ON THE KATĚTOV AND STONE-ČECH EXTENSIONS

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Introduction. In general topology, one knows several standard extension spaces defined for one class of spaces or another and it is a natural question concerning any two such extensions which are defined for the same space whether they can ever be equal to each other. In the following, this problem will be considered for the Stone-Čech compactification βE of a completely regular non-compact Hausdorff space E [4] and Katětov's maximal Hausdorff extension κE of E [5]. It will be shown that $\beta E \neq \kappa E$ always holds or, what amounts to the same, that κE can never be compact. As an application of this it will be proved that any completely regular Hausdorff space is dense in some non-compact space in which the Stone-Weierstrass approximation theorem holds.

All topological concepts will be used in the sense of N. Bourbaki [3].

1. On the Katětov extension. A filter $\mathfrak{F}$ on a space E which has a basis consisting of open sets will be called an open filter. By Zorn's lemma one readily sees that any open filter $\mathfrak{F}$ is contained in some maximal open filter $\mathcal{M}$. Such filters $\mathcal{M}$ have, in virtue of their maximality, the following property. If $X \subseteq E$ is open and $X \notin \mathcal{M}$, then there exists a $Y \in \mathcal{M}$ such that $X \cap Y = \emptyset$, the void set. This implies that any maximal open filter contains all everywhere dense open sets of the space E.

The extension κE of a Hausdorff space E which is to be studied here can be defined in terms of the set Σ of all non-convergent maximal open filters on E as follows. With any $\mathcal{M} \in \Sigma$ one associates a point $x_{\mathcal{M}} \notin E$ such that the $x_{\mathcal{M}}$ are distinct for distinct $\mathcal{M}$. Then, the set $E \cup \{ x_{\mathcal{M}} \mid \mathcal{M} \in \Sigma \}$ is given a topology by assigning to each $x \in E$ the same neighbourhoods as in E and to each $x_{\mathcal{M}}$ the neighbourhoods $\{x_{\mathcal{M}}\} \cup M$, $M \in \mathcal{M}$. E is then obviously a dense subspace of this new space κE.

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PROPOSITION 1. The space $\varkappa E$ can never be compact.

Proof. We assume that $\varkappa E$ is compact and deduce a contradiction. This will be done by constructing, for any $\mathcal{M} \in \Sigma$, a well-ordered descending sequence of sets in $\mathcal{M}$ which forms a basis for $\mathcal{M}$ and by then showing that the existence of such a basis is incompatible with the maximality of $\mathcal{M}$.

First, the assumed compactness of $\varkappa E$ has the following two consequences which will be needed later. (1) For any open $X \in \mathcal{M}$ there exists an open $Y \in \mathcal{M}$ such that $\overline{Y} \subset X$, where $\overline{Y}$ denotes the closure of Y in E. This follows from the regularity of compact spaces and the fact that E is a subspace of $\varkappa E$. (2) E is completely regular and $\beta E = \varkappa E$, since subspaces of compact spaces are always completely regular and since for any completely regular E βE is the continuous image of $\varkappa E$ by a mapping which induces the identity mapping on E [5].

Next, consider the neighbourhood $\{x_{\mathcal{M}}\} \cup E$ of $x_{\mathcal{M}}$ in $\varkappa E$. By the regularity of $\varkappa E$, there exists an open $X_o \in \mathcal{M}$ such that the closure of $\{x_{\mathcal{M}}\} \cup X_o$ in $\varkappa E$ is contained in $\{x_{\mathcal{M}}\} \cup E$. This closure, however, is $\overline{X}_o \cup \{x_{\mathcal{M}'} \mid X_o \in \mathcal{M}'\}$ and it follows from $\overline{X}_o \cup \{x_{\mathcal{M}'} \mid X_o \in \mathcal{M}'\} \subseteq E \cup \{x_{\mathcal{M}}\}$ that $x_{\mathcal{M}}$ is the only $x_{\mathcal{M}'}$ such that $X_o \in \mathcal{M}'$. This implies that for any $X \subseteq X_o$ the closure of $X \cup \{x_{\mathcal{M}}\}$ in $\varkappa E$ is just $\overline{X} \cup \{x_{\mathcal{M}}\}$.

Now, let $\mathcal{M}_o$ be the set of all open $X \subseteq X_o$ in $\mathcal{M}$ and denote by $X < Y$ the transitive relation $\overline{X} \subset Y$ on $\mathcal{M}_o$. From a well-known maximum principle one can then infer that $\mathcal{M}_o$ contains a maximal descending well-ordered ($<$)-chain $\mathcal{W}$. The intersection $W = \bigcap X \; (X \in \mathcal{W})$ is closed since $\bigcap \overline{X} = \bigcap X \; (X \in \mathcal{W})$ in virtue of the meaning of the relation $X < Y$. Suppose first that $W = \emptyset$. One then has $\{x_{\mathcal{M}}\} = \bigcap \{x_{\mathcal{M}}\} \cup \overline{X} \; (X \in \mathcal{W})$ and since the $\{x_{\mathcal{M}}\} \cup \overline{X}$ are closed neighbourhoods of $x_{\mathcal{M}}$ in $\varkappa E$ it follows from a property of compact spaces [1] that they form a fundamental system of neighbourhoods of $x_{\mathcal{M}}$ in $\varkappa E$. This means that $\mathcal{W}$ is a basis of $\mathcal{M}$. Next, assume $W \neq \emptyset$. Then, it follows that $\mathbb{I}W \cup \complement W$, where $\mathbb{I}W$ denotes the interior and $\complement W$ the complement of W in E, belongs to $\mathcal{M}$ because it is an everywhere dense open set in E. $\mathbb{I}W \cup \complement W \in \mathcal{M}$ implies that either $\mathbb{I}W$ or $\complement W$ is in $\mathcal{M}$, where the former is, of course, only possible if $\mathbb{I}W \neq \emptyset$. Now, if $\mathbb{I}W \in \mathcal{M}$ one can take an open $Y \in \mathcal{M}$ such that $Y < \mathbb{I}W$; but then, $\mathcal{W} \cup \{Y\}$ would be a prolongation of the well-ordered descending $(<)$-chain $\mathcal{W}$ in $\mathcal{M}_o$ and this contradicts the maximality of $\mathcal{W}$. Thus, it is seen that $\complement W \in \mathcal{M}$ and then the collection of sets $X \cap \complement W$, $X \in \mathcal{W}$, belonging to $\mathcal{M}$ has void intersection; it follows that for any $X_1 \in \mathcal{M}$ with $\overline{X}_1 \subseteq \complement W$ one has $\bigcap \overline{X \cap X_1} \subseteq \bigcap (\overline{X} \cap \complement W) = (\bigcap X) \cap \complement W = \emptyset$ $(X \in \mathcal{M})$ and as before, this implies that the sets $X \cap X_1$ form a basis of $\mathcal{M}$.

In all, it is now established that $\mathcal{M}$ possesses a basis $\mathcal{B}$ consisting of open sets which is at least a descending well-ordered chain with respect to set inclusion. This $\mathcal{B}$ will contain cofinal subchains which are even descending well-ordered ($<$)-chains since for any $X \in \mathcal{B}$ there exists a $Y \in \mathcal{B}$ such that $Y < X$, in virtue of the property (1) of $\mathcal{M}$ mentioned above and by the fact that $\mathcal{M}$ has $\mathcal{B}$ as a basis. Let, then, $\mathcal{C} \subseteq \mathcal{B}$ be such a ($<$)-chain and assume that $\mathcal{C}$ contains no subchain of smaller ordinal type which is a basis of $\mathcal{M}$, i.e., which is cofinal with $\mathcal{B}$. Let, further, C_α, $\alpha < \eta$, be an indexing of $\mathcal{C}$ by all ordinals $\alpha < \eta$ such that $C_\alpha < C_\beta$ if and only if $\alpha > \beta$. The ordinal η must be the supremum of the limit numbers $\lambda < \eta$: otherwise, there would be no limit number $\lambda < \eta$, i.e., $\eta = \omega$, or one would have a largest limit number $\lambda_0 < \eta$ in which case the chain $C_{\lambda_0} > C_{\lambda_0 + 1} > \ldots$ would be cofinal with $\mathcal{C}$, i.e., again $\eta = \omega$ by the choice of $\mathcal{C}$; the equation $\eta = \omega$, however, would mean that $x_{\mathcal{M}}$ is a G_δ in $\kappa E = \beta E$, and this is excluded according to Čech [4].

To arrive at the desired contradiction, put $D_\alpha = C_\alpha - C_{\alpha+1}$ and $L = \bigcup D_\lambda$, λ the limit numbers less than η. The open set L obviously meets every set of $\mathcal{M}$; however, it does not belong to $\mathcal{M}$, for if ν is the first limit number $\lambda \geqslant \alpha$ $(\alpha < \eta)$ and $X = C_{\nu+1} - C_{\nu+2}$ then $C_\alpha \subseteq L$ would imply $X = C_\alpha \cap X \subseteq L \cap X = \emptyset$ whereas $X \neq \emptyset$. Thus, the existence of L contradicts the maximality of $\mathcal{M}$, and with this the proof is completed.

The space κE is absolutely closed [5] and would therefore be compact if it were regular [1]. Thus follows the

COROLLARY. The space κE can never be regular.

2. <u>On the Stone-Weierstrass approximation theorem</u>. This theorem is said to hold in a space E if the ring C(E) of all bounded continuous real functions on E separates the points of E (i.e., if $x \neq y$, then $fx \neq fy$ for some $f \in C(E)$) and if any subring of C(E) which contains the constant functions and separates the points of E is dense in C(E) with respect to the topology of uniform convergence on E. This is always the case when the space E is compact, but, as proved in [2], there also exist non-compact spaces with this property. It will now be shown by means of the result in section 1 that there exist in fact a great number of such non-compact spaces.

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PROPOSITION 2. Any completely regular Hausdorff space is dense in some non-compact space in which the Stone-Weierstrass approximation theorem holds.

Proof. Let E be the space and $\varphi: \kappa E \to \beta E$ the continuous mapping onto βE which induces the identity mapping on E. Now, for any $u \in \beta E - E$ let one point $u^* \in \varphi^{-1} u$ be chosen and call the subspace $E \cup \{u^* \mid u \in \beta E - E\}$ of κE E^*. E is then dense in E^*. Further, E^* is not compact. If it were it would be closed in κE; but since $E \subseteq E^*$ and E is dense in κE, this would give $E^* = \kappa E$, whereas κE cannot be compact.

It remains to be shown that the Weierstrass-Stone theorem holds in E^*. First, $C(E^*)$ separates the points of E^*, for $C(\beta E)$ separates the points of βE and φ induces a one-to-one continuous mapping of E^* onto βE. Next, take any bounded continuous real function f on E^* and let g be its restriction to E. This g has a unique continuous extension $\bar{g}$ to βE and since the function $\bar{g}\varphi$ on E^* coincides with f on E, a dense subspace, one has the decomposition $f = \bar{g}\varphi$. Now, if a subring $R \subseteq C(E^*)$ contains the constant functions and separates the points of E^*, then by the correspondence $f \to \bar{g}$, R determines a similar subring S of $C(\beta E)$, for if f separates u_1^* and u_2^* in E^* then its $\bar{g}$ separates $y_1 = \varphi y_1^*$ and $y_2 = \varphi y_2^*$ because of $f = \bar{g}\varphi$, and by definition of E^*, any $y \in \beta E$ is the image of a $Y^* \in E^*$ under φ. Since βE is compact this shows that S is dense in $C(\beta E)$ in the sense of uniform convergence on βE. But then, again, because of the decomposition $f = \bar{g}\varphi$, R is also dense in $C(E^*)$ with respect to uniform convergence on E^*.

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