THE FOURIER MULTIPLIER PROBLEM FOR SPACES OF CONTINUOUS FUNCTIONS WITH p-SUMMABLE TRANSFORMS

Dedicated to the memory of Hanna Neumann

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1. Introduction

In this paper we consider spaces A^p, $p \in [1,2]$, and multipliers (A^p, A^q), $p \in [1,2]$, $q \in [1,2]$. In 4.4 and 6.1 we identify (A^p, A^q) for $p \in [1,2]$, $q \in [p,2]$, and in 7.3 we identify (A^2, A^1). In 7.1 we give a sufficient condition, and in 7.5 a necessary condition, for membership of (A^p, A^q), $p \in (1,2)$, $q \in [1,p)$. We give, in 7.2, a necessary condition for membership of (A^2, A^q), $q \in [1,2)$. We include constructive proofs of some strict inclusion results for A^p, $p \in [1,2]$, (3.1 and 3.2), and also, in 5.3, for (A^p, A^p), $p \in [1,2]$.

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2. Preliminaries

2.1 We consider functions on the circle group T, and write

$$A^p = \{f \in C(T) : \hat{f} \in l^p(Z)\}, \qquad p \in [1, \infty);$$

compare here the author's paper [1]. It is known that A^p is a Banach space under the norm

$$N_p : h \mapsto \|h\|_\infty + \|\hat{h}\|_p = \|h\|_\infty + M_p(h).$$

We define e_ν to be the function $e^{it} \mapsto e^{i\nu t}$ on T and note that, for $h \in A^p$,

$$N_p(e_\nu h) = N_p(h); \; M_p(e_\nu h) = M_p(h). \tag{2.1}$$

The spectrum of $h \in L^1(T)$ is defined by

$$\mathrm{sp}(h) = \{n \in Z : \hat{h}(n) \neq 0\}.$$

If ϕ, ψ are positive functions on $\{0, 1, 2, \cdots\}$, we write $\phi \sim \psi$ if and only if $0 < \inf \phi^{-1}\psi \leqq \sup \phi^{-1}\psi < \infty$.

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2.2 In [4], p. 33, the Rudin-Shapiro polynomials P_m $(m = 0,1,2,\cdots)$ are defined by

$$P_m = \sum_{n=0}^{2^m-1} \varepsilon_m(n)e_n,$$

where the $\varepsilon_m(n) \in \{-1,1\}$ are chosen in such a way that

$$(2.2) \qquad |P_m| \leqq 2^{(m+1)/2}; \; M_p(P_m) = 2^{m/p}, \qquad m = 0,1,2,\cdots.$$

2.3 By a multiplier from A^p to A^q, $p \in [1,2]$, $q \in [1,2]$, we mean a continuous linear operator $T : A^p \to A^q$ which commutes with translations. As can be seen from [2], 16.3.1, to each multiplier $T : A^p \to A^q$ there corresponds a unique distribution ϕ such that T is (the restriction to A^p of) the operator T_ϕ defined by

$$(2.3) \qquad T_\phi f = \phi * f.$$

We denote the space of such distributions ϕ by (A^p, A^q) and refer to $\phi \in (A^p, A^q)$ as a multiplier from A^p to A^q. A distribution ϕ belongs to (A^p, A^q) if and only if

$$(2.4) \qquad N_q(\phi * f) \leqq \text{const. } N_p(f), \; \forall f \in TP,$$

where TP denotes the space of trigonometric polynomials on T. In particular, a distribution ϕ belongs to $(A^p, C) = (A^p, A^2)$ if and only if

$$(2.5) \qquad \|\phi * f\|_\infty \leqq \text{const. } N_p(f), \; \forall f \in TP;$$

or, what is equivalent, if and only if

$$(2.6) \qquad |\phi * f(1)| \leqq \text{const. } N_p(f), \; \forall f \in TP.$$

2.4 We denote by PM the space of pseudomeasures on T, and those pseudomeasures having Fourier transforms in $l^k, k \in (0,\infty]$, we denote by PM^k. PM^1 is identifiable with $A = A^1$, PM^2 with L^2, and PM^∞ with PM. We denote by M the space of Radon measures on T, and by M^k those measures having Fourier transforms in l^k, $k \in (0,\infty]$. M^2 is identifiable with L^2.

2.5 We write p' for the conjugate exponent of $p \in [1,\infty)$. p' is such that $1/p + 1/p' = 1$, $p \in (1,\infty)$, and $p' = \infty$ if $p = 1$.

2.6 We define $(A^p)'$ to be the set of linear functionals l on TP such that

$$(2.7) \qquad |l(f)| \leqq \text{const. } N_p(f), \quad \forall f \in TP.$$

Since TP is dense in A^p, the restriction from A^p to TP gives a $1-1$ map of the dual of A^p onto $(A^p)'$.

2.7 For $a \in T$, we define translation operators τ_a by

$$(2.8) \qquad \tau_a f : x \mapsto f(ax), \; \forall x \in T.$$

3. Strict inclusion results for A^p, $p \in [1,2]$

In this section we will prove constructively the following strict inclusions:

$$\bigcup_{p \in [1,q)} A^p \subsetneqq A^q \text{ if } q \in (1,2], \tag{3.1}$$

and

$$A^q \subsetneqq \bigcap_{p \in (q,2]} A^p \text{ if } q \in [1,2). \tag{3.2}$$

CONSTRUCTION 3.1 The strict inclusion (3.1).
Consider a given $q \in (1,2]$. Define $f_k \in TP$ by

$$f_k = \beta_{k,q} P_k e_{\nu_k}, \qquad k = 0,1,2,\cdots, \tag{3.3}$$

where the sequences $(\beta_{k,q})$ and (ν_k) will be chosen appropriately, the latter in such a way to ensure that the $S_k = \mathrm{sp}(f_k)$ are disjoint. Now, from (2.1), (2.2) and (3.3) we have

$$N_q(f_k) \sim \beta_{k,q} 2^{k/q}, \; q \in (1,2], \tag{3.4}$$

and

$$N_p(f_k) \sim \beta_{k,q} 2^{k/p}, \; p \in [1,q). \tag{3.5}$$

Define $f = \sum_{k=0}^{\infty} f_k$. Since $N_q(f) \leqq \sum_{k=0}^{\infty} N_q(f_k)$ it follows from (3.4) that a sufficient condition for $f \in A^q$ is that

$$\sum_{k=0}^{\infty} \beta_{k,q} 2^{k/q} < \infty, \; q \in (1,2]. \tag{3.6}$$

Choose

$$\beta_{k,q} = (k+1)^{-2} 2^{-k/q}, \qquad k = 0,1,2,\cdots. \tag{3.7}$$

Then (3.6) is satisfied since

$$\sum_{k=0}^{\infty} \beta_{k,q} 2^{k/q} = \sum_{k=0}^{\infty} (k+1)^{-2} < \infty.$$

We will now show that, with $(\beta_{k,q})$ as in (3.7), $f \notin \bigcup_{p \in [1,q)} A^p$. Since the series defining f converges in A^q,

$$\hat{f}(n) = \begin{cases} \hat{f}_k(n) & n \in S_k, \quad k = 0,1,2,\cdots, \\ 0 & n \notin \bigcup_k S_k, \end{cases} \tag{3.8}$$

where

$$\hat{f}_k(n) = \begin{cases} \varepsilon_k(n)(k+1)^{-2} 2^{-k/q} & n \in S_k \\ 0 & n \notin S_k. \end{cases} \tag{3.9}$$

Also, for f_k defined as in (3.3),

$$S_k = \{n \in Z : \nu_k \leqq n \leqq \nu_k + 2^k - 1\}, \tag{3.10}$$

so each S_k is a finite set with cardinality

$$|S_k| = 2^k. \tag{3.11}$$

Thus, making use of (3.8), (3.9) and (3.11),

$$\begin{aligned} M_p^p(f) &= \sum_{n \in Z} |\hat{f}(n)|^p \\ &= \sum_{k=0}^{\infty} \sum_{n \in S_k} |\hat{f}_k(n)|^p \\ &= \sum_{k=0}^{\infty} (k+1)^{-2p} 2^{-kp/q} 2^k \\ &= \sum_{k=0}^{\infty} (k+1)^{-2p} 2^{k(1-p/q)} \\ &= \infty \text{ for } q \in (1,2],\ p \in [1,q), \end{aligned}$$

and so $f \notin \bigcup_{p \in [1,q)} A^p$.

We still need to choose (ν_k) appropriately. It is sufficient to choose (ν_k) to be a strictly monotonic increasing sequence such that

$$\nu_{k+1} > \nu_k + 2^k - 1, \qquad k = 0,1,2,\cdots, \tag{3.12}$$

to ensure that the S_k, $k = 0,1,2,\cdots$, are disjoint. (3.12) is satisfied by the choice

$$\nu_k = 2^{k+1}, \qquad k = 0,1,2,\cdots, \tag{3.13}$$

and our construction is completed.

CONSTRUCTION 3.2. The strict inclusion (3.2).

The method employed here is the same as in 3.1. Similar reasoning shows that, given $q \in [1,2)$,

$$f = \sum_{k=0}^{\infty} f_k,$$

where

$$f_k = (k+1)^{-1/q} 2^{-k/q} P_k e_{2^{k+1}}, \qquad k = 0,1,2,\cdots,$$

is such that $f \notin A^q$, but $f \in \bigcap_{p \in (q,2]} A^p$.

4. The multipliers (A^p, A^p), $p \in [1,2]$

LEMMA 4.1. $(A^p, A^p) = (A^p, C)$, $p \in [1,2]$.

PROOF. Since, for $p \in [1,2]$, $A^p \subseteq C$ with a continuous injection,

$$(A^p, A^p) \subseteq (A^p, C). \tag{4.1}$$

Conversely, suppose $\phi \in (A^p, C)$, $p \in [1,2]$. Then

$$\|\phi * f\|_\infty \leqq \text{const. } \|f\|_\infty, \forall f \in TP. \tag{4.2}$$

Also, using (2.5),

$$|\hat{\phi}(n) \cdot \hat{f}(n)| \leqq \|\phi * f\|_\infty \leqq \text{const. } N_p(f), \forall f \in TP, \forall n \in Z. \tag{4.3}$$

Put $f = e_n$ in (4.3) to get

$$|\hat{\phi}(n)| \leqq \text{const.}, \forall n \in Z. \tag{4.4}$$

Thus $\phi \in PM$, and so

$$\|\hat{\phi} \cdot \hat{f}\|_p \leqq \|\hat{\phi}\|_\infty \|\hat{f}\|_p < \infty, \qquad \forall f \in A^p. \tag{4.5}$$

Hence, combination of (4.2) and (4.5) shows that

$$N_p(\phi * f) \leqq \text{const. } N_p(f)$$

and so by (2.4) $\phi \in (A^p, A^p)$. Thus

$$(A^p, C) \subseteq (A^p, A^p). \tag{4.6}$$

Combination of (4.1) and (4.6) completes our proof.

4.2 In view of what was said in 2.3 and 2.6, there is a $1-1$ correspondence $l \leftrightarrow \phi$ between $(A^p)'$ and (A^p, C) under which

$$l(f) = \phi * f(1), \quad \forall f \in TP. \tag{4.7}$$

This is equivalent to

$$l(\tau_x f) = \phi * f(x), \quad \forall f \in TP, \quad \forall x \in T. \tag{4.8}$$

LEMMA 4.3. *To every $l \in (A^p)'$ corresponds $\mu \in M$ and $\sigma \in PM^{p'}$ such that*

$$l(f) = \mu * f(1) + \sigma * f(1), \quad \forall f \in TP. \tag{4.9}$$

The converse is also true.

PROOF. Consider the mapping $f \mapsto (f, \hat{f})$, $f \in TP$, and define

$$S = \{(f, \hat{f}) \in C \times l^p : f \in TP\}. \tag{4.10}$$

Take $l \in (A^p)'$ and define a map l' on S by

$$l' : (f, \hat{f}) \mapsto l(f). \tag{4.11}$$

l' is well-defined since

$$((f, \hat{f}) = (g, \hat{g})) \Rightarrow (f = g).$$

l' is clearly linear; and since

(4.12) $$|l'((f,\hat{f}))| = |l(f)| \leqq \text{const. } N_p(f) = \text{const. } (\|f\|_\infty + \|\hat{f}\|_p),$$

l' is continuous on S as a subspace of $C \times l^p$. Thus, by the Hahn-Banach Theorem, l' can be extended to a continuous linear functional on the whole of $C \times l^p$. Denote this extension by l' also. We can now write

(4.13) $$l(f) = l'((f,0)) + l'((0,\hat{f})), \quad \forall f \in TP.$$

The mapping $f \mapsto l'((f,0))$ is a continuous linear functional on C, so it can be represented by a measure, $\mu \in M$, such that

(4.14) $$l'((f,0)) = \langle \check{\mu}, f \rangle = \mu * f(1), \quad \forall f \in C.$$

Also, $\theta \mapsto l'((0,\theta))$ is a continuous linear functional on l^p, $p \in [1,2]$, so it can be represented by an element, $\alpha \in l^{p'}$, such that

(4.15) $$l'((0,\theta)) = \sum_{n \in Z} \alpha(n)\theta(n).$$

Define $\sigma \in PM^{p'}$ by

(4.16) $$\hat{\sigma}(n) = \alpha(n), \quad \forall n \in Z.$$

Then, for $f \in TP$,

(4.17) $$\sigma * f = \sum_{n \in Z} \hat{\sigma}(n)\hat{f}(n)e_n.$$

Thus, by (4.15), we can write

(4.18) $$\sigma * f(1) = \sum_{n \in Z} \alpha(n)\hat{f}(n) = l'((0,\hat{f})), \quad \forall f \in TP.$$

Combination of (4.13), (4.14) and (4.18) gives

$$l(f) = \mu * f(1) + \sigma * f(1), \quad \forall f \in TP,$$

where $\mu \in M$ and $\sigma \in PM^{p'}$.

Conversely, suppose $\mu \in M$ and $\sigma \in PM^{p'}$. Consider the map $l : f \mapsto \mu * f(1) + \sigma * f(1)$ on TP. We see that, for every $f \in TP$,

$$\begin{aligned} |l(f)| \leqq |\mu * f(1)| + |\sigma * f(1)| &\leqq \|\mu * f\|_\infty + \|\sigma * f\|_\infty \\ &\leqq \|\mu\| \|f\|_\infty + \|\hat{\sigma}\|_{p'} \|\hat{f}\|_p \\ &\leqq \text{const. } N_p(f). \end{aligned}$$

Thus $l \in (A^p)'$.

THEOREM 4.4. $(A^p, C) = (A^p, A^p) = M + PM^{p'}, \quad p \in [1,2]$.

PROOF. By 4.1, $(A^p, A^p) = (A^p, C)$ for $p \in [1,2]$. By 4.2, $\phi \in (A^p, C)$ if and only if ϕ is such that

$$\phi * f(x) = l(\tau_x f), \quad \forall f \in TP, \quad \forall x \in T,$$

for some $l \in (A^p)'$. Thus, by 4.3, $\phi \in (A^p, C)$ if and only if there exist $\mu \in M, \sigma \in PM^{p'}$ such that, for every $f \in TP$ and every $x \in T$,

$$\begin{aligned}\phi * f(x) = l(\tau_x f) &= \mu * \tau_x f(1) + \sigma * \tau_x f(1) \\ &= \mu * f(x) + \sigma * f(x).\end{aligned}$$

This signifies that $\phi = \mu + \sigma$.

5. Strict inclusion results for (A^p, A^p), $p \in [1,2]$

Firstly, we will prove the following strict inclusion results:

$$(5.1) \qquad (A^q)' \subsetneqq \bigcap_{p \in [1,q)} (A^p)' \text{ if } q \in (1,2],$$

and

$$(5.2) \qquad \bigcup_{p \in (q,2]} (A^p)' \subsetneqq (A^q)' \text{ if } q \in [1,2).$$

We note here that the wide inclusion "$\subseteq$" in (5.1) and (5.2) is trivial, since N_r is stronger than N_s if $r < s$.

CONSTRUCTION 5.1. The strict inclusion (5.1).

Consider a given $q \in (1,2]$. We wish to construct a linear functional, l say, on the space TP, such that

$$(5.3) \qquad l(f) = \sum_{n \in Z} c_n \hat{f}(n), \quad f \in TP,$$

where (c_n) is chosen so that l is not continuous in the topology induced by A^q, but l is continuous in the topology induced by A^p, for every $p \in [1,q)$. For $p \in (1,q)$ it is sufficient to choose $(c_n) \in l^{p'}$, for then, for every $f \in TP$,

$$|l(f)| \leqq \Big|\sum_{n \in Z} c_n \hat{f}(n)\Big| \leqq \|(c_n)\|_{p'} M_p(f) \leqq \|(c_n)\|_{p'} N_p(f).$$

Now define

$$(5.4) \qquad f_k = \beta_{k,q} P_k e_{v_k}, \qquad k = 0,1,2,\cdots,$$

where $(\beta_{k,q})$ and (v_k) will be chosen appropriately, the latter to ensure that the $S_k = \mathrm{sp}(f_k)$ are disjoint. We have

$$(5.5) \qquad \hat{f}_k(n) = \begin{cases} \varepsilon_k(n)\beta_{k,q} & n \in S_k \\ 0 & n \notin S_k. \end{cases}$$

Put

$$(5.6) \qquad c_n = \begin{cases} b_{k,q} \operatorname{sgn} \hat{f}_k(n) \, |\hat{f}_k(n)|^{q-1} & n \in S_k \\ 0 & n \notin \bigcup_k S_k \end{cases}$$

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where $(b_{k,q})$ is a sequence of positive terms which will be chosen appropriately. Then $(c_n) \in l^{p'}$ if and only if

$$\|(c_n)\|_{p'}^{p'} = \sum_{n \in Z} |c_n|^{p'} = \sum_{k=0}^{\infty} b_{k,q}^{p'} \sum_{n \in S_k} |\hat{f}_k(n)|^{(q-1)p'} < \infty ;$$

which is equivalent to

$$\|(c_n)\|_{p'}^{p'} = \sum_{k=0}^{\infty} b_{k,q}^{p'} \beta_{k,q}^{(q-1)p'} \, |S_k| < \infty . \tag{5.7}$$

To ensure that l is not continuous in the topology induced by A^q, we seek to arrange that

$$\sup_k \left(\frac{|l(f_k)|}{N_q(f_k)} \right) = \infty . \tag{5.8}$$

By (2.1), (2.2) and (5.4),

$$N_q(f_k) \sim \beta_{k,q} 2^{k/q} .$$

Also,

$$|l(f_k)| = \Big| \sum_{n \in Z} c_n \hat{f}_k(n) \Big| = \sum_{n \in S_k} b_{k,q} |\hat{f}_k(n)|^q = b_{k,q} \beta_{k,q}^q |S_k| .$$

Thus (5.8) can be replaced by the condition

$$\sup_k \left(\frac{b_{k,q} \beta_{k,q}^q \, | S_k |}{\beta_{k,q} 2^{k/q}} \right) = \infty ;$$

that is, by

$$\sup_k \, (b_{k,q} \beta_{k,q}^{(q-1)} 2^{-k/q} \, | S_k |) = \infty . \tag{5.9}$$

(3.10) and (3.11) apply here, so (5.9) becomes

$$\sup_k (b_{k,q} \beta_{k,q}^{(q-1)} 2^{-k/q} 2^k) = \infty . \tag{5.10}$$

Choose

$$b_{k,q} = (k+1)^{-1/q'} ; \beta_{k,q} = (k+1)^{2/q} 2^{-k/q}, \qquad k = 0, 1, 2, \cdots . \tag{5.11}$$

Then (5.7) is satisfied, since

$$\begin{aligned} \|(c_n)\|_{p'}^{p'} &= \sum_{k=0}^{\infty} (k+1)^{-p'/q'} (k+1)^{2(q-1)p'/q} \, 2^{-k(q-1)p'/q} 2^k \\ &= \sum_{k=0}^{\infty} (k+1)^{p'/q'} 2^{-k(p'/q'-1)} \\ &< \infty \qquad \text{for } p \in (1,q) . \end{aligned}$$

Also, (5.9) is satisfied, since

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$$\sup_k ((k+1)^{-1/q'}(k+1)^{2(q-1)/q}2^{-k(q-1)/q}2^{-k/q}2^k)$$
$$= \sup_k [(k+1)^{1/q'}] = \infty \qquad \text{for } q \in (1,2].$$

As in 3.1 we can choose (v_k) such that

$$v_k = 2^{k+1}, \qquad k = 0,1,2,\cdots. \tag{5.12}$$

For the case $q \in (1,2]$ and $p = 1$, it is sufficient to have $(c_n) \in l^\infty$. From (5.6) and (5.11),

$$\|(c_n)\|_\infty = \sup_{n \in Z} |c_n| = \sup_k ((k+1)^{1/q'} 2^{-k/q'}).$$

Since

$$(k+1)^{1/q'} 2^{-k/q'} \leqq 1, \quad \forall k \geqq 0, \quad q \in (1,2],$$

we see that $\|(c_n)\|_\infty = 1 < \infty$, and so $(c_n) \in l^\infty$, and our construction is completed.

CONSTRUCTION 5.2. The strict inclusion (5.2).

Consider a given $q \in (1,2)$. The method employed here is the same as in 5.1, and similar reasoning shows that the linear functional, l, on TP, defined by

$$l(f) = \sum_{n \in Z} c_n \hat{f}(n), \tag{5.13}$$

where

$$c_n = \begin{cases} b_{k,q} \operatorname{sgn} \hat{f}_k(n) |\hat{f}_k(n)| & n \in S_k \\ 0 & n \notin \bigcup_k S_k, \end{cases} \tag{5.14}$$

$$f_k = \beta_{k,q} P_k e_{v_k}, \qquad k = 0,1,2,\cdots, \tag{5.15}$$

$$b_{k,q} = (k+1)^{-1/q'}; \ \beta_{k,q} = (k+1)^{-1} 2^{-k/q'}, \qquad k = 0,1,2,\cdots, \tag{5.16}$$

is such that l is continuous in the topology induced by $A^q, q \in (1,2)$, but l is not continuous in the topology induced by A^p for every $p \in (q,2]$.

We now consider the case $q = 1$. We want to construct a suitable linear functional, l, on TP, of the form given in (5.13). $(c_n) \in l^\infty$ is a sufficient condition for l to be continuous in the topology induced by $A^1 = A$. Choose

$$c_n = \begin{cases} b_k \operatorname{sgn} \hat{f}_k(n) & n \in S_k \\ 0 & n \notin \bigcup_k S_k, \end{cases} \tag{5.17}$$

where (b_k) is a sequence of positive terms which we will choose appropriately, and

$$f_k = P_k e_{v_k}, \qquad k = 0,1,2,\cdots. \tag{5.18}$$

To ensure that l is not continuous in the topology induced by A^p, $p \in (1,2]$, we seek to arrange that

$$\sup_k \left(\frac{|l(f_k)|}{N_p(f_k)} \right) = \infty, \qquad \forall p \in (1,2]. \tag{5.19}$$

By (2.1), (2.2) and (5.18),

$$N_p(f_k) \sim 2^{k/p};$$

and

$$|l(f_k)| = \left| \sum_{n \in Z} c_n \hat{f}_k(n) \right| = \sum_{n \in S_k} b_k |\hat{f}_k(n)| = b_k |S_k|,$$

so we can replace (5.19) by

$$\sup_k (b_k |S_k| 2^{-k/p}) = \infty, \qquad \forall p \in (1,2]. \tag{5.20}$$

(3.10) and (3.11) apply here, so (5.20) becomes

$$\sup_k \, [b_k 2^{k(1-1/p)}] = \infty, \qquad \forall p \in (1,2]. \tag{5.21}$$

Choose

$$b_k = 1, \; k = 0, 1, 2, \cdots. \tag{5.22}$$

Then $(c_n) \in l^\infty$ since

$$\|(c_n)\|_\infty = \sup_{n \in Z} |c_n| = 1 < \infty;$$

and (5.21) is satisfied since

$$\sup_k (2^{k(1-1/p)}) = \infty, \quad \forall p \in (1,2].$$

Again, as in 3.1, we can choose (v_k) such that

$$v_k = 2^{k+1}, \; k = 0, 1, 2, \cdots,$$

and our construction is completed.

THEOREM 5.3. *The following strict inclusions hold*:

$$(A^q, A^q) \subsetneqq \bigcap_{p \in [1,q)} (A^p, A^p) \text{ if } q \in (1,2], \tag{5.23}$$

and

$$\bigcup_{p \in (q,2]} (A^p, A^p) \subsetneqq (A^q, A^q) \text{ if } q \in [1,2). \tag{5.24}$$

PROOF. By 5.1, if $q \in (1,2]$, then $\exists l \in \bigcap_{p \in [1,q)} (A^p)'$, $l \notin (A^q)'$. Let ϕ correspond to l as in 4.2. Then $\phi \in (A^p, C)$, $\forall p \in [1,q)$ and $\phi \notin (A^q, C)$. Use of 4.1 gives the result (5.23).

Similar resoning can be used to derive (5.24) from 5.2.

6. The multipliers (A^p, A^q), $p \in [1,2]$, $q \in [p,2]$

THEOREM 6.1. $(A^p, A^q) = (A^p, A^p) = M + PM^{p'}$, $p \in [1,2]$, $q \in [p,2]$.

PROOF. For $q \in [p,2]$, $A^p \subseteq A^q$ with continuous injection, so

(6.1) $$(A^p, A^q) \supseteq (A^p, A^p),\ p \in [1,2],\ q \in [p,2].$$

Conversely, since $A^q \subseteq C$ with continuous injection,

(6.2) $$(A^p, A^q) \subseteq (A^p, C),\ p \in [1,2],\ q \in [p,2].$$

Use of 4.1 with (6.2) gives

(6.3) $$(A^p, A^q) \subseteq (A^p, A^p),\ p \in [1,2],\ q \in [p,2].$$

Combine (6.1) and (6.3) and then use 4.4 to deduce the required result.

7. The multipliers (A^p, A^q), $p \in [1,2]$, $q \in [1,p)$.

THEOREM 7.1. $M^{pq/(p-q)} \subseteq (A^p, A^q)$, $p \in [1,2]$, $q \in [1,p)$.

PROOF. Consider $\mu \in M^{pq/(p-q)}$. Then, since $\mu \in M$,

(7.1) $$\|\mu * f\|_\infty \leqq \|\mu\| \, \|f\|_\infty,\ \forall f \in TP,\ p \in [1,2].$$

Also, Hölder's inequality gives for every $f \in TP$

$$\sum_{n \in Z} |\hat{\mu}(n)\hat{f}(n)|^q \leqq \left(\sum_{n \in Z} |\hat{f}(n)|^{qs}\right)^{1/s} \left(\sum_{n \in Z} |\hat{\mu}(n)|^{qs'}\right)^{1/s'}.$$

For $s = p/q$, $s' = p/(p-q)$, this becomes

(7.2) $$\sum_{n \in Z} |\hat{\mu}(n)\hat{f}(n)|^q \leqq \left(\sum_{n \in Z} |\hat{f}(n)|^p\right)^{q/p} \leqq \left(\sum_{n \in Z} |\hat{\mu}(n)|^{pq/(p-q)}\right)^{(p-q)/pq}, \quad \forall f \in TP.$$

By (7.1) and (7.2), we have for $f \in TP$

$$\begin{aligned} N_q(\mu * f) &= \|\mu * f\|_\infty + \|\widehat{\mu * f}\|_q \\ &\leqq \|\mu\| \, \|f\|_\infty + \|\hat{\mu}\|_{pq/(p-q)} \|\hat{f}\|_p \leqq \text{const. } N_p(f). \end{aligned}$$

Now refer to (2.4).

THEOREM 7.2. $\phi \in (A^2, A^q) \Rightarrow \phi \in l^{2q/(2-q)}$, $q \in [1,2)$.

PROOF. From [3], Corollary 2.3, p. 468 it follows that, if $\hat{\phi} \cdot \hat{f} \in l^q(Z)$, $q \in [1,2)$, for each $f \in C(T)$, then $\hat{\phi} \in l^{2q/(2-q)}$. Since $A^2 = C$, our result follows directly.

THEOREM 7.3. $(C, A) = L^2$.

PROOF. From 7.1,

$$L^2 = M^2 \subseteq (A^2, A^1) = (C, A). \tag{7.3}$$

Conversely, suppose $\phi \in (C, A)$. Then, by 7.2, $\hat{\phi} \in l^2$, and so $\phi \in L^2$. Thus $(C, A) \subseteq L^2$.

7.4 We now establish preliminary results leading to a necessary condition for $\phi \in (A^p, A^q)$, $p \in [1, 2]$, $q \in [1, p)$.

Consider

$$S = \{(c_n) \in l^p \colon \sum_{n \in Z \setminus \{0\}} |c_n|^2 \log^{1+\varepsilon} |n| < \infty,\ \varepsilon > 0\}. \tag{7.4}$$

Then, from [2], 14.3.6, p. 205, for $(c_n) \in S$, almost all the series

$$\sum_{n \in Z} r_{|n|}(t) c_n e_n$$

are the Fourier series of continuous functions. (In fact, of functions in A^p). If $\phi \in (A^p, A^q)$, $p \in [1, 2]$, $p \in [1, 2]$, $q \in [1, p)$, then

$$\phi * f \in A^q,\ \forall f \in A^p,$$

and so

$$\left(\sum_{n \in Z} |\hat{\phi}(n) c_n|^q \right)^{1/q} < \infty,\ \forall c = (c_n) \in S.$$

Define a map $Q_\phi \colon S \to l^q$ by

$$Q_\phi \colon (c_n) \mapsto (\hat{\phi}(n) c_n).$$

Q_ϕ is clearly linear. It is not hard to see that S is a Banach space under the norm

$$\| \cdot \|_S \colon (c_n) \mapsto \left(\sum_{n \in Z} |c_n|^p \right)^{1/p} + \left(\sum_{n \in Z \setminus \{0\}} |c_n|^2 \log^{1+\varepsilon} |n| \right)^{1/2}.$$

An application of the Closed Graph Theorem shows that Q_ϕ is a continuous map from S to l^q, so we have

$$\left(\sum_{n \in Z} |\hat{\phi}(n) c_n|^q \right)^{1/q} \leqq K \left[\left(\sum_{n \in Z} |c_n|^p \right)^{1/p} + \left(\sum_{n \in Z \setminus \{0\}} |c_n|^2 \log^{1+\varepsilon} |n| \right)^{1/2} \right], \tag{7.5}$$

where $K = K(\phi, \varepsilon)$ is a constant.

THEOREM 7.5. *If $\phi \in (A^p, A^q)$, $p \in [1, 2]$, $q \in [1, p)$, then for every $\varepsilon > 0$,*

$$\sum_{|n| \leqq R} |\hat{\phi}(n)|^{pq/(p-q)} \leqq \max[1, (K(1 + J_R))^{pq/(p-q)}],$$

where $K = K(\phi, \varepsilon)$ is independent of R and

$$J_R = \max_{0 < |n| \leqq R} (|\hat{\phi}(n)|^{(2-p)q/(p-q)} \log^{1+\varepsilon} |n|). \tag{7.6}$$

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PROOF. Suppose $\phi \in (A^p, A^q)$, $p \in [1, 2]$, $q \in [1, p)$. Then (7.5) holds. Choose $(c_n) \in S$ such that

$$c_n = \begin{cases} |\hat{\phi}(n)|^{q/(p-q)} & |n| \leqq R \\ 0 & |n| > R, \end{cases}$$

and consider J_R as defined in (7.6). Then, on writing $\sigma_R = \sum_{|n| \leqq R} |\hat{\phi}(n)|^{pq/(p-q)}$, (7.5) yields

$$\sigma_R^{1/q} \leqq K(\sigma_R^{1/p} + J_R \sigma_R^{1/2});$$

that is,

$$\sigma_R^{1/q - 1/p} \leqq K(1 + J_R \sigma_R^{1/2 - 1/p}).$$

It follows that

$$\sigma_R \leqq \max[1, (K(1 + J_R))^{pq/(p-q)}].$$

COROLLARY 7.6. *If* $\phi \in (A^p, A^q)$, $p \in [1, 2]$, $q \in [1, p)$, *then*

$$\sum_{|n| \leqq R} |\hat{\phi}(n)|^{pq/(p-q)} = O\{(\log R)^{\Delta}\},$$

where $\Delta = (1 + \varepsilon)pq/2(p - q)$, *and* $\varepsilon > 0$.

PROOF. Suppose $\phi \in (A^p, A^q)$, $p \in [1, 2]$, $q \in [1, p)$. Then $\phi \in (A, C) = PM$, and it follows that $J_R = O\{(\log R)^{(1+\varepsilon)/2}\}$, where J_R is defined in (7.6). Application of 7.5 now gives the desired result.

COROLLARY 7.7. *If* $\phi \in (A^p, A^q)$, $p \in [1, 2]$, $q \in [1, p)$, *and* $J_R = O(1)$, *where* J_R *is defined in* (7.6), *then* $\hat{\phi} \in l^{pq/(p-q)}$.

PROOF. This result follows directly from 7.5.

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