)] u_\gamma{}^{kn(\tau)} u_\tau .$$

Lemma 3. 6 implies that $\tau(\zeta_i^k) = \zeta_i{}^{kn(\tau)}$ so that $u_\tau P = P u_\tau$ if and only if $\bar{f}(\tau, \gamma^k) = \bar{f}(\gamma^{kn(\tau)}, \tau)$ for every τ in G and every integer k for which $\sum_{i=1}^{t} \tau(\zeta_i^k)$ is non-zero. Prop. 3. 8 now implies that P is in $\operatorname{ctr} \Delta(\bar{f}, \bar{S}, G)$ if and only if P is in the subring $\Delta(\bar{f}, \bar{S}, H_f)$. Therefore $\Delta(\bar{f}, \bar{S}, G)$ has precisely as many simple components as $\Delta(\bar{f}, \bar{S}, H_f)$ and this is equal to the order of H_f since $\bar{f} = 1$ on $H_f \times H_f$.

The main theorem of this section follows at once from Prop. 3. 9.

THEOREM 3. 10. *The number of maximal two-sided ideals in a* Π*-principal hereditary order is equal to the order of its conductor group.*

Appendix. Cohomology. In this appendix we shall study the second cohomology group $H^2(G, U(S))$ where S is a wildly ramified inertial extension of a complete discrete rank one valuation ring R for which the Galois group G of the quotient field extension is an elementary Abelian p-group. The results are used in Section 2 of this paper.

We first prove two preliminary facts which may be presented in a more general context.

LEMMA A. 1. *Let* G *be a finite group,* A *a left* G*-module, and* (τ) *the cyclic group generated by the element* τ *of* G. *Let* f *denote an element of* $Z^2(G, A)$ *such that the image of* f *under the restriction map* $Z^2(G, A) \to Z^2((\tau), A)$ *is normalized in the sense of cyclic groups. Then*

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$$\prod_{i=1}^{n} [f(\tau, \sigma)/f(\sigma, \tau)]^{\tau^{n-i}} = f(\tau^{-1}, \tau)/f^{\sigma}(\tau^{-1}, \tau)$$

for each σ in G commuting with τ where n denotes the order of τ.

Proof. From the associativity property of the 2-cocycle f we obtain at once the equalities $f(\sigma\tau^{-1}, \tau)f(\sigma, \tau^{-1}) = f^{\sigma}(\tau^{-1}, \tau)$, $f(\tau^{-1}\sigma, \tau)f(\tau^{-1}, \sigma) = f(\tau^{-1}, \sigma\tau)f^{\tau^{-1}}(\sigma, \tau)$ and $f(\tau^{-1}, \tau\sigma)f^{\tau^{-1}}(\tau, \sigma) = f(\tau^{-1}, \tau)$ which together imply that

$$f^{\tau^{n-1}}(\tau, \sigma)f(\tau^{n-1}, \sigma)/f^{\tau^{n-1}}(\sigma, \tau)f(\sigma, \tau^{n-1}) = f(\tau^{-1}, \tau)/f^{\sigma}(\tau^{-1}, \tau).$$

We next obtain an expression for $f(\tau^{n-1}, \sigma)$. Consider $f(\tau^{n-i-1}, \sigma)$ for $1 \leq i \leq n-1$. From the associativity property of f together with the fact that f is normalized on $(\tau) \times (\tau)$ in the sense of cyclic groups we obtain that $f(\tau^{n-i-1}, \tau\sigma) f^{\tau^{n-i-1}}(\tau, \sigma) = f(\tau^{n-i}, \sigma)$ and $f(\tau^{n-i-1}, \sigma\tau) f^{\tau^{n-i-1}}(\sigma, \tau) = f(\tau^{n-i-1}\sigma, \tau)f(\tau^{n-i-1}, \sigma)$. Together these equalities imply that

$$f(\tau^{n-i-1}, \sigma) = [f(\tau, \sigma)/f(\sigma, \tau)]^{\tau^{n-i-1}} f(\sigma\tau^{n-i-1}, \tau)f(\tau^{n-i-1}, \sigma).$$

Combining these equalities we finally obtain that

$$f(\tau^{n-1}, \sigma) = \prod_{i=2}^{n} [f(\tau, \sigma)/f(\sigma, \tau)]^{\tau^{n-i}} f(\sigma\tau^{n-i}, \tau).$$

On the other hand, by combining the equalities $f(\sigma, \tau^{n-i}) = f(\sigma\tau^{n-i-1}, \tau)f(\sigma, \tau^{n-i-1})$ for $1 \leq i \leq n-1$ we obtain that $f(\sigma, \tau^{n-1}) = \prod_{i=2}^{n} f(\sigma\tau^{n-i}, \tau)$.

Substituting these expressions for $f(\tau^{n-1}, \sigma)$ and $f(\sigma, \tau^{n-1})$ into the equality established in the first paragraph of the proof we conclude that $\prod_{i=1}^{n} [f(\tau, \sigma)/f(\sigma, \tau)]^{\tau^{n-i}} = f(\tau^{-1}, \tau)/f^{\sigma}(\tau^{-1}, \tau)$.

DEFINITION. Let $G = E_1 \times \cdots \times E_t$ be a decomposition of an Abelian group G into a direct product of cyclic groups, and A a left G-module. An element f of $Z^2(G, A)$ which is of the form $f = f_1 \cdots f_t$ where each element f_i of $Z^2(E_i, A)$ is normalized in the sense of cyclic groups is said to be *normalized in the sense of Abelian groups*; i.e. f is normalized in the sense of Abelian groups if and only if $f(\sigma_1 \cdots \sigma_t, \omega_1 \cdots \omega_t) = f(\sigma_1, \omega_1) \cdots f(\sigma_t, \omega_t)$ where σ_i and ω_i are in E_i.

LEMMA A. 2. *Let $G = E_1 \times \cdots \times E_t$ denote a decomposition of an Abelian group G into a direct product of cyclic groups, and A a left G-module. For each* *element f of $Z^2(G, A)$ there exists a 2-cocycle g of $Z^2(G, A)$ cohomologous to f such that*

1) $g(\sigma_i, \sigma_j) = 1$ *for all elements* σ_i *in* E_i *and* σ_j *in* E_j *with* $i < j$

2) *the restriction of* g *to* $E_i \times E_i$ *is normalized in the sense of cyclic groups for* $1 \leq i \leq t$.

Proof. An argument similar to that of Lemma 2. 2 shows that f is cohomologous to a 2-cocycle h satisfying assertion 2). Now define a map $\phi : G \to A$ by setting $\phi(\tau) = h(\sigma_i, \sigma_j)$ if τ is an element of the form $\tau = \sigma_i \sigma_j$ with σ_i in E_i and σ_j in E_j and $i < j$, and $\phi(\tau) = 1$ otherwise. It is easy to verify that the 2-cocycle g defined by $g(\tau, \rho) = h(\tau, \rho)\phi(\tau)\phi^{\tau}(\rho)/\phi(\tau\rho)$ has the desired properties.

Now we proceed to establish results concerning cohomology and wild ramification.

PROPOSITION A. 3. *Let S be a wildly ramified inertial extension of a complete discrete rank one valuation ring R such that the Galois group G of the quotient field extension is an elementary Abelian p-group, and let $\tilde{f}$ denote the image of an element f of $Z^2(G, U(S))$ under the natural map $Z^2(G, U(S)) \to Z^2(G, U(S/\Pi^2 S))$. If f is normalized in the sense of Lemma* A. 2, *then $\tilde{f}$ is normalized in the sense of Abelian groups.*

Proof. Observe first of all that the action of G on $S/\Pi^2 S$ induced by the action of G on S is trivial because G is the first ramification group of S over R.

The proof of this proposition is facilitated by choosing judiciously a decomposition of the elementary Abelian p-group G into a direct product of cyclic groups. Let G_2 denote the second ramification group of S over R, i.e. G_2 is the set of all elements σ of G such that $\sigma(s) \equiv s \bmod \Pi^3 S$ for all s in S. An elementary p-group is completely reducible. Therefore G_2 is a direct factor of G according to the theorem on p. 148 of [8], from which it follows that G is isomorphic to $G/G_2 \times G_2$ in a natural way. Let $G/G_2 = Q_1 \times \cdots \times Q_s$ be a decomposition of G/G_2 into a direct product of cyclic groups, and let $G_2 = Q_{s+1} \times \cdots \times Q_t$ be such a decomposition of G_2, so that $G = Q_1 \times \cdots \times Q_t$.

For $1 \leq i \leq t$ define S_i to be the fixed ring of Q_i, and let Π_i denote a prime element of S_i. If $1 \leq i \leq s$ then the second ramification group

$G_2^{(i)}$ of S over S_i vanishes. For, an element σ of $G_2^{(i)}$ has the property that $\sigma(s) \equiv s \mod \Pi^3 S$ for each s in S, and therefore σ is in G_2. Since $G/G_2 \cap G_2 = (1)$ we conclude that $\sigma = 1$. On the other hand, for $s+1 \leq i \leq t$ it is easy to see that $G_2^{(i)} = Q_i$.

Let $N_i : S \to S_i$ denote the norm function from S into S_i. We next observe that for elements σ_i of Q_i and σ_j of Q_j with $i < j$, the congruences $N_i(f(\sigma_j, \sigma_i)) \equiv 1 \mod \Pi_i{}^2 S_i$ and $N_j(f(\sigma_j, \sigma_i)) \equiv 1 \mod \Pi_j{}^2 S_j$ hold. For the assumption on f together with Lemma A. 1 implies that $N_j(f(\sigma_j, \sigma_i)) = f(\sigma_j^{-1}, \sigma_j)/f^{\sigma_i}(\sigma_j^{-1}, \sigma_j)$. Now $f(\sigma_j^{-1}, \sigma_j)$ is in S_j (see p. 82 of [1]). Therefore $f^{\sigma_i}(\sigma_j^{-1}, \sigma_j) \equiv f(\sigma_j^{-1}, \sigma_j) \mod \Pi_j{}^2 S_j$ since the Galois group of the quotient field extension of $S_j \supset R$ is G/Q_j, and hence $N_j(f(\sigma_j, \sigma_i)) \equiv 1 \mod \Pi_j{}^2 S_j$. A similar application of Lemma A. 1 shows that $N_i(f(\sigma_j, \sigma_i)) \equiv 1 \mod \Pi_i{}^2 S_i$.

We show next that $f(\sigma_j, \sigma_i) \equiv 1 \mod \Pi^2 S$ for all σ_j in Q_j and σ_i in Q_i with $i < j$. Consider the filtration $U(S)^i$ of $U(S)$ defined on p. 74 of [7], and observe that $f(\sigma_j, \sigma_i) \equiv 1 \mod \Pi S$ according to Prop. A. 1 of [11] so that $f(\sigma_j, \sigma_i)$ is in $U(S)^1$. If $s < j$ then the second ramification group of S over S_j is non-vanishing. Therefore the map $N_{j,1} : U(S)^1/U(S)^2 \to U(S_j)^1/U(S_j)^2$ is an injection according to Cor. 1 on p. 93 of [7], and so $f(\sigma_j, \sigma_i) \equiv 1 \mod \Pi^2 S$. On the other hand, if $i < j \leq s$ then the second ramification group of S over S_j vanishes. Therefore the sequence

$$(0) \longrightarrow Q_j \xrightarrow{\theta_{1,j}} U(S)^1/U(S)^2 \xrightarrow{N_{1,j}} U(S_j)^1/U(S_j)^2$$

is exact according to Cor. 1 on p. 93 of [7] where $\theta_{1,j}$ is induced by the map $\sigma \to \Pi^\sigma/\Pi$ of Q_j into $U(S)^1$. The fact that $N_j(f(\sigma_j, \sigma_i)) \equiv 1 \mod \Pi_j{}^2 S_j$ now implies that $f(\sigma_j, \sigma_i) \equiv \Pi^{\omega_j}/\Pi \mod U(S)^2$ for some element ω_j of Q_j. In a similar way, the fact that $N_i(f(\sigma_j, \sigma_i)) \equiv 1 \mod \Pi_i{}^2 S_i$ implies that $f(\sigma_j, \sigma_i) \equiv \Pi^{\omega_i}/\Pi \mod U(S)^2$ for some element ω_i of Q_i. Together these congruences imply that $\Pi^{\omega_j}/\Pi^{\omega_i}$ is in $U(S)^2$ from which it follows that $\Pi^{\omega_j \omega_i^{-1}} - \Pi$ is in $\Pi^3 S$ and so $\omega_j \omega_i^{-1}$ is in the second ramification group G_2 of S over R. But ω_i and ω_j are elements of G/G_2. The fact that $G/G_2 \cap G_2 = (1)$ implies that $\omega_j = \omega_i$, and so $\omega_j = 1$ since $Q_i \cap Q_j = (1)$. This completes the proof of the fact that $f(\sigma_j, \sigma_i) \equiv 1 \mod \Pi^2 S$.

We have shown that $\tilde{f}(\sigma_j, \sigma_i) = \tilde{f}(\sigma_i, \sigma_j) = 1$ for all σ_i in Q_i and σ_j in Q_j when $i \neq j$. A computation similar to that of Cor. A. 2 of [11] shows that this is sufficient to guarantee that $\tilde{f}$ is normalized in the sense of Abelian groups.

PROPOSITION A. 4. *Let S denote a wildly ramified inertial extension of a complete discrete rank one valuation ring R such that the Galois group G of the quotient field extension is an elementary Abelian p-group, and f an element of $Z^2(G, U(S))$. Then the crossed product $\Delta(\tilde{f}, S/\Pi^2 S, G)$ is a commutative ring where $\tilde{f}$ denotes the image of f under the natural map $Z^2(G, U(S)) \to Z^2(G, U(S/\Pi^2 S))$.*

Proof. The 2-cocycle f is cohomologous to an element g of $Z^2(G, U(S))$ which is normalized in the sense of Lemma A. 2. The fact that $\tilde{g}$ is normalized in the sense of Abelian groups (Prop. A. 3) together with the fact that G acts trivially on $S/\Pi^2 S$ implies that the crossed product $\Delta(\tilde{g}, S/\Pi^2 S, G)$ is a commutative ring. Since $\tilde{f}$ is cohomologous to $\tilde{g}$ it follows that $\Delta(\tilde{f}, S/\Pi^2 S, G)$ is isomorphic to $\Delta(\tilde{g}, S/\Pi^2 S, G)$ and this completes the proof of the proposition.

PROPOSITION A. 5. *Let S denote a wildly ramified inertial extension of R such that the Galois group G of the quotient field extension is an elementary Abelian p-group, and let $G = Q_1 \times \cdots \times Q_t$ be a decomposition of G into a direct product of cyclic p-groups. Let f be an element of $Z^2(G, U(S))$ with the property that the restriction f_i of f to $Q_i \times Q_i$ is normalized in the sense of cyclic groups for each i. Then there exists an element a_i in $U(R)$ such that $f(\sigma_i, \sigma_i^{-1}) = a_i \mod \Pi^p S$ for each i where σ_i denotes a generator of E_i.*

Proof. Let S_i denote the fixed ring of Q_i and Π_i a prime element of S_i. Recall that $S_i = R[\Pi_i]$ according to Cor. 3-3-2 of [9] where the brackets denote ring adjunction. Therefore the element $f(\sigma_i, \sigma_i^{-1})$ of S_i may be written in the form $f(\sigma_i, \sigma_i^{-1}) = b_0 + b_1 \Pi_i + \cdots + b_{m-1} \Pi_i^{m-1}$ with coefficients in R, where m denotes the order of G/Q_i. Since $\Pi_i \equiv 0 \mod \Pi^p S$ it suffices to choose $a_i = b_0$ to prove the proposition.

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