The number n is multiply perfect if and only if $\sigma_1(n) \equiv 0 \pmod{n}$. By (1) this is equivalent to

$$(2) \qquad T_1(n) \equiv S_1(n) - \varphi_1(n) + 1 \pmod{n}.$$

The right hand side of (2) is congruent to

$$-\sum_{d|n, d>1} \mu(d) d S_1(n/d) + 1 \equiv -\sum_{d|n, d>1} \mu(d) n \tfrac{1}{2}(1+n/d) + 1 \pmod{n}.$$

If n is odd, each $1 + n/d$ is even and $n | n\frac{1}{2}(1+n/d)$. Thus an odd n is multiply perfect if and only if $T_1(n) \equiv 1 \pmod{n}$.

Now let $n = \prod_{p|n} p^{\alpha}$ be even. Correcting the statement of our problem we have to assume $n \neq 2$. We wish to show that n is multiply perfect if and only if $T_1(n) \equiv 1 + n/2 \pmod{n}$. Thus we have to show $\sum_{d|n, d>1} \mu(d) n\frac{1}{2}(1+n/d) \equiv n/2 \pmod{n}$ or $\sum_{d|n, d>1} \mu(d)(1+n/d) + 1 \equiv 0 \pmod{2}$. This is equivalent to

$$(4) \qquad 2 | \Sigma = \sum_{d|n} \mu(d)(1+n/d).$$

But $\Sigma = \sum_{d|n} \mu(d)(n/d) + \sum_{d|n} \mu(d) = \sum_{d|n} \mu(d)(n/d)$

$$= \varphi(n) = \prod_{p|n} (p^{\alpha} - p^{\alpha - 1}).$$

Thus Σ is even unless $n = 2$. This proves (4).

P 3. Let F be a finite field of characteristic p. Let V_n be an n-dimensional vector space over F. In V_n a symmetric bilinear form (a,b) is given. Let $n \geq 2$ if $p = 2$ and $n \geq 3$ if p is odd. Show that there is a vector $a \neq 0$ in V_n such that $(a,a) = 0$.

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Solution by the proposer. Let $F = \{\xi, \eta, \ldots\}$ be a finite field of characteristic p. Let G denote the multiplicative group of all the squares $\neq 0$. If $p = 2$, $\xi^2 = \eta^2$ if and only if $\xi = \eta$. Thus the mapping of the elements $\neq 0$ of F onto G is one-one and G is the multiplicative group of F. If $p > 2$, this mapping is two-one and G is a subgroup of index two in the multiplicative group of F. Let $\bar{G}$ denote the complement of G in this group.

If $1 + G = G$, $1 \in G$ would successively imply $2, 3, \ldots, p-1 \in G$ and finally $p = 0 \in G$. Thus

$$(1) \qquad 1 + G \neq G.$$