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ON THE RESTRICTED CESÀRO SUMMABILITY OF MULTIPLE ORTHOGONAL SERIES

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1. Introduction. We actually treat double orthogonal series in detail, simply for the sake of brevity in notations. Multiple orthogonal series will be shortly indicated in the concluding Section 8.

Let $(X, \mathscr{F}, \mu)$ be an arbitrary positive measure space and $\{\phi_{ik}(x):i, k = 0, 1, \dots\}$ an orthonormal system defined on X. We consider the double orthogonal series

$$(1.1) \quad \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}\phi_{ik}(x)$$

where $\{a_{ik}:i, k = 0, 1, \dots\}$ is a double sequence of real numbers (coefficients), for which

$$(1.2) \quad \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 < \infty.$$

By the extended Riesz-Fischer theorem, there exists a function $f(x) \in L^2(X, \mathscr{F}, \mu) = L^2$ such that (1.1) is the generalized Fourier series of $f(x)$ with respect to $\{\phi_{ik}(x)\}$ and the rectangular partial sums

$$s_{mn}(x) = \sum_{i=0}^{m} \sum_{k=0}^{n} a_{ik}\phi_{ik}(x) \quad (m, n = 0, 1, \dots)$$

converge to $f(x)$ in L^2-metric:

$$\int [s_{mn}(x) - f(x)]^2 \, d\mu(x) \to 0 \quad \text{as } \min(m, n) \to \infty.$$

Here and in the sequel, the integrals are taken over the entire space X.

Let α and β be real numbers, $\alpha > -1$ and $\beta > -1$. We remind the reader that the (C, α, β)-means of series (1.1) are defined as follows (for single series, see e.g. [**5**, p. 77]):

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$$\sigma_{mn}^{\alpha\beta}(x) = \frac{1}{A_m^\alpha A_n^\beta} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{\alpha-1} A_{n-k}^{\beta-1} s_{ik}(x)$$

$$= \frac{1}{A_m^\alpha A_n^\beta} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{\alpha} A_{n-k}^{\beta} a_{ik}\phi_{ik}(x) \quad (m, n = 0, 1, \dots)$$

where

$$A_m^\alpha = \binom{m+\alpha}{m} = \begin{cases} (\alpha + 1)(\alpha + 2)\dots(\alpha + m)/m! & \text{for } m = 1, 2, \dots; \\ 1 & \text{for } m = 0. \end{cases}$$

In the case $\alpha = \beta = 0$ we have $s_{mn}(x) = \sigma_{mn}^{00}(x)$, while the case $\alpha = \beta = 1$ gives the first arithmetic means:

$$\sigma_{mn}^{11}(x) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{k=0}^{n} s_{ik}(x).$$

2. Preliminary results. We begin with the following convention. Given a system $\{f_p(x)\}$ of functions in L^2 and a sequence $\{\lambda_p\}$ of positive numbers, we write

$$f_p(x) = o_x\{\lambda_p\} \text{ a.e. (as } p \to \infty)$$

if

$$f_p(x)/\lambda_p \to 0 \text{ a.e. as } p \to \infty$$

and, in addition, there exists a function $F(x) \in L^2$ such that

$$\sup_p |f_p(x)|/\lambda_p \leqq F(x) \text{ a.e.}$$

Here p ranges over either $0, 1, \dots$ or $1, 2, \dots$.

Theorems A, B, C and D below are proved in [**2**]. The first of them is a Kolmogorov type result for double orthogonal series (see [**1**, pp. 118-119] concerning single orthogonal series).

THEOREM A. ([**2**, Lemma 2]). *Under condition* (1.2),

(2.1) $$s_{2^p,2^p}(x) - \sigma_{2^p,2^p}^{11}(x) = o_x\{1\} \text{ a.e.}$$

Analyzing the proof, a slightly stronger conclusion can be drawn: for every $\theta \geqq 1$

(2.1′) $$\max_{q:\theta^{-1} \leqq 2^q/2^p \leqq \theta} |s_{2^p,2^q}(x) - \sigma_{2^p,2^q}^{11}(x)| = o_x\{1\} \text{ a.e.}$$

The second theorem is a Kaczmarz type result (cf. [**1**, pp. 119-120]).

THEOREM B. ([**2**, Lemmas 3 and 4 and formula (4.6)]). *Under condition* (1.2), *for every* $\theta \geqq 1$

$$(2.2)\qquad \max_{2^p \leqq m \leqq 2^{p+1}} \max_{n:\theta^{-1} \leqq n/m \leqq \theta} |\sigma^{11}_{mn}(x) - \sigma^{11}_{2^p,2^p}(x)| = o_x\{1\} \text{ a.e.}$$

as $p \to \infty$.

Actually, somewhat more is proved in [**2**]: for every $\theta \geqq 1$

$$(2.2')\qquad \max_{2^p \leqq m \leqq 2^{p+1}} \max_{n:\theta^{-1}2^p \leqq n \leqq \theta 2^{p+1}} |\sigma^{11}_{mn}(x) - \sigma^{11}_{2^p,2^p}(x)| = o_x\{1\} \text{ a.e.}$$

Comparing Theorems A and B yields that under condition (1.2) the a.e. convergence of $\{s_{2^p,2^p}(x)\}$ as $p \to \infty$ and the a.e. convergence of $\{\sigma_{mn}(x)\}$ as $\min(m, n) \to \infty$ in such a way that $\theta^{-1} \leqq n/m \leqq \theta$ with a fixed $\theta \geqq 1$, are equivalent to one another. The latter property may be called a.e. restricted $(C, 1, 1)$-summability (and in the same sense we can speak about a.e. restricted (C, α, β)-summability).

Applying a Rademacher-Menšov type result to the subsequence $\{s_{2^p,2^p}(x):p = 0, 1, \dots\}$ (see [**2**, Lemma 1]), we can conclude a Menšov-Kaczmarz type result (cf. [**1**, pp. 125-126]).

THEOREM C. ([**2**, Theorem 1]). *Under the condition*

$$(2.3)\qquad \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 [\log \log(\max(i, k) + 4)]^2 < \infty,$$

for every $\theta \geqq 1$

$$\max_{n:\theta^{-1} \leqq n/m \leqq \theta} |\sigma^{11}_{mn}(x) - f(x)| = o_x\{1\} \text{ a.e. } as\ m \to \infty.$$

In this paper the logarithms are to the base 2.

Assuming only (1.2), the order of magnitude of $\sigma^{11}_{mn}(x)$ can be estimated in the case where m and n tend restrictedly to ∞.

THEOREM D ([**2**, Theorem 2]). *Under condition* (1.2), *for every* $\theta \geqq 1$

$$\max_{n:\theta^{-1} \leqq n/m \leqq \theta} |\sigma^{11}_{mn}(x)| = o_x\{\log \log(m + 4)\} \text{ a.e.}$$

3. Main results. We prove that condition (2.3) is also sufficient for the a.e. restricted $(C, \alpha > 0, \beta > 0)$-summability of series (1.1).

THEOREM 1. *If* $\alpha > 0$, $\beta > 0$, $\theta \geqq 1$ *and condition* (2.3) *is satisfied, then*

$$(3.1)\qquad \sup_{n:\theta^{-1} \leqq n/m \leqq \theta} |\sigma^{\alpha\beta}_{mn}(x) - f(x)| = o_x\{1\} \text{ a.e. } as\ m \to \infty.$$

The next theorem extends the validity of Theorem D.

THEOREM 2. *If* $\alpha > 0$, $\beta > 0$, $\theta \geqq 1$ *and condition* (1.2) *is satisfied, then*

$$\max_{n:\theta^{-1} \leqq n/m \leqq \theta} |\sigma_{mn}^{\alpha\beta}(x)| = o_x\{\log \log(m + 4)\} \text{ a.e.} \tag{3.2}$$

The following theorem plays a key role in the proofs of Theorems 1 and 2.

THEOREM 3. *If* $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$, $\theta \geqq 1$ *and condition* (1.2) *is satisfied, then*

$$\left\{\frac{1}{(M + 1)^2} \sum_{m=0}^{M} \sum_{n=0}^{\theta M} [\sigma_{mn}^{\alpha-1,\beta-1}(x) - \sigma_{mn}^{\alpha\beta}(x)]^2\right\}^{1/2} = o_x\{1\} \text{ a.e. as } M \to \infty. \tag{3.3}$$

By $\sum_{n=0}^{\theta M}$ we mean that the summation is carried out for all integer values of n such that $0 \leqq n \leqq \theta M$.

On the other hand, taking Theorems 1, 2 and 3 for granted, we can immediately deduce two corollaries ensuring the so-called strong (C, α, β)-summability of series (1.1) in the restricted case.

COROLLARY 1. *If* $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$, $\theta \geqq 1$ *and condition* (2.3) *is satisfied, then*

$$\left\{\frac{1}{(M + 1)^2} \sum_{m=0}^{M} \sum_{n=0}^{\theta M} [\sigma_{mn}^{\alpha-1,\beta-1}(x) - f(x)]^2\right\}^{1/2} = o_x\{1\} \text{ a.e. as } M \to \infty.$$

COROLLARY 2. *If* $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$, $\theta \geqq 1$ *and condition* (1.2) *is satisfied, then*

$$\left\{\frac{1}{(M + 1)^2} \sum_{m=0}^{M} \sum_{n=0}^{\theta M} [\sigma_{mn}^{\alpha-1,\beta-1}(x)]^2\right\}^{1/2} = o_x\{\log \log(M + 4)\} \text{ a.e.}$$

We note that in the special case $\alpha = \beta = 1$ similar but not comparable statements were derived in [**2**, Theorems 3 and 4] using another method.

Analyzing the proofs of Theorems 1 and 2 given in Sections 5 and 6, we can gain the following byproduct, interesting in itself.

COROLLARY 3. *If* $\alpha > 0$, $\beta > 0$, $\theta \geqq 1$ *and condition* (1.2) *is satisfied, then the convergence of* $\{\sigma_{mn}^{\alpha\beta}(x)\}$ *on a measurable set as* $\min(m, n) \to \infty$ *in such a way that* $\theta^{-1} \leqq n/m \leqq \theta$ *is equivalent for all* $\alpha > 0$ *and* $\beta > 0$, *up to a set of measure zero.*

In other words, if series (1.1) with (1.2) is restrictedly (C, α_0, β_0)-summable on a measurable subset Y of X for a given pair of $\alpha_0 > 0$ and $\beta_0 > 0$, then it is also restrictedly (C, α, β)-summable a.e. on Y for each pair $\alpha > 0$ and $\beta > 0$.

4. Auxiliary results. In this section we consider numerical series

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} u_{ik}$$

of real numbers. Now the (C, α, β)-means are defined by

$$\sigma_{mn}^{\alpha\beta} = \frac{1}{A_m^\alpha A_n^\beta} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{\alpha} A_{n-k}^{\beta} u_{ik}$$

$$(m, n = 0, 1, \ldots; \alpha > -1, \beta > -1).$$

We remind the reader of some identities and inequalities well-known in the literature. For all α and γ

$$A_m^{\alpha+\gamma+1} = \sum_{i=0}^{m} A_i^{\alpha} A_{m-i}^{\gamma} \tag{4.1}$$

(see, e.g. [**5**, p. 77, formula (1.10)]). Hence the representations

$$\alpha_{mn}^{\alpha+\gamma,\beta} = \frac{1}{A_m^{\alpha+\gamma}} \sum_{i=0}^{m} A_{m-i}^{\gamma-1} A_i^{\alpha} \sigma_{in}^{\alpha\beta} \ (\alpha + \gamma > -1) \tag{4.2}$$

and

$$\alpha_{mn}^{\alpha,\beta+\delta} = \frac{1}{A_n^{\beta+\delta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1} A_k^{\beta} \sigma_{mk}^{\alpha\beta} \ (\beta + \delta > -1) \tag{4.3}$$

easily follow.

We also need the following estimate: There exist two positive constants C_1 and C_2 depending only on α such that

$$C_1 \leqq A_m^{\alpha}/m^{\alpha} \leqq C_2 \quad (m = 1, 2, \ldots; \alpha > -1) \tag{4.4}$$

(see [**1**, p. 69, formula (25)] or [**5**, p. 77, formula (1.18)]).

In the next Tauberian result $\{\lambda_M : M = 0, 1, \ldots\}$ is a non-decreasing sequence of positive numbers.

LEMMA 1. *If* $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$, $\epsilon > 0$, $\eta > 0$, $\theta \geqq 1$ *and*

$$\frac{1}{\lambda_M} \left\{ \frac{1}{(M+1)^2} \sum_{m=0}^{M} \sum_{n=0}^{\theta M} [\sigma_{mn}^{\alpha\beta}]^2 \right\}^{1/2} \to 0 \text{ as } M \to \infty, \tag{4.5}$$

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then

$$(4.6)\qquad \frac{1}{\lambda_M}\left[\max_{N:\theta^{-1}\leqq N/M\leqq\theta}|\sigma_{MN}^{\alpha+1/2+\epsilon,\beta+1/2+\eta}|\right]\to 0 \text{ as } M\to\infty.$$

Furthermore, if

$$\frac{1}{\lambda_M}\left\{\frac{1}{(M+1)^2}\sum_{m=0}^{M}\sum_{n=0}^{\theta M}[\sigma_{mn}^{\alpha\beta}]^2\right\}^{1/2}\leqq B\quad (M=0,1,\dots)$$

with a positive number B, then there exists a constant C depending only on α, β, ϵ, η *and* θ *such that*

$$\frac{1}{\lambda_M}\left[\max_{N:\theta^{-1}\leqq N/M\leqq\theta}|\sigma_{MN}^{\alpha+1/2+\epsilon,\beta+1/2+\eta}|\right]\leqq CB\quad (M=0,1,\dots).$$

In case $M = 0$ the condition $\theta^{-1}\leqq N/M\leqq\theta$ is meant to be satisfied by $N = 0$.

Proof. The basic idea goes back to Zygmund [**4**, pp. 360-361]. By (4.2),

$$\sigma_{MN}^{\alpha+1/2+\epsilon,\beta+1/2+\eta} = \frac{1}{A_M^{\alpha+1/2+\epsilon}}\sum_{m=0}^{M}A_{M-m}^{-1/2+\epsilon}A_m^{\alpha}\sigma_{mN}^{\alpha,\beta+1/2+\eta}.$$

Hence, via the Cauchy inequality,

$$(4.7)\qquad \max_{N:\theta^{-1}\leqq N/M\leqq\theta}|\sigma_{MN}^{\alpha+1/2+\epsilon,\beta+1/2+\eta}|$$

$$\leqq \frac{1}{A_M^{\alpha+1/2+\epsilon}}\sum_{m=0}^{M}A_{M-m}^{-1/2+\epsilon}A_m^{\alpha}[\max_{N:\theta^{-1}\leqq N/M\leqq\theta}|\sigma_{mN}^{\alpha,\beta+1/2+\eta}|]$$

$$\leqq \frac{1}{A_M^{\alpha+1/2+\epsilon}}\left\{\sum_{m=0}^{M}[A_{M-m}^{-1/2+\epsilon}A_m^{\alpha}]^2\right.$$

$$\left.\times\sum_{m=0}^{M}[\max_{N:\theta^{-1}\leqq N/M\leqq\theta}|\sigma_{mN}^{\alpha,\beta+1/2+\eta}|]^2\right\}^{1/2}.$$

Taking into account (4.1) and (4.4), it is not hard to check that

$$(4.8)\qquad \frac{1}{A_M^{\alpha+1/2+\epsilon}}\left\{\sum_{m=0}^{M}[A_{M-m}^{-1/2+\epsilon}A_m^{\alpha}]^2\right\}^{1/2} = O\left\{\frac{1}{(M+1)^{1/2}}\right\}.$$

Repeating the above reasoning, this time starting with $\sigma_{mN}^{\alpha,\beta+1/2+\eta}$, by (4.3), (4.1) and (4.4) we get that

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$$(4.9)\quad |\sigma_{mN}^{\alpha,\beta+1/2+\eta}| = \left|\frac{1}{A_N^{\beta+1/2+\eta}} \sum_{n=0}^{N} A_{N-n}^{-1/2+\eta} A_n^{\beta} \sigma_{mn}^{\alpha\beta}\right|$$

$$\leqq \frac{1}{A_N^{\beta+1/2+\eta}} \left\{ \sum_{n=0}^{N} [A_{N-n}^{-1/2+\eta} A_n^{\beta}]^2 \sum_{n=0}^{N} [\sigma_{mn}^{\alpha\beta}]^2 \right\}^{1/2}$$

$$= O\{1\}\left\{\frac{1}{N+1} \sum_{n=0}^{N} [\sigma_{mn}^{\alpha\beta}]^2\right\}^{1/2}.$$

Combining (4.7), (4.8) and (4.9) (the latter taken for each N such that $\theta^{-1} \leqq N/M \leqq \theta$) yields

$$\max_{N:\theta^{-1}\leqq N/M\leqq\theta} |\sigma_{MN}^{\alpha+1/2+\epsilon,\beta+1/2+\eta}|$$

$$= O\{1\}\left\{\frac{1}{M+1} \sum_{m=0}^{M} \left[\max_{N:\theta^{-1}\leqq N/M\leqq\theta}\left(\frac{1}{N+1} \sum_{n=0}^{N} [\sigma_{mn}^{\alpha\beta}]^2\right)\right]\right\}^{1/2}$$

$$= O\{1\}\left\{\frac{1}{M+1} \sum_{m=0}^{M} \frac{1}{\theta^{-1}M+1} \sum_{n=0}^{\theta M} [\sigma_{mn}^{\alpha\beta}]^2\right\}^{1/2}$$

$$= O\{1\}\left\{\frac{1}{(M+1)^2} \sum_{m=0}^{M} \sum_{n=0}^{\theta M} [\sigma_{mn}^{\alpha\beta}]^2\right\}^{1/2} = o\{\lambda_M\} \text{ as } M \to \infty.$$

The last step is due to assumption (4.5). The estimate obtained is (4.6) to be proved.

The second part of Lemma 1 can be verified in a similar manner.

We will make use of the following representations, too:

$$(4.10)\quad \sigma_{mn}^{\alpha-1,\beta} - \sigma_{mn}^{\alpha\beta}$$

$$= \frac{1}{\alpha A_m^{\alpha} A_n^{\beta}} \sum_{i=1}^{m} \sum_{k=0}^{n} A_{m-i}^{\alpha-1} A_{n-k}^{\beta} i u_{ik} \quad (\alpha > 0, \beta > -1)$$

and

$$(4.11)\quad \sigma_{mn}^{\alpha-1,\beta-1} - \sigma_{mn}^{\alpha-1,\beta} - \sigma_{mn}^{\alpha,\beta-1} + \sigma_{mn}^{\alpha\beta}$$

$$= \frac{1}{\alpha\beta A_m^{\alpha} A_n^{\beta}} \sum_{i=1}^{m} \sum_{k=1}^{n} A_{m-i}^{\alpha-1} A_{n-k}^{\beta-1} i k u_{ik} \quad (\alpha > 0, \beta > 0).$$

Both easily follow through the identities

$$A_m^{\alpha-1} = \frac{\alpha}{\alpha+m} A_m^{\alpha} \quad\text{and}\quad A_{m-i}^{\alpha} = \frac{\alpha+m-i}{\alpha} A_{m-i}^{\alpha-1}.$$

Finally, we present two more inequalities:

$$(4.12)\quad \sum_{m=i}^{\infty}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2 = O\left\{\frac{1}{i}\right\} \quad (i = 1, 2, \ldots ; \alpha > \tfrac{1}{2})$$

and

$$(4.13)\quad \sum_{m=i}^{\infty}\frac{1}{m}\left[1 - \frac{A_{m-i}^{\alpha}}{A_m^{\alpha}}\right]^2 = O\{1\} \quad (i = 1, 2, \ldots ; \alpha > 0).$$

The first inequality is well-known (see, e.g. [**1**, p. 110]), while the second one was proved in [**3**, formula (4.9)].

5. Proof of theorem 1. This is done on the basis of Theorem 3, which will be proved in Section 7, and on the following consequence of Lemma 1.

COROLLARY 4. *If* $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$, $\epsilon > 0$, $\eta > 0$, $\theta \geqq 1$ *and*

$$(5.1)\quad \left\{\frac{1}{(M+1)^2}\sum_{m=0}^{M}\sum_{n=0}^{\theta M}[\sigma_{mn}^{\alpha\beta}(x) - f(x)]^2\right\}^{1/2} = o_x\{1\} \text{ a.e.}$$

as $M \to \infty$,

then

$$(5.2)\quad \max_{N:\theta^{-1}\leqq N/M\leqq\theta}|\sigma_{MN}^{\alpha+1/2+\epsilon,\beta+1/2+\eta}(x) - f(x)| = o_x\{1\} \text{ a.e.}$$

as $M \to \infty$.

In fact, setting $\lambda_M \equiv 1 \quad (M = 0, 1, \ldots)$,

$$u_{00} = a_{00}\phi_{00}(x) - f(x) \quad \text{and}$$

$$u_{ik} = a_{ik}\phi_{ik}(x) \quad (i^2 + k^2 > 0),$$

Corollary 4 immediately follows from Lemma 1.

After these preliminaries the proof of (3.1) is quite simple. By Theorem C, (3.1) holds for $\alpha = \beta = 1$. Hence, by Theorem 3, we get (5.1) for $\alpha = \beta = 0$. Thus, by Corollary 4, we obtain (5.2) also for $\alpha = \beta = 0$. Using again Theorem 3, we find (5.1) for $\alpha = -\frac{1}{2} + \epsilon$ and $\beta = -\frac{1}{2} + \eta$. Hence, by Corollary 4, we get (5.2) for the same pair α and β, i.e., (3.1) for $\alpha = 2\epsilon$ and $\beta = 2\eta$. Since ϵ and η are arbitrary positive numbers, Theorem 1 is completely proved.

6. Proof of theorem 2. The proof relies again on Theorem 3 and on the following consequence of Lemma 1.

COROLLARY 5. *If* $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$, $\epsilon > 0$, $\eta > 0$, $\theta \geqq 1$ *and*

$$(6.1)\quad \left\{\frac{1}{(M+1)^2}\sum_{m=0}^{M}\sum_{n=0}^{\theta M}[\sigma_{mn}^{\alpha\beta}(x)]^2\right\}^{1/2} = o_x\{\log\log(M+4)\} \text{ a.e.,}$$

then

$$(6.2)\quad \max_{N:\theta^{-1}\leqq N/M\leqq\theta} |\sigma_{MN}^{\alpha+1/2+\epsilon,\beta+1/2+\eta}(x)| = o_x\{\log\log(M+4)\} \text{ a.e.}$$

This time we set

$$\lambda_M = \log\log(M+4) \quad (M = 0, 1, \dots)$$

and

$$u_{ik} = a_{ik}\phi_{ik}(x) \quad (i, k = 0, 1, \dots)$$

in Lemma 1.

Now, by Theorem D, (3.2) holds for $\alpha = \beta = 1$. Hence, by Theorem 3, we conclude (6.1) for $\alpha = \beta = 0$. Thus by Corollary 5, we obtain (6.2) for $\alpha = \beta = 0$. Using again Theorem 3, we find (6.1) for $\alpha = -\frac{1}{2} + \epsilon$ and $\beta = -\frac{1}{2} + \eta$. Hence, by Corollary 5, we get (6.2) also for $\alpha = -\frac{1}{2} + \epsilon$ and $\beta = -\frac{1}{2} + \eta$. But the latter estimate coincides with (3.2) for $\alpha = 2\epsilon$ and $\beta = 2\eta$. Since $\epsilon > 0$ and $\eta > 0$ are arbitrary, (3.2) is proved for all $\alpha > 0$ and $\beta > 0$.

7. **Proof of theorem 3.** By the triangle inequality, the left-hand side of estimate (3.3) to be proved can be estimated as follows

$$\left\{\frac{1}{(M+1)^2}\sum_{m=0}^{M}\sum_{n=0}^{\theta M}[\sigma_{mn}^{\alpha-1,\beta-1}(x) - \alpha_{mn}^{\alpha\beta}(x)]^2\right\}^{1/2}$$

$$\leqq \left\{\frac{1}{(M+1)^2}\sum_{m=0}^{M}\sum_{n=0}^{\theta M}[\sigma_{mn}^{\alpha-1,\beta-1}(x) - \sigma_{mn}^{\alpha-1,\beta}(x) - \sigma_{mn}^{\alpha,\beta-1}(x) + \sigma_{mn}^{\alpha\beta}(x)]^2\right\}^{1/2}$$

$$+ \left\{\frac{1}{(M+1)^2}\sum_{m=0}^{M}\sum_{n=0}^{\theta M}[\sigma_{mn}^{\alpha-1,\beta}(x) - \sigma_{mn}^{\alpha\beta}(x)]^2\right\}^{1/2}$$

$$+ \left\{\frac{1}{(M+1)^2}\sum_{m=0}^{M}\sum_{n=0}^{\theta M}[\sigma_{mn}^{\alpha,\beta-1}(x) - \sigma_{mn}^{\alpha\beta}(x)]^2\right\}^{1/2}.$$

According to this estimate, Theorem 3 will be a consequence of Lemmas 2-4 below.

LEMMA 2. *If* $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$, $\theta \geqq 1$ *and condition* (1.2) *is satisfied, then*

(7.1) $$ {}^2\delta^{\alpha\beta}_{M,\theta}(x) = \left\{ \frac{1}{(M+1)^2} \sum_{m=0}^{M} \sum_{n=0}^{\theta M} [\sigma^{\alpha-1,\beta-1}_{mn}(x) - \sigma^{\alpha-1,\beta}_{mn}(x) - \sigma^{\alpha,\beta-1}_{mn}(x) + \sigma^{\alpha\beta}_{mn}(x)]^2 \right\}^{1/2} = o_x\{1\} \text{ a.e. } as\ M \to \infty. $$

Proof. Let $M \geqq 1$. Then there exists an integer $p \geqq 0$ such that $2^{p-1} < M \leqq 2^p$. Clearly,

$$ {}^2\delta^{\alpha\beta}_{M,\theta}(x) \leqq 2\ {}^2\delta^{\alpha\beta}_{2^p,\theta}(x). $$

Thus, in order to prove (7.1) it suffices to derive

(7.2) $$ {}^2\delta^{\alpha\beta}_{2^p,\theta}(x) = o_x\{1\} \text{ a.e. as } p \to \infty. $$

To this goal, we define

$$ F^{\alpha\beta}_{2,\theta}(x) = \left\{ \sum_{p=0}^{\infty} [{}^2\delta^{\alpha\beta}_{2^p,\theta}(x)]^2 \right\}^{1/2} $$

and prove $F^{\alpha\beta}_{2,\theta}(x) \in L^2$. In fact, representation (4.11) and inequality (4.12) help obtain

$$ \begin{aligned} &\int [F^{\alpha,\beta}_{2,\theta}(x)]^2\, d\mu(x) \\ &= \sum_{p=0}^{\infty} \frac{1}{(2^p+1)^2} \sum_{m=0}^{2^p} \sum_{n=0}^{\theta 2^p} \sum_{i=1}^{m} \sum_{k=1}^{n} \left[\frac{A^{\alpha-1}_{m-\nu^0}}{\alpha A^{\alpha}_{m}}\right]^2 \left[\frac{A^{\beta-1}_{n-k}}{\beta A^{\beta}_{n}}\right]^2 i^2k^2a^2_{ik} \\ &= \sum_{p=0}^{\infty} \frac{1}{(2^p+1)^2} \sum_{i=1}^{2^p} \sum_{k=1}^{\theta 2^p} i^2k^2a^2_{ik} \sum_{m=1}^{2^p} \left[\frac{A^{\alpha-1}_{m-\nu^0}}{\alpha A^{\alpha}_{m}}\right]^2 \sum_{n=k}^{\theta 2^p} \left[\frac{A^{\beta-1}_{n-k}}{\beta A^{\beta}_{n}}\right]^2 \\ &= O\{1\} \sum_{p=0}^{\infty} \frac{1}{(2^p+1)^2} \sum_{i=1}^{2^p} \sum_{k=1}^{\theta 2^p} ika^2_{ik} \\ &= O\{1\} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} ika^2_{ik} \sum_{p:2^p \to \max(i,k/\theta)} \frac{1}{(2^p+1)^2} \\ &= O\{1\} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a^2_{ik} < \infty. \end{aligned} $$

Hence B. Levi's theorem implies (7.2).

LEMMA 3. *If* $\alpha > \frac{1}{2}$, $\beta > 0$, $\theta \geqq 1$ *and condition* (1.2) *is satisfied, then*

(7.3) $$ {}^3\delta^{\alpha\beta}_{M,\theta}(x) = \left\{ \frac{1}{(M+1)^2} \sum_{m=0}^{M} \sum_{n=0}^{\theta M} [\sigma^{\alpha-1,\beta}_{mn}(x) - \sigma^{\alpha\beta}_{mn}(x)]^2 \right\}^{1/2} $$

$$= o_x\{1\} \text{ a.e. } as\ M \to \infty.$$

Proof. Since again

$${}^3\delta^{\alpha\beta}_{M,\theta}(x) \leqq 2\ {}^3\delta^{\alpha\beta}_{2^p,\theta}(x)$$

for $2^{p-1} < M \leqq 2^p$ with $p \geqq 0$, instead of (7.3) it is enough to prove

$$(7.4) \quad {}^3\delta^{\alpha\beta}_{2^p,\theta}(x) = o_x\{1\} \text{ a.e. as } p \to \infty.$$

Now we define

$$F^{\alpha\beta}_{3,\theta}(x) = \left\{ \sum_{p=0}^{\infty} [{}^3\delta^{\alpha\beta}_{2^p,\theta}(x)]^2 \right\}^{1/2}.$$

We will prove that $F^{\alpha\beta}_{3,\theta}(x) \in L^2$, whence via B. Levi's theorem (7.4) follows.

To this end, using representation (4.10) we can estimate as follows:

$$F^{\alpha\beta}_{3,\theta}(x) \leqq \left\{ \sum_{p=0}^{\infty} \frac{1}{(2^p+1)^2} \sum_{m=0}^{2^p} \sum_{n=0}^{\theta 2^p} \left[\sum_{i=1}^{m} \sum_{k=0}^{n} \frac{A^{\alpha-1}_{m-i}}{\alpha A^{\alpha}_m} i a_{ik} \phi_{ik}(x) \right]^2 \right\}^{1/2}$$

$$+ \left\{ \sum_{p=0}^{\infty} \frac{1}{(2^p+1)^2} \sum_{m=0}^{2^p} \sum_{n=0}^{\theta 2^p} \left[\sum_{i=1}^{m} \sum_{k=1}^{n} \frac{A^{\beta-1}_{m-i}}{\alpha A^{\alpha}_m} \right.\right.$$

$$\left.\left. \times \left(1 - \frac{A^{\beta}_{n-k}}{A^{\beta}_n}\right) i a_{ik} \phi_{ik}(x) \right]^2 \right\}^{1/2}$$

$$= F^{\alpha\beta}_{4,\theta}(x) + F^{\alpha\beta}_{5,\theta}(x),$$

say. If we prove that both $F^{\alpha\beta}_{4,\theta}(x)$ and $F^{\alpha\beta}_{5,\theta}(x)$ belong to L^2, then we are done.

First, we deal with $F_{4,\theta}(x)$ by using (4.12):

$$(7.5) \quad \int [F^{\alpha\beta}_{4,\theta}(x)]^2 d\mu(x)$$

$$= \sum_{p=0}^{\infty} \frac{1}{(2^p+1)^2} \sum_{m=0}^{2^p} \sum_{n=0}^{\theta 2^p} \sum_{i=1}^{m} \sum_{k=0}^{n} \left[\frac{A^{\alpha-1}_{m-i}}{\alpha A^{\alpha}_m}\right]^2 i^2 a^2_{ik}$$

$$= \sum_{p=0}^{\infty} \frac{1}{(2^p+1)^2} \sum_{i=1}^{2^p} \sum_{k=0}^{\theta 2^p} i^2 a^2_{ik} \sum_{m=i}^{2^p} \left[\frac{A^{\alpha-1}_{m-i}}{\alpha A^{\alpha}_m}\right]^2 \sum_{n=k}^{\theta 2^p} 1$$

$$= O\{1\} \sum_{p=0}^{\infty} \frac{1}{2^p+1} \sum_{i=1}^{2^p} \sum_{k=0}^{\theta 2^p} i a^2_{ik}$$

$$= O\{1\} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} i a_{ik}^2 \sum_{p:2^p \geqq \max(i,k/\theta)} \frac{1}{2^p + 1}$$

$$= O\{1\} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 < \infty.$$

Second, we treat $F_{5,\theta}^{\alpha\beta}(x)$ with the help of (4.12) and (4.13). Proceeding as in the case of (7.5), we get

$$\int [F_{5,\theta}^{\alpha\beta}(x)]^2 d\mu(x)$$

$$= \sum_{p=0}^{\infty} \frac{1}{(2^p + 1)^2} \sum_{m=0}^{2^p} \sum_{n=0}^{\theta 2^p} \sum_{i=1}^{m} \sum_{k=1}^{n} \left[\frac{A_{m-i}^{\alpha-1}}{\alpha A_m^{\alpha}}\right]^2$$

$$\times \left[1 - \frac{A_{n-k}^{\beta}}{A_n^{\beta}}\right]^2 i^2 a_{ik}^2$$

$$\leqq \sum_{p=0}^{\infty} \frac{\theta}{2^p + 1} \sum_{i=1}^{2^p} \sum_{k=1}^{\theta 2^p} i^2 a_{ik}^2 \sum_{m=1}^{2^p} \left[\frac{A_{m-i}^{\alpha-1}}{\alpha A_m^{\alpha}}\right]^2 \sum_{n=k}^{\theta 2^p} \frac{1}{n} \left[1 - \frac{A_{n-k}^{\beta}}{A_n^{\beta}}\right]^2$$

$$= O\{1\} \sum_{p=0}^{\infty} \frac{\theta}{2^p + 1} \sum_{i=1}^{2^p} \sum_{k=1}^{\theta 2^p} i a_{ik}^2 < \infty.$$

The next "almost symmetric" counterpart of Lemma 3 can be derived in a similar way. Therefore, its proof will be omitted.

LEMMA 4. *If* $\alpha > 0$, $\beta > \frac{1}{2}$, $\theta \geqq 1$ *and condition* (1.2) *is satisfied, then*

$$\left\{\frac{1}{(M + 1)^2} \sum_{m=0}^{M} \sum_{n=0}^{\theta M} [\sigma_{mn}^{\alpha,\beta-1}(x) - \sigma_{mn}^{\alpha\beta}(x)]^2\right\}^{1/2} = o_x\{1\} \text{ a.e. as } M \to \infty.$$

8. Extension to multiple case. Let Z_+^d be the set of d-tuple, $k = (k_1, \ldots, k_d)$ with nonnegative integers for coordinates, where d is a fixed positive integer. Let $\{\phi_k(x):k \in Z_+^d\}$ be an orthonormal system on the measure space $(X, \mathcal{F}, \mu)$. We consider the d-multiple orthogonal series

$$(8.1) \qquad \sum_{k \in Z_+^d} a_k \phi_k(x) = \sum_{k_1=0}^{\infty} \ldots \sum_{k_d=0}^{\infty} a_{k_1,\ldots,k_d} \phi_{k_1,\ldots,k_d}(x)$$

where $\{a_k:k \in Z_+^d\}$ is a d-multiple sequence of real numbers for which

$$(8.2) \qquad \sum_{k \in Z_+^d} a_k^2 < \infty.$$

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By the extended Riesz-Fischer theorem there exists a function $f(x) \in L^2$ such that the rectangular partial sums of (8.1) defined by

$$s_n(x) = \sum_{k_1=0}^{n_1} \dots \sum_{k_d=0}^{n_d} a_{k_1,\dots,k_d} \phi_{k_1,\dots,k_d}(x)$$

$$(n = (n_1, \dots, n_d) \in Z_+^d)$$

converge to $f(x)$ in L^2-metric:

$$\int [s_n(x) - f(x)]^2 d\mu(x) \to 0 \text{ as } \min_{1 \leqq j \leqq d} n_j \to \infty.$$

Let $\alpha_1, \dots, \alpha_d$ be real numbers, $\alpha_j > -1$ for each $j = 1, \dots, d$. The $(C, \alpha_1, \dots, \alpha_d)$ – means of series (8.1) are defined by

$$\sigma_{n_1,\dots,n_d}^{\alpha_1,\dots,\alpha_d}(x) = \sum_{k_1=0}^{n_1} \dots \sum_{k_d=0}^{n_d} \left(\prod_{j=1}^{d} \frac{A_{n_j-k_j}^{\alpha_j-1}}{A_{n_j}^{\alpha_j}} \right) s_{k_1,\dots,k_d}(x)$$

$$= \sum_{k_1=0}^{n_1} \dots \sum_{k_d=0}^{n_d} \left(\prod_{j=1}^{d} \frac{A_{n_j-k_j}^{\alpha_j}}{A_{n_j}^{\alpha_j}} \right) a_{k_1,\dots,k_d} \phi_{k_1,\dots,k_d}(x).$$

The extensions of Theorems A and B, Theorem 1 and 2, and Corollary 1, for instance, read as follow.

THEOREM A′. *Under condition* (8.2),

$$s_{2^p,\dots,2^p}(x) - \sigma_{2^p,\dots,2^p}^{1,\dots,1}(x) = o_x\{1\} \text{ a.e.}$$

THEOREM B′. *Under condition* (8.2), *for every* $\theta \geqq 1$

$$\max_{2^p \leqq n_1 \leqq 2^{p+1}} \max_{n_2:\theta^{-1} \leqq n_2/n_1 \leqq \theta} \dots \max_{n_d:\theta^{-1} \leqq n_d/n_1 \leqq \theta} \left| \sigma_{n_1,\dots,n_d}^{1,\dots,1}(x) - \sigma_{2^p,\dots,2^p}^{1,\dots,1}(x) \right| = o_x\{1\} \text{ a.e. } as\ p \to \infty.$$

THEOREM 1′. *If* $\alpha_j > 0$ *for each* $j = 1, \dots, d$, $\theta \geqq 1$ *and the condition*

$$(8.3) \quad \sum_{k_1=0}^{\infty} \dots \sum_{k_d=0}^{\infty} a_{k_1,\dots,k_d}^2 [\log \log (\max_{1 \leqq j \leqq d} k_j + 4)]^2 < \infty$$

is satisfied, then

$$\max_{n_2:\theta^{-1} \leqq n_2/n_1 \leqq \theta} \dots \max_{n_d:\theta^{-1} \leqq n_d/n_1 \leqq \theta} |\sigma_{n_1,\dots,n_d}^{\alpha_1 \dots \alpha_d}(x) - f(x)| = o_x\{1\} \text{ a.e. } as\ n_1 \to \infty.$$

THEOREM 2′. *If* $\alpha_j > 0$ *for each* $j = 1, \ldots, d$, $\theta \geqq 1$ *and condition* (8.2) *is satisfied, then*

$$\max_{n_2:\theta^{-1}\leqq n_2/n_1\leqq\theta} \ldots \max_{n_d:\theta^{-1}\leqq n_d/n_1\leqq\theta} |\sigma^{\alpha_1\ldots\alpha_d}_{n_1,\ldots,n_d}(x)| = o_x\{\log\log(n_1 + 4)\} \text{ a.e.}$$

COROLLARY 1′. *If* $\alpha_j > 0$ *for each* $j = 1, \ldots, d$, $\theta \geqq 1$ *and condition* (8.3) *is satisfied, then*

$$\left\{\frac{1}{(M+1)^d} \sum_{n_1=0}^{M} \sum_{n_2=0}^{\theta M} \ldots \sum_{n_d=0}^{\theta M} [\sigma^{\alpha_1\alpha_2\ldots\alpha_d}_{n_1,n_2,\ldots,n_d}(x) - f(x)]^2\right\}^{1/2} = o_x\{1\} \text{ a.e. as } M \to \infty.$$

Of course, the corresponding extensions of Theorem 3 and Corollaries 2 and 3 are also true.

The proofs of these extensions can be carried out in a similar fashion to those of Theorems A, B, 1, 2, 3 and Corollaries 1, 2, 3, but the technical details become more complicated.

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