ON THE SPECTRUM OF n-TUPLES OF p-HYPONORMAL OPERATORS

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1. Introduction. Let $B(H)$ denote the algebra of operators (i.e., bounded linear transformations) on the Hilbert space H. $A \in B(H)$ is said to be p-hyponormal $(0<p\leq 1)$, if $(AA^*)^p \leq (A^*A)^p$. (Of course, a 1-hyponormal operator is hyponormal.) The p-hyponormal property is monotonic decreasing in p and a p-hyponormal operator is q-hyponormal operator for all $0<q\leq p$. Let A have the polar decomposition $A = U|A|$, where U is a partial isometry and $|A|$ denotes the (unique) positive square root of A^*A. If A has equal defect and nullity, then the partial isometry U may be taken to be unitary. Let $\mathscr{H}U(p)$ denote the class of p-hyponormal operators for which U in $A = U|A|$ is unitary. $\mathscr{H}U(1/2)$ operators were introduced by Xia and $\mathscr{H}U(p)$ operators for a general $0<p<1$ were first considered by Aluthge (see [**1, 14**]); $\mathscr{H}U(p)$ operators have since been considered by a number of authors (see [**3, 4, 5, 9, 10**] and the references cited in these papers). Generally speaking, $\mathscr{H}U(p)$ operators have spectral properties similar to those of hyponormal operators. Indeed, let $A \in \mathscr{H}U(p)$, $(0<p<1/2)$, have the polar decomposition $A = U|A|$, and define the $\mathscr{H}U(p+1/2)$ operator $\hat{A}$ by $\hat{A} = |A|^{1/2}U|A|^{1/2}$. Let $\hat{A} = V|\hat{A}|$ with V unitary and $\tilde{A}$ be the hyponormal operator defined by $\hat{A} = |\hat{A}|^{1/2}V|\hat{A}|^{1/2}$. Then we have the following result.

LEMMA 0. $\sigma_s(A) = \sigma_s(\tilde{A})$, *where* σ_s *denotes either of the following: point spectrum, approximate point spectrum, eigenvalues of finite multiplicity, spectrum, Weyl spectrum, and essential spectrum.*

Recall that an n-tuple $\mathscr{A} = (A_1, A_2, \ldots, A_n)$ of operators is said to be doubly commuting if $A_iA_j - A_jA_1 = 0$ and $A_i^*A_j - A_jA_i^* = 0$, for all $1 \leq i \neq j \leq n$. Doubly commuting n-tuples $\mathscr{A}$ of operators in $\mathscr{H}U(p)$ have been considered by Muneo Cho in [**3**], where it is shown that a weak Putnam theorem holds for $\mathscr{A}$ and that $\mathscr{A}$ is jointly normaloid. In this note we study the relationship between the spectral properties of $\mathscr{A}$ and $\tilde{\mathscr{A}} = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$, and prove that $\sigma_s(\mathscr{A}) = \sigma_s(\tilde{\mathscr{A}})$, where σ_s is either the joint point spectrum or the joint approximate point spectrum or the joint (Taylor) spectrum. This then leads us to:

(a) $\|\mathscr{A}\| = \|\tilde{\mathscr{A}}\|$;
(b) if $\sigma(\mathscr{A}) \in \mathscr{R}^n$, then A_i is self-adjoint, for all $1 \leq i \leq n$.

We show that the (Cho-Takaguchi) joint Weyl spectrum of $\mathscr{A}$ is contained in the (Taylor) spectrum $\sigma(\mathscr{A})$ of $\mathscr{A}$ minus the set of isolated points of $\sigma(\mathscr{A})$ which are joint eigenvalues of finite multiplicity, and that $\mathscr{A}$ and $\tilde{\mathscr{A}}$ have the same (Harte) essential spectrum. We conclude this note with a result (in the spirit of Dash [**8**, Corollary 4.6]) on the joint eigenvalues of $\mathscr{A}$ in the Calkin algebra.

We assume henceforth, without loss of generality, that $0<p<1/2$. Most of the notation that we use in this note is standard (and usually explained at the first instance of

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occurence). The following theorem, the n-tuple version of the Berberian extension theorem, will play an important role in the sequel.

THEOREM B. *If* $\mathscr{A}=(A_1, A_2, \ldots, A_n)$ *is an* n*-tuple of commuting operators on* H, *then there exists a Hilbert space* $H^0 \supset H$ *and an isometric *-isomorphism* $A_i \rightarrow A_i^0$, $(1 \leq i \leq n)$, *preserving order such that* $\sigma_\pi(A_i)=\sigma_\pi(A_i^0)=\sigma_p(A_i^0)$ *and* $\sigma_\pi(\mathscr{A})=\sigma_\pi(A_1, A_2, \ldots, A_n)=\sigma_\pi(A_1^0, A_2^0, \ldots, A_n^0)=\sigma_p(A_1^0, A_2^0, \ldots, A_n^0)=\sigma_p(\mathscr{A}^0)$. (*Here* $\sigma_p(\mathscr{A})$ *and* $\sigma_\pi(\mathscr{A})$ *denote, respectively, the joint spectrum and the joint approximate point spectrum (defined below) of* $\mathscr{A}$.)

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2. Results. Throughout the following $\mathscr{A}=(A_1, A_2, \ldots, A_2)$ will denote a doubly commuting (i.e., $A_iA_j-A_jA_i=0$ and $A_iA_j^*-A_j^*A_i=0$, for all $1 \leq i \neq j \leq n$)$n-$tuple of $\mathscr{H}U(p)$ operators A_i $(1 \leq i \leq n)$. Given $A_i=U_i\,|A_i|$, define $\hat{A}_i$ by $\hat{A}_i=|A_i|^{1/2}\,U_i\,|A_i|^{1/2}$; also, letting $\hat{A}_i$ have the polar decomposition $\hat{A}_i=V_i\,|\hat{A}_i|$, define $\tilde{A}_i$ by

$$\tilde{A}_i=|\hat{A}_i|^{1/2}\,V_i\,|\hat{A}_i|^{1/2} \quad (1 \leq i \leq n).$$

The n-tuples $\hat{\mathscr{A}}$ and $\tilde{\mathscr{A}}$ are then defined by $\hat{\mathscr{A}}=(\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n)$ and $\tilde{\mathscr{A}}_1=(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$.

LEMMA 1. $\mathscr{A}$ *is doubly commuting* $\Rightarrow \hat{\mathscr{A}}$ *is doubly commuting* $\Rightarrow \tilde{\mathscr{A}}$ *is doubly commuting. Also,* $\mathscr{A}$ *is doubly commuting* $\Rightarrow [A_i, |\hat{A}_j|]=0=[\tilde{A}_i, |\hat{A}_j|]=0$, *for* $1 \leq i \neq j \leq n$, *where* $[A, B]$ *denotes the commutator* $AB-BA$ *of* A *and* B.

Proof. Given $A_i=U_i\,|A_i|$ and $\hat{A}_i=V_i\,|\hat{A}_i|$, the doubly commuting hypothesis on $\mathscr{A}$ implies that

$$[U_i, U_j]=[|A_i|, |A_j|]=[|A_i|, U_j]=0,$$

for all $1 \leq i \neq j \leq n$. (See [**11**, Theorems 2 and 4].) Consequently, $\hat{\mathscr{A}}$ is doubly commuting and so

$$[V_i, V_j]=[|\hat{A}_i|, |\hat{A}_j|]=[|\hat{A}_i|, V_j]=0,$$

for all $1 \leq i \neq j \leq n$. This implies that $\tilde{\mathscr{A}}$ is doubly commuting. The argument above also implies that $[A_i, \hat{A}_j]=[A_i, \hat{A}_j^*]=[\hat{A}_i, \tilde{A}_j]=[\hat{A}_i, \tilde{A}_j^*]=0$, for all $1 \leq i \neq j \leq n$. Hence, also, $[A_i, |\hat{A}_j|]=[\tilde{A}_i, |\hat{A}_j|]=0$, for all $1 \leq i \neq j \leq n$.

In the following we shall denote the Taylor joint spectrum of $\mathscr{A}$ by $\sigma(\mathscr{A})$. (See [**13**] for the definition of Taylor spectrum of a commuting n-tuple of operators.) We say that $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $(\lambda_i \in \mathbb{C}$ for all $1 \leq i \leq n)$, is in the joint approximate point spectrum $\sigma_\pi(\mathscr{A})$ of $\mathscr{A}$ if there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\|(A_i-\lambda_i)x_k\| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,$$

for all $1 \leq i \leq n$; $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{C}$ for all $1 \leq i \leq n$, is in the joint point spectrum $\sigma_p(\mathscr{A})$ of $\mathscr{A}$ if there exists a non-trivial vector $x \in H$ such that

$$(A_i-\lambda_i)x=0, \quad \text{for all} \quad 1 \leq i \leq n.$$

We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is in the *normal point spectrum* $\sigma_{np}(\mathscr{A})$ of $\mathscr{A}$ if there exists a non-trivial vector $x \in H$ such that $(A_i - \lambda_i)x = 0 \Leftrightarrow (A_i - \lambda_i)^* x = 0$, for all $1 \le i \le n$.

LEMMA 2. $\sigma_p(\mathscr{A}) = \sigma_{np}(\mathscr{A}) = \sigma_{np}(\tilde{\mathscr{A}}) = \sigma_p(\tilde{\mathscr{A}})$.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma_p(\mathscr{A})$ and let $x \in H$ be such that $x \ne 0$ and $(A_i - \lambda_i)x = 0$, for all $1 \le i \le n$. It is easily seen that $\tilde{A}_i |\hat{A}_i|^{1/2} |A_i|^{1/2} = |\hat{A}_i|^{1/2} |A_i|^{1/2} A_i$; hence

$$\tilde{A}_i |\hat{A}_i|^{1/2} |A_i|^{1/2} x = \lambda_i |\hat{A}_i|^{1/2} |A_i|^{1/2} x,$$

for all $1 \le i \le n$. Let

$$y = \prod_{t=1}^{n'} |\hat{A}_t|^{1/2} |A_t|^{1/2} x,$$

where "$'$" on the product "$\prod_{t=1}^{n}$" denotes that only those $|A_t|$ s, (and so also $|\hat{A}_t|$ s), appear in the product for which λ_t in $A_t x = \lambda_t x$ does not equal 0. Then y is non-trivial, and

$$\tilde{A}_i y = \lambda_i y, \quad \text{for all} \quad i = 1, 2, \ldots, n \quad \text{for which} \quad \lambda_i \ne 0.$$

If $\lambda_i = 0$, i.e. $A_i x = 0$, then $|A_i|^{1/2} x = 0$. This implies that $\hat{A}_i x = 0$. Since this in turn implies that $|\hat{A}_i|^{1/2} x = 0$, we conclude that $\tilde{A}_i x = 0$. Since $[A_i, \tilde{A}_j] = 0$ for all $1 \le i \ne j \le n$, we have that $\tilde{A}_i y = 0$. Consequently, $\lambda \in \sigma_p(\tilde{\mathscr{A}})$ and $\sigma_p(\mathscr{A}) \subseteq \sigma_p(\tilde{\mathscr{A}})$.

If, on the other hand, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma_p(\tilde{\mathscr{A}})$, then there is a non-trivial $x \in H$ such that $(\tilde{A}_i - \lambda_i)x = 0$ and $(\tilde{A}_i^* - \bar{\lambda}_i)x = 0$ for all $1 \le i \le n$. Since $A_i^* |A_i|^{1/2} |\hat{A}_i|^{1/2} = |A_i|^{1/2} |\hat{A}_i|^{1/2} \tilde{A}_i^*$,

$$A_i^* |A_i|^{1/2} |\hat{A}_i|^{1/2} x = \bar{\lambda}_i |A_i|^{1/2} |\hat{A}_i|^{1/2} x$$

for all $1 \le i \le n$. Defining $(0 \ne) y$ by

$$y = \prod_{t=1}^{n'} |A_t|^{1/2} |\hat{A}_t|^{1/2} | x,$$

where $\prod_{t=1}^{n}{}'$ has meaning similar to that above, we have $A_i^* y = \bar{\lambda}_i y$, for all $i = 1, 2, \ldots, n$ such that $\lambda_i \ne 0$. Since $\lambda_i \in \sigma_p(\tilde{A}_i)$ implies $\lambda_i \in \sigma_p(A_i) = \sigma_{np}(A_i)$ (see Lemma 0), $A_i y = \lambda_i y$ for all $i = 1, 2, \ldots, n$ such that $\lambda_i \ne 0$. Now if $\tilde{A}_i x = 0$, then $0 \in \sigma_p(\hat{A}_i) = \sigma_p(A_i)$ and

$$\begin{aligned} \tilde{A}_i^* x = 0 &\Rightarrow |\hat{A}_i|^{1/2} V_i^* |\hat{A}_i|^{1/2} x = 0 \\ &\Rightarrow \hat{A}_i^* |\hat{A}_i|^{1/2} x = 0 \Leftrightarrow \hat{A}_i |\hat{A}_i|^{1/2} x = 0 \\ &\Rightarrow |\hat{A}_i|^{1/2} x = 0 \Rightarrow \hat{A} x = 0 \Leftrightarrow \hat{A}^* x = 0 \\ &\Rightarrow A_i^* |A_i|^{1/2} x = 0 \Leftrightarrow A_i |A_i|^{1/2} x = 0 \\ &\Rightarrow |A_i|^{1/2} x = 0 \Rightarrow A_i x = 0 \Leftrightarrow A_i^* x = 0. \end{aligned}$$

(Line 2 follows since $0 \in \sigma_p(A_i)$. Line 4 follows because $0 \in \sigma_p(A_i) = \sigma_{np}(A_i)$.) Consequently, $A_i y = 0$ for such an i. Hence $\sigma_p(\tilde{\mathscr{A}}) \subseteq \sigma_p(\mathscr{A})$. Since $\sigma_p(A_i) = \sigma_{np}(A_i)$ and $\sigma_p(\tilde{A}_i) = \sigma_{np}(\tilde{A}_i)$, for all $1 \le i \le n$, this completes the proof.

LEMMA 3. $\sigma_\pi(\mathscr{A}) = \sigma_{n\pi}(\mathscr{A}) = \sigma_{n\pi}(\tilde{\mathscr{A}}) = \sigma_\pi(\tilde{\mathscr{A}})$.

Proof. Letting $A^0 = (A_1^0, A_2^0, \ldots, A_n^0)$ denote the Berberian extension of $\mathscr{A}$ (see Theorem B), it follows from Lemma 2 that

$$\sigma_\pi(\mathscr{A}) = \sigma_0(\mathscr{A}^0) = \sigma_{np}(\mathscr{A}^0) = \sigma_{np}(\tilde{\mathscr{A}}^0) = \sigma_p(\tilde{\mathscr{A}}^0) = \sigma_\pi(\tilde{\mathscr{A}}).$$

We are now in a position to prove the equality of the (Taylor) spectra of $\mathscr{A}$ and $\tilde{\mathscr{A}}$.

THEOREM 1. $\sigma(\mathscr{A}) = \sigma(\tilde{\mathscr{A}})$.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(\mathscr{A})$. Then there exists a partition

$$\{i_1, \ldots, i_m\} U \{j_1, \ldots, j_s\} \quad \text{of} \quad \{1, 2, \ldots, n\}$$

and a sequence $\{x_k\}$ of unit vectors in H such that

$$(A_{i_r} - \lambda_{i_r})x_k \to 0 \quad \text{and} \quad (A_{j_t}^* - \bar{\lambda}_{j_t})x_k \to 0 \quad \text{as} \quad k \to \infty,$$

for all $1 \le r \le m$ and $1 \le t \le s$. (See [**7**, Corollary 3.3].) Let $\mathscr{A}^0$ denote the Berberian extension $(A_{i_1}^0, \ldots, A_{im}^0, A_{j_1}^0, \ldots, A_{j_s}^0)$ of $\mathscr{A}$, and let $\mathscr{B} = (A_{i_1}^0, \ldots, A_{i_m}^0, A_{j_1}^{0*}, \ldots, A_{j_s}^{0*})$. Then

$$(\lambda_{i_1}, \ldots, \lambda_{i_m}, \bar{\lambda}_{j_1}, \ldots, \bar{\lambda}_{j}) \in \sigma_p(\mathscr{B}).$$

Since $\sigma_p(A_{i_r}^0) = \sigma_p(\tilde{A}_{i_r}^0) = \sigma_{np}(\tilde{A}_{i_r}^0)$, for all $1 \le r \le m$, and since

$$\tilde{A}_{j_t}^* |\hat{A}_{j_t}|^{1/2} V_{j_t}^* |A_{j_t}|^{1/2} U_{j_t}^* = |\hat{A}_{j_t}|^{1/2} V_{j_t}^* |A_{j_t}|^{1/2} U_{j_t}^* A_{j_t}^*,$$

it follows (from an argument similar to that used in the proof of Lemma 2) that $\sigma_p(\mathscr{B}) \subseteq \sigma_p(\tilde{\mathscr{B}})$ and

$$\bar{\lambda} \in \sigma_p(\tilde{\mathscr{A}}^{0*}) = \sigma_\pi(\tilde{\mathscr{A}}^*) \subseteq \sigma(\tilde{\mathscr{A}}^*).$$

Hence $\lambda \in \sigma(\tilde{\mathscr{A}})$, and $\sigma(\mathscr{A}) \subseteq \sigma(\tilde{\mathscr{A}})$.

Conversely, if $\lambda \in \sigma(\tilde{\mathscr{A}})$, then (from an argument similar to that above) $\bar{\lambda} \in \sigma_p(\tilde{\mathscr{A}}^{0*})$. This implies that $\bar{\lambda} \in \sigma_\pi(\mathscr{A}^*) \subseteq \sigma(\mathscr{A}^*)$, $\lambda \in \sigma(\mathscr{A})$ and $\sigma(\tilde{\mathscr{A}}) \subseteq \sigma(\mathscr{A})$. Hence $\sigma(\mathscr{A}) = \sigma(\tilde{\mathscr{A}})$, and the proof is complete.

The joint spectral radius $r(\mathscr{T})$ and the joint operator norm $\|\mathscr{T}\|$ of an n-tuple $\mathscr{T} = (T_1, T_2, \ldots, T_n)$ are defined by

$$r(\mathscr{T}) = \sup\left\{|\lambda| = \left(\sum_{i=1}^n |\lambda_i|^2\right)^{1/2} : \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(\mathscr{T})\right\}$$

and

$$\|\mathscr{T}\| = \sup\left\{\left(\sum_{i=1}^n \|T_i x\|^2\right)^{1/2} : x \in H, \ \|x\| = 1\right\}.$$

See [**6**]. The operators $\mathscr{A}$ and $\tilde{\mathscr{A}}$ being jointly normaloid (see [**3**, Theorem 9] and [**6**, Theorem 3.4]), $r(\mathscr{A}) = \|\mathscr{A}\|$ and $r(\tilde{\mathscr{A}}) = \|\tilde{\mathscr{A}}\|$. Theorem 1 thus implies the following result.

COROLLARY 1. $\|\mathscr{A}\| = \|\hat{\mathscr{A}}\| = \|\tilde{\mathscr{A}}\|$.

That $\|A\| = \|\tilde{A}\|$ for a single operator $A \in \mathscr{H}U(p)$ has been proved by M. Fujii *et al.* in [**10**].

Given a semi-normal (i.e., hyponormal or co-hyponormal) operator $T = X + iY$, a well known result of Putnam [**12**] states that if a real number $r \in \sigma(X)$ (or $r + is \in \sigma(T)$, for some real numbers r and s), then there exists a real number s such that $r + is \in \sigma(T)$ (resp., $r \in \sigma(X)$ and $s \in \sigma(Y)$). This result extends to doubly commuting n-tuples of hyponormal operators [**4**]. Does a similar result hold (for $A \in \mathscr{H}U(p)$ and) doubly commuting n-tuples in $\mathscr{H}U(p)$? The technique of this paper (seemingly) does not lend to a proof of this. We do however have the following analogue for $\mathscr{H}U(p)$ operators of a result on n-tuples of doubly commuting hyponormal operators with spectrum in $\mathbb{R}^n$. (See [**4**, Corollary].)

COROLLARY 2. *If* $\sigma(\mathscr{A}) \subseteq \mathbb{R}^n$, *then* A_i *is self-adjoint, for all* $1 \le i \le n$.

Proof. Since $\sigma(\tilde{\mathscr{A}}) = \sigma(\mathscr{A}) \subseteq \mathbb{R}^n$, $\tilde{A}_i$ is self-adjoint, for all $1 \le i \le n$, by [**4**]. Recall that A_i is normal if and only if $\tilde{A}_i$ is normal [**9**, Corollary 2]; hence A_i is self-adjoint, for all $1 \le i \le n$.

Following Chō [**2**], we define the joint Weyl spectrum $\sigma_\omega(\mathscr{T})$ of a commuting n-tuple $\mathscr{T}$ by

$$\sigma_\omega(\mathscr{T}) = \bigcap \{\sigma(\mathscr{T} + \mathscr{K});\ \mathscr{K} \text{ is an } n\text{-tuple of compact operators and } (\mathscr{T} + \mathscr{K}) \text{ is a commuting } n\text{-tuple}\}.$$

Let $\sigma_{00}(\mathscr{T})$ denote the set of isolated points of $\sigma(\mathscr{T})$ which are joint eigen-values of finite multiplicity of $\mathscr{T}$. It is clear from Theorem 1 that, if λ is an isolated point of $\sigma(\mathscr{A})$, then λ is an isolated point of $\sigma(\tilde{\mathscr{A}})$. The operator $\tilde{\mathscr{A}}$ being a doubly commutitive n-tuple of hyponormal operators, an isolated point λ of $\sigma(\tilde{\mathscr{A}})$ is a point of $\sigma_p(\tilde{\mathscr{A}})$. Hence by Lemma 2 we have the following result.

COROLLARY 3. *If* λ *is an isolated point of* $\sigma(\mathscr{A})$, *then* $\lambda \in \sigma_p(\mathscr{A})$.

Recall that if A is p-hyponormal, then $\sigma_\omega(A) = \sigma(A) - \sigma_{00}(A)$ by [9] and if $\mathscr{T}$ is a doubly commuting n-tuple of hyponormal operators, then $\sigma_\omega(\mathscr{T}) \subseteq \sigma(\mathscr{T}) - \sigma_{00}(\mathscr{T})$ by [**2**].

THEOREM 2. $\sigma_\omega(\mathscr{A}) \subseteq \sigma(\mathscr{A}) - \sigma_{00}(\mathscr{A})$.

Proof. Suppose $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \sigma_{00}(\mathscr{A})$, and let $N = \ker\left\{\sum_{i=1}^{n} (A_i - \lambda_i)^*(A_i - \lambda_i)\right\}$. Since $\lambda \in \sigma_p(\mathscr{A})$ if and only if $0 \in \sigma_p\left(\sum_{i=1}^{n} (A_i - \lambda)^*(A_i - \lambda_i)\right)$, N is finite dimensional. By Lemma 2, $\sigma_p(\mathscr{A}) = \sigma_{np}(\mathscr{A})$; hence N reduces $\mathscr{A}$,$\mathscr{A}_0 = \mathscr{A} \mid N = (A_1 \mid N, A_2 \mid N, \dots, A_n \mid N)$ is normal and $\mathscr{A}_1 = \mathscr{A} \mid N^\perp = (A_1 \mid N^\perp, A_2 \mid N^\perp, \dots, A_n \mid N^\perp)$ is a doubly commuting n-tuple of $\mathscr{H}U(p)$ operators. Let P be the orthogonal projection of H onto N. P is then a compact operator which satisfies $[A_i, P] = [A_i^*, P] = 0$, for all $i = 1, 2, \dots, n$. The operator

$$\mathscr{A} + \mathscr{P} = \left(A_1 + \frac{1}{\sqrt{n}} P, A_2 + \frac{1}{\sqrt{n}} P, \dots, A_n + \frac{1}{\sqrt{n}} P\right)$$

is a doubly commuting n-tuple. Let

$$\mathcal{R} = (\mathcal{A} + \mathcal{P}) \mid N = \left(\left(A_1 + \frac{1}{\sqrt{n}}P\right) \Big| N, \left(A_2 + \frac{1}{\sqrt{n}}P\right) \Big| N, \ldots, \left(A_n + \frac{1}{\sqrt{n}}\right)P \mid N\right),$$

$$\mathcal{S} = (\mathcal{A} + \mathcal{P}) \mid N^{\perp}.$$

$\mathcal{R}$ and $\mathcal{S}$ are then doubly commuting n-tuples such that $\sigma(\mathcal{A} + \mathcal{P}) = \sigma(\mathcal{R}) U \sigma(\mathcal{S})$.

Suppose that $\lambda \in \sigma(\mathcal{A} + \mathcal{P})$. Then $\lambda \notin \sigma(\mathcal{R})$ and so λ must be an isolated point of $\sigma(\mathcal{S})$. There exists a partition $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_s\}$ of $\{1, 2, \ldots, n\}$ and a sequence $\{x_k\}$ of unit vectors in $N^{\perp}$ such that

$$\left(A_{i_r} - \lambda_{i_r} + \frac{1}{\sqrt{n}}P\right)x_k \to 0 \quad \text{and} \quad \left(A_{j_t}^* - \bar{\lambda}_{j_t} + \frac{1}{\sqrt{n}}P\right)x_k \to 0 \quad \text{as } k \to \infty.$$

But then $\lambda \in \sigma(\mathcal{A}_1)$ and hence (by Corollary 3) $\lambda \in \sigma_p(\mathcal{A}_1)$. Thus there exists an $x \in N^{\perp}$ such that $(A_i - \lambda_i)x = 0$, for all $i = 1, 2, \ldots, n$. Since this is a contradiction, we must have $\lambda \notin \sigma_\omega(\mathcal{A})$.

Remarks. (i) the Taylor–Weyl spectrum of $\mathcal{T}$, $\sigma_{T\omega}(\mathcal{T})$, is defined to be the set of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $(\mathcal{T} - \lambda)$ is not Taylor–Weyl (where $\mathcal{T} - \lambda$ is said to be *Taylor–Weyl* if $\mathcal{T} - \lambda$ is Fredholm and index $(\mathcal{T} - \lambda) = 0$). Theorem 2 implies that $\sigma(\mathcal{A}) \setminus \sigma_{T\omega}(\mathcal{A}) \supseteq \sigma_{00}(\mathcal{A})$. The inclusion $\sigma(\mathcal{A}) \setminus \sigma_{T\omega}(\mathcal{A}) \subseteq \sigma_{00}(\mathcal{A})$ does not hold (even for hyponormal $\mathcal{A}$).

(ii) Given a p-hyponormal operator A, $\sigma_\omega(A) = \sigma_\omega(\tilde{A})$ by [**9**]. Does $\sigma_\omega(\mathcal{A}) = \sigma_\omega(\tilde{\mathcal{A}})$?

The *Harte spectrum* $\sigma_H(\mathcal{T})$ of the commutative n-tuple $\mathcal{T}$ is defined to be $\sigma_H(\mathcal{T}) = \sigma^l(\mathcal{T}) \cup \sigma^r(\mathcal{T})$, where $\sigma^l(\mathcal{T})$ (respectively, $\sigma^r(\mathcal{T})$) is the set of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\{T_i - \lambda_i\}_{1 \le i \le n}$ generates a proper left (resp., right) ideal in $B(H)$. The (Harte) essential spectrum $\sigma_e(\mathcal{T})$ is defined by $\sigma_e(\mathcal{T}) = \sigma(a)$, where $a = (a_1, a_2, \ldots, a_n) = \pi(\mathcal{T})$ and π is the canonical homomoprhism of $B(H)$ onto the Calkin algebra $B(H)/K(H)$; $K(H)$ is the algebra of compact operators on H. For a single linear operator, the (Harte) essential spectrum coincides with the essential spectrum; the following extends the conclusion $\sigma_e(A) = \sigma_e(\tilde{A})$ of Lemma 0 to $\sigma_e(\mathcal{A})$.

Theorem 3. $\sigma_e(\mathcal{A}) = \sigma_e(\tilde{\mathcal{A}})$.

Proof. Suppose $\lambda \in \sigma_e(\tilde{\mathcal{A}})$. Then, $\tilde{\mathcal{A}}$ being a hyponormal n-tuple, there exists a sequence $\{x_k\}$ of unit vectors converging weakly to 0 in H such that

$$\|(\tilde{A}_i - \lambda_i)^* x_k\| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for all } 1 \le i \le n,$$

by [**8**, Theorem 2.6]. Let $\{y_k\}$ be the sequence defined by

$$y_k = \left(\prod_{i=1}^{n}{}' |A_i|^{1/2} |\tilde{A}_i|^{1/2} x_k \right) \Bigg/ \left\| \prod_{i=1}^{n}{}' |A_i|^{1/2} |\hat{A}_i|^{1/2} x_k \right\|,$$

where "$'$" on the product $\prod_{i=1}^{n}$ denotes that only those $|A_i|$ s and $|\hat{A}_i|$ s appear in the product for which $\lambda_i \neq 0$. (Notice that if $\||A_i|^{1/2}x_k\|$ or $\||A_i|^{1/2}|\hat{A}_i|^{1/2}x_k\| \to 0$ as $k \to \infty$, for some i with $1 \leq i \leq n$, then $\||\hat{A}_i|^{1/2}x_k\|$ and $\|\tilde{A}_i x_i\| \to 0$ as $k \to \infty$.) Since $(x_k, h) \to 0$ as $k \to \infty$ for all $h \in H$, $(y_k, h) \to 0$ as $k \to \infty$ and

$$\|(A_j - \lambda_j)^* y_k\| = \left\| \frac{\prod_{i=1}^{n}{}' |A_i|^{1/2} |\hat{A}_i|^{1/2}}{\left\| \prod_{i=1}^{n}{}' |A_i|^{1/2} |\hat{A}_i|^{1/2} x_k \right\|} (\tilde{A}_j - \lambda_j)^* x_k \right\| \to 0 \quad \text{as } k \to \infty,$$

for all $1 \leq j \leq n$. Thus $\lambda \in \sigma_e(\mathscr{A})$ and $\sigma_e(\tilde{\mathscr{A}}) \subseteq \sigma_e(\mathscr{A})$.

Consider now $\lambda \in \sigma_e(\mathscr{A}) = \sigma_e^l(\mathscr{A}) \cup \sigma_e^r(\mathscr{A})$. Suppose that $\lambda \in \sigma_e^l(\mathscr{A})$; then there exists a sequence $\{x_k\}$ of unit vectors converging weakly to 0 in H such that $\|(A_i - \lambda_i)x_i\| \to 0$ as $k \to \infty$, for all $1 \leq i \leq n$. Defining the sequence $\{y_k\}$ by

$$y_k = \frac{\left(\prod_{i=1}^{n}{}' |\hat{A}_i|^{1/2} |A_i|^{1/2} x_k \right)}{\left\| \prod_{i=1}^{n}{}' |\hat{A}_i|^{1/2} |A_i|^{1/2} x_k \right\|},$$

$\left(\text{where } \prod_{i=1}^{n}{}' \text{ has a meaning similar to that above}\right)$, an argument similar to that above shows that $\{y_k\}$ is a sequence of unit vectors converging weakly to 0 in H such that

$$\|(\tilde{A}_j - \lambda_j)yk\| \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for all } 1 \leq i \leq n.$$

Hence $\lambda \in \sigma_e^l(\tilde{\mathscr{A}})$. A similar argument shows that if $\lambda \in \sigma_e^r(\mathscr{A})$ then $\lambda \in \sigma_e^r(\tilde{\mathscr{A}})$. Thus $\sigma_e(\mathscr{A}) \subseteq \sigma_e(\tilde{\mathscr{A}})$, and the proof is complete.

Corollary 4. $\sigma_e(\mathscr{A}) = \sigma_e^r(\mathscr{A})$.

Proof. The argument of the proof of Theorem 3 implies that

$$\sigma_e^r(\mathscr{A}) \subseteq \sigma_e(\mathscr{A}) = \sigma_e(\tilde{\mathscr{A}}) = \sigma_e^r(\tilde{\mathscr{A}}) \subseteq \sigma_e^r(\mathscr{A}).$$

Corollary 5. $\sigma_H(\mathscr{A}) = \sigma_e(\mathscr{A}) \cup \sigma_p(\mathscr{A}^*)^*$.

Proof. Let $\sigma_\delta(\mathscr{A}) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$: there exists a sequence $\{x_k\}$ of unit vectors in H such that $\|(A_i - \lambda_i)^* x_k\| \to 0$ as $k \to \infty$, for all $i = 1, 2, \ldots, n\}$ denote the joint approximate defect spectrum of $\mathscr{A}$. Then

$$\sigma_H(\mathscr{A}) = \sigma^l(\mathscr{A}) \cup \sigma^r(\mathscr{A}) = \sigma_\pi(\mathscr{A}) \cup \sigma_\delta(\mathscr{A}).$$

By Lemma 3, $\sigma_\pi(\mathscr{A}) = \sigma_\pi(\tilde{\mathscr{A}})$; applying an argument similar to that used in the proof of Lemma 2 to A_i^{0*} it is seen that $\sigma_\delta(\mathscr{A}) = \sigma_\delta(\tilde{\mathscr{A}})$. We have

$$\sigma_H(\mathscr{A}) = \sigma_\pi(\mathscr{A}) \cup \sigma_\delta(\mathscr{A}) = \sigma_\pi(\tilde{\mathscr{A}}) \cup \sigma_\delta(\tilde{\mathscr{A}}) = \sigma_\delta(\tilde{\mathscr{A}}) = \sigma_H(\tilde{\mathscr{A}}),$$

since $\tilde{\mathscr{A}}$ is a hyponormal n-tuple. Also, since

$$\sigma_H(\tilde{\mathscr{A}}) = \sigma_\delta(\tilde{\mathscr{A}}) = \sigma_e(\tilde{\mathscr{A}}) \cup \sigma_p(\tilde{\mathscr{A}}^*)^* = \sigma_e(\mathscr{A}) \cup \sigma_p(\mathscr{A}^*)^*,$$

the proof is complete.

The n-tuple $(A_1, A_2, \ldots, A_n)$ is said to be *essentially doubly commuting* (resp., *essentially* $\mathscr{H}U(p)$) if the n-tuple $(a_1, a_2, \ldots, a_n)$, where $a_i = \pi(A_i)$ for all $1 \le i \le n$, and $\pi : B(H) \to B(H) \setminus K(H)$, is doubly commuting (resp., $\mathscr{H}U(p)$). We close this note with the following result.

THEOREM 4. Suppose $(A_1, A_2, \ldots, A_n)$ is an n-tuple of essentially doubly commuting essentially $\mathscr{H}U(p)$ operators. Then $A_1, A_2, \ldots, A_n$ have a common reducing subspace "modulo the compact operators".

Proof. The hypotheses imply that $a_i \in \mathscr{H}U(p)$ for all $1 \le i \le n$ and that the a_is are doubly commuting. Since $\sigma_e^l(\mathscr{A}) \cap \sigma_e^r(\mathscr{A})$ is not empty (this is consequence of the definition of essential spectrum—see [8, Lemma 4.2]), there exists $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma_e^l(\mathscr{A}) \cap \sigma_e^r(\mathscr{A})$ and a non-zero projection q in (the Calkin algebra) $B(H)/K(H)$ such that

$$a_i q = \lambda_i q \quad (1 \le i \le n).$$

Since $\sigma_p(a_i) = \sigma_{np}(a_i)$, this implies that $a_i^* q = \bar{\lambda}_i q$ $(1 \le i \le n)$. Consequently $a_i q = (\bar{\lambda}_i q)^* = (a_i^* q)^* = q a_i$ $(1 \le i \le n)$, or, letting $\pi(Q) = q$, $(A_i Q - Q A_i)$ is a compact operator, for all $1 \le i \le n$. This completes the proof.

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