Sixth Meeting, 8*th May*, 1903.

Mr Charles Tweedie in the Chair.

On the convergents to a recurring continued fraction, with application to finding integral solutions of the equation $x^2 - Cy^2 = (-1)^n D_n$.

By Alexander Holm, M.A.

1. Let $$\frac{E+\sqrt{C}}{D} = a_1 + \frac{1}{a_2+} \cdots\cdots \frac{1}{a_k+} \frac{\dot{1}}{\dot{a}_1+} \frac{1}{a_2+} \cdots\cdots \frac{1}{\dot{a}_c+} \cdots\cdots \quad (1)$$

where D and E are integers, and C a positive integer not a perfect square, k being the number of partial quotients in the non-recurring part of the continued fraction, and c the number in the cycle,

and let $$\frac{E'+\sqrt{C}}{D'} = \dot{a}_1 + \frac{1}{a_2+} \cdots\cdots \frac{1}{\dot{a}_c+} \cdots\cdots \qquad - \quad - \quad (2)$$

D′ and E′ being integers.

If $\frac{p_s}{q_s}$ and $\frac{P_s}{Q_s}$ are the sth convergents to the continued fractions (1) and (2),

then $$\frac{p_{k+mc+s}}{q_{k+mc+s}} = a_1 + \frac{1}{a_2+} \cdots \frac{1}{a_k+} \frac{1}{a_1+} \frac{1}{a_2+} \cdots \frac{1}{a_{mc}+} \frac{1}{a_1+} \frac{1}{a_2+} \cdots \frac{1}{a_s}$$

$$= a_1 + \frac{1}{a_2+} \cdots \frac{1}{a_k+} \frac{1}{a_1+} \frac{1}{a_2+} \cdots \frac{1}{a_{mc}+} \frac{1}{\frac{P_s}{Q_s}}$$

$$= \frac{\frac{P_s}{Q_s} p_{k+mc} + p_{k+mc-1}}{\frac{P_s}{Q_s} q_{k+mc} + q_{k+mc-1}}.$$

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Put $r = k + mc$;

$$\therefore \quad \frac{p_{r+s}}{q_{r+s}} = \frac{p_r P_s + p_{r-1} Q_s}{q_r P_s + q_{r-1} Q_s}.$$

The numerator of this fraction is prime to the denominator; *

$$\therefore \quad \left. \begin{aligned} p_{r+s} &= p_r P_s + p_{r-1} Q_s \\ q_{r+s} &= q_r P_s + q_{r-1} Q_s \end{aligned} \right\} \quad \text{- - - - -} \quad (3).$$

Again

$$\frac{E + \sqrt{C}}{D} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_k +} \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots \frac{1}{a_{mc} +} \frac{1}{\dot{a}_1 +} \frac{1}{a_2 +} \cdots \frac{1}{\dot{a}_c +} \cdots$$

$$= a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_k +} \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots \frac{1}{a_{mc} +} \frac{1}{\dfrac{E' + \sqrt{C}}{D'}}$$

$$= \frac{\dfrac{E' + \sqrt{C}}{D'} p_{k+mc} + p_{k+mc-1}}{\dfrac{E' + \sqrt{C}}{D'} q_{k+mc} + q_{k+mc-1}};$$

$$\therefore \quad \frac{E + \sqrt{C}}{D} = \frac{\dfrac{E' + \sqrt{C}}{D'} p_r + p_{r-1}}{\dfrac{E' + \sqrt{C}}{D'} q_r + q_{r-1}}.$$

From this eliminate p_{r-1} and q_{r-1} by means of (3).

$$\therefore \quad \frac{E + \sqrt{C}}{D} = \frac{\dfrac{E' + \sqrt{C}}{D'} p_r Q_s + p_{r+s} - p_r P_s}{\dfrac{E' + \sqrt{C}}{D'} q_r Q_s + q_{r+s} - q_r P_s}$$

$$= \frac{p_{r+s} - p_r \left(P_s - \dfrac{E' + \sqrt{C}}{D'} Q_s \right)}{q_{r+s} - q_r \left(P_s - \dfrac{E' + \sqrt{C}}{D'} Q_s \right)};$$

$$\therefore \quad p_{r+s} - \frac{E + \sqrt{C}}{D} q_{r+s} = \left(p_r - \frac{E + \sqrt{C}}{D} q_r \right) \left(P_s - \frac{E' + \sqrt{C}}{D'} Q_s \right).$$

The above is similar to Serret, pp. 60–61, except that the period

* See Serret, *Alg. Sup.*, 4me éd. t. i., p. 61.

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has been made to begin at the first quotient of the cycle instead of at any quotient.

Now $r = k + mc$;

$$\therefore \quad p_{k+mc+s} - \frac{E+\sqrt{C}}{D} q_{k+mc+s}$$

$$= \left(p_{k+mc} - \frac{E+\sqrt{C}}{D} q_{k+mc}\right)\left(P_s - \frac{E'+\sqrt{C}}{D'} Q_s\right) \quad (4)$$

and in particular when $m = 0$,

$$p_{k+s} - \frac{E+\sqrt{C}}{D} q_{k+s} = \left(p_k - \frac{E+\sqrt{C}}{D} q_k\right)\left(P_s - \frac{E'+\sqrt{C}}{D'} Q_s\right) \quad (5).$$

Then let $s = c$ in (4).

$$\therefore \quad p_{k+(m+1)c} - \frac{E+\sqrt{C}}{D} q_{k+(m+1)c}$$

$$= \left(p_{k+mc} - \frac{E+\sqrt{C}}{D} q_{k+mc}\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)$$

$$= \left(p_{k+(m-1)c} - \frac{E+\sqrt{C}}{D} q_{k+(m-1)c}\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^2 \text{ similarly.}$$

....

$$= \left(p_k - \frac{E+\sqrt{C}}{D} q_k\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^{m+1};$$

or writing m for $m+1$,

$$p_{k+mc} - \frac{E+\sqrt{C}}{D} q_{k+mc} = \left(p_k - \frac{E+\sqrt{C}}{D} q_k\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m \quad (6).$$

Substitute this in (4).

$$\therefore \quad p_{k+mc+s} - \frac{E+\sqrt{C}}{D} q_{k+mc+s}$$

$$= \left(p_k - \frac{E+\sqrt{C}}{D} q_k\right)\left(P_s - \frac{E'+\sqrt{C}}{D'} Q_s\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m$$

$$= \left(p_{k+s} - \frac{E+\sqrt{C}}{D} q_{k+s}\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m \quad \text{by (5).}$$

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Now put $k+s=n$ so that $n \nless k$;

$$\therefore \; p_{mc+n} - \frac{E+\sqrt{C}}{D} q_{mc+n} = \left(p_n - \frac{E+\sqrt{C}}{D} q_n\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m \quad (7)$$

where $m = 1, 2, 3, \ldots\ldots$ and $n \nless k$.

In this way we can dispense with the tedious demonstration of Serret, § 28, to prove that what corresponds to $P_c - \frac{E'+\sqrt{C}}{D'} Q_c$ in (7) is constant, no matter at what quotient of the cycle the period is made to begin.

Example : $\frac{123+\sqrt{37}}{28} = 4 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\dot{1}+}\,\frac{1}{3+}\,\frac{1}{\dot{2}+}\,\ldots\ldots$

$c=3$, and the first five convergents are $\frac{4}{1}, \frac{5}{1}, \frac{9}{2}, \frac{14}{3}, \frac{23}{5}$,

and $\frac{3+\sqrt{37}}{7} = \dot{1} + \frac{1}{3+}\,\frac{1}{\dot{2}+}\,\ldots\ldots$

the first three convergents being $\frac{1}{1}, \frac{4}{3}, \frac{9}{7}$.

Putting $n=5$, $m=2$ in (7) we have

$$p_{11} - \frac{123+\sqrt{37}}{28} q_{11} = \left(23 - \frac{123+\sqrt{37}}{28} \times 5\right)\left(9 - \frac{3+\sqrt{37}}{7} \times 7\right)^2$$

$$= \frac{4337 - 713\sqrt{37}}{28}$$

$$\therefore \; q_{11} = 713$$

$$\text{and} \quad p_{11} = \frac{123}{28} \times 713 + \frac{4337}{28} = 3287.$$

2. **Particular case of a pure recurring continued fraction.**

Let $\qquad \frac{E'+\sqrt{C}}{D'} = \dot{a}_1 + \frac{1}{a_2+} \cdots\cdots \frac{1}{\dot{a}_c+} \cdots\cdots.$

Here $\qquad D = D', \quad E = E',$

$\qquad p_s = P_s, \quad q_s = Q_s$;

$\therefore$ by (7) we have

$$P_{mc+n} - \frac{E'+\sqrt{C}}{D'} Q_{mc+n} = \left(P_n - \frac{E'+\sqrt{C}}{D'} Q_n\right)\left(P_c - \frac{E'+\sqrt{C}}{D'} Q_c\right)^m \quad (8).$$

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In particular let $n=c$, and then write m for $m+1$.

$$\therefore \quad P_{mc}-\frac{E'+\sqrt{C}}{D'}Q_{mc}=\left(P_c-\frac{E'+\sqrt{C}}{D'}Q_c\right)^m \qquad (9).$$

3. To connect the convergents of the continued fraction representing $-\frac{E'-\sqrt{C}}{D'}$ with those of the continued fraction representing $\frac{E'+\sqrt{C}}{D'}$, when the latter is a pure recurring continued fraction.

Let $$x=\frac{E'+\sqrt{C}}{D'}=\dot{a}_1+\frac{1}{a_2+}\cdots\cdots\frac{1}{\dot{a}_c+}\cdots\cdots$$

$$\therefore \quad x=a_1+\frac{1}{a_2+}\cdots\cdots\frac{1}{a_{mc}+}\frac{1}{x};$$

$$\therefore \quad x=\frac{xP_{mc}+P_{mc-1}}{xQ_{mc}+Q_{mc-1}};$$

$$\therefore \quad Q_{mc}x^2-(P_{mc}-Q_{mc-1})x-P_{mc-1}=0 \qquad (10)$$

$\therefore \quad \frac{E'+\sqrt{C}}{D'}$ is one of the roots of this quadratic equation.

The other root $$\frac{E'-\sqrt{C}}{D'}=-\frac{1}{\dot{a}_c+}\frac{1}{a_{c-1}+}\cdots\cdots\frac{1}{a_2+}\frac{1}{\dot{a}_1+}\cdots\cdots \qquad (11).$$

(See Serret, p. 49.)

$$\therefore \quad -\frac{E'-\sqrt{C}}{D'}=\frac{1}{\dot{a}_{mc}+}\frac{1}{a_{mc-1}+}\cdots\cdots\frac{1}{a_2+}\frac{1}{\dot{a}_1+}\cdots\cdots.$$

Let $\frac{P'_r}{Q'_r}$ be the rth convergent.

Then if $n<mc$, $$\frac{P'_{mc-n}}{Q'_{mc-n}}=\frac{1}{a_{mc}+}\frac{1}{a_{mc-1}+}\cdots\cdots\frac{1}{a_{n+2}+}\frac{1}{a_{n+1}};$$

$$\therefore \quad \frac{Q'_{mc-n}}{Q'_{mc-n-1}}=a_{n+1}+\frac{1}{a_{n+2}+}\cdots\cdots\cdots\cdots\frac{1}{a_{mc}},$$

and $$\frac{P'_{mc-n}}{P'_{mc-n-1}}=a_{n+1}+\frac{1}{a_{n+2}+}\cdots\cdots\cdots\cdots\frac{1}{a_{mc-1}}.$$

Now $$\frac{P_{mc}}{Q_{mc}} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n +} \frac{1}{a_{n+1} +} \frac{1}{a_{n+2} +} \cdots\cdots \frac{1}{a_{mc}}$$

$$= a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_n +} \frac{1}{\dfrac{Q'_{mc-n}}{Q'_{mc-n-1}}}$$

$$= \frac{\dfrac{Q'_{mc-n}}{Q'_{mc-n-1}} P_n + P_{n-1}}{\dfrac{Q'_{mc-n}}{Q'_{mc-n-1}} Q_n + Q_{n-1}}$$

$$= \frac{P_n Q'_{mc-n} + P_{n-1} Q'_{mc-n-1}}{Q_n Q'_{mc-n} + Q_{n-1} Q'_{mc-n-1}}.$$

The numerator is prime to the denominator.

$$\therefore \quad P_n Q'_{mc-n} + P_{n-1} Q'_{mc-n-1} = P_{mc}$$

$$\text{and} \quad Q_n Q'_{mc-n} + Q_{n-1} Q'_{mc-n-1} = Q_{mc}.$$

Eliminate Q'_{mc-n}.

$$\therefore \quad (P_n Q_{n-1} - P_{n-1} Q_n) Q'_{mc-n-1} = P_n Q_{mc} - Q_n P_{mc};$$

$$\therefore \quad (-1)^n Q'_{mc-n-1} = P_n Q_{mc} - Q_n P_{mc}.$$

Again $$\frac{P_{mc-1}}{Q_{mc-1}} = a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_n +} \frac{1}{a_{n+1} +} \frac{1}{a_{n+2} +} \cdots\cdots \frac{1}{a_{mc-1}}.$$

$$= a_1 + \frac{1}{a_2 +} \cdots\cdots\cdots \frac{1}{a_n +} \frac{1}{\dfrac{P'_{mc-n}}{P'_{mc-n-1}}},$$

and in the same way as above we find

$$(-1)^n P_{mc-n-1} = P_n Q_{mc-1} - Q_n P_{mc-1};$$

$$\therefore \quad (-1)^n \left(P'_{mc-n-1} + \frac{E' + \sqrt{C}}{D} Q'_{mc-n-1}\right)$$

$$= P_n \left(Q_{mc-1} + \frac{E' + \sqrt{C}}{D'} Q_{mc}\right) - Q_n \left(P_{mc-1} + \frac{E' + \sqrt{C}}{D'} P_{mc}\right)$$

Now

$$\frac{E + \sqrt{C}}{D'} + \frac{E' - \sqrt{C}}{D} = \text{sum of roots of the quadratic (10)} = \frac{P_{mc} - Q_{mc-1}}{Q_{mc}};$$

$$\therefore \quad Q_{mc-1} + \frac{E' + \sqrt{C}}{D} Q_{mc} = P_{mc} - \frac{E' - \sqrt{C}}{D} Q_{mc}$$

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and $\frac{E'+\sqrt{C}}{D}\cdot\frac{E'-\sqrt{C}}{D}=$ product of roots of the quadratic $=\frac{-P_{mc-1}}{Q_{mc}}$;

$$\therefore\quad P_{mc-1}+\frac{E'+\sqrt{C}}{D}P_{mc}=\frac{E'+\sqrt{C}}{D'}\left(P_{mc}-\frac{E\quad\sqrt{C}}{D}Q_{mc}\right);$$

$$\therefore\quad(-1)^n\left(P'_{mc-n-1}+\frac{E+\sqrt{C}}{D}Q'_{mc-n-1}\right)$$

$$=\left(P_n-\frac{E'+\sqrt{C}}{D}Q_n\right)\left(P_{mc}-\frac{E'-\sqrt{C}}{D}Q_{mc}\right).$$

Again $\left(P_{mc}-\frac{E'+\sqrt{C}}{D'}Q_{mc}\right)\left(P_{mc}-\frac{E'-\sqrt{C}}{D'}Q_{mc}\right)$

$$=P_{mc}^2-\left(\frac{E'+\sqrt{C}}{D}+\frac{E'-\sqrt{C}}{D'}\right)P_{mc}Q_{mc}+\frac{E'+\sqrt{C}}{D'}\cdot\frac{E'-\sqrt{C}}{D'}Q_{mc}^2$$

$$=P_{mc}^2-\frac{P_{mc}-Q_{mc-1}}{Q_{mc}}\cdot P_{mc}Q_{mc}-\frac{P_{mc-1}}{Q_{mc}}\cdot Q_{mc}^2$$

$$=P_{mc}Q_{mc-1}-P_{mc-1}Q_{mc}$$

$$=(-1)^{mc};$$

$\therefore$ by (9) we have

$$\left(P_c-\frac{E'+\sqrt{C}}{D'}Q_c\right)^m\left(P_{mc}-\frac{E'-\sqrt{C}}{D'}Q_{mc}\right)=(-1)^{mc};$$

$$\therefore\quad P_{mc}-\frac{E'-\sqrt{C}}{D'}Q_{mc}=(-1)^{mc}\left(P_c-\frac{E'+\sqrt{C}}{D'}Q_c\right)^{-m};$$

$$\therefore\quad P'_{mc-n-1}+\frac{E'+\sqrt{C}}{D}Q_{mc-n-1}$$

$$=(-1)^{mc+n}\left(P_n-\frac{E'+\sqrt{C}}{D}Q_n\right)\left(P_c-\frac{E+\sqrt{C}}{D}Q_c\right)^{-m}\qquad(12).$$

The relations connecting the convergents of $-\frac{E-\sqrt{C}}{D}$ with those of $\frac{E'+\sqrt{C}}{D'}$ are thus established directly.

Comparing (12) with (8) we see that the power of $P_c-\frac{E'+\sqrt{C}}{D'}Q_c$ is $-m$ instead of m.

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Example, $\frac{3+\sqrt{37}}{7} = \dot{1} + \frac{1}{3+}\ \frac{1}{\dot{2}+}\ \cdots\cdots$

$c = 3$, and the first three convergents are $\frac{1}{1}, \frac{4}{3}, \frac{9}{7}$.

Taking $n = 2$, $m = 3$ in (12) we have

$$P'_6 + \frac{3+\sqrt{37}}{7}Q'_6 = (-1)^{11}\left(4 - \frac{3+\sqrt{37}}{7}\times 3\right)\left(9 - \frac{3+\sqrt{37}}{7}\times 7\right)^{-3}$$

$$= \frac{663 + 109\sqrt{37}}{7};$$

$$\therefore \quad Q'_6 = 109;$$

$$\therefore \quad P'_6 \quad = -\frac{3}{7}\times 109 + \frac{663}{7} = 48.$$

Thus $\frac{48}{109}$ is the sixth convergent to

$$-\frac{3-\sqrt{37}}{7} = \frac{1}{2+}\ \frac{1}{3+}\ \frac{1}{\dot{1}+}\ \cdots\cdots.$$

4. For a pure quadratic surd, let

$$\frac{\sqrt{C}}{D} = a + \frac{1}{\dot{a}_1+}\ \frac{1}{a_2+}\ \cdots\cdots \frac{1}{a_{c-1}+}\ \frac{1}{\dot{2a}+}\ \cdots\cdots$$

$\frac{p_r}{q_r}$ being the rth convergent.

Then $\frac{aD+\sqrt{C}}{D} = a + \frac{\sqrt{C}}{D} = \dot{2a} + \frac{1}{a_1+}\ \frac{1}{a_2+}\ \cdots\cdots \frac{1}{\dot{a}_{c-1}+}\ \cdots\cdots$

$\therefore\ \frac{aD+\sqrt{C}}{D}$ is represented by a pure recurring continued fraction,

and if $\frac{P_r}{Q_r}$ is the rth convergent, by (8) we have

$$P_{mc+n} - \frac{aD+\sqrt{C}}{D}Q_{mc+n} = \left(P_n - \frac{aD+\sqrt{C}}{D}Q_n\right)\left(P_c - \frac{aD+\sqrt{C}}{D}Q_c\right)^m.$$

Now $\frac{P_r}{Q_r} = 2a + \frac{1}{a_1+}\ \frac{1}{a_2+}\ \cdots\cdots \frac{1}{a_{r-1}} = a + \frac{p_r}{q_r} = \frac{p_r + aq_r}{q_r}.$

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The numerator is prime to the denominator.

$$\therefore \quad P_r = p_r + aq_r \text{ and } Q_r = q_r;$$

$$\therefore \quad p_{mc+n} + aq_{mc+n} - \frac{aD + \sqrt{C}}{D} q_{mc+n}$$

$$= \left(p_n + aq_n - \frac{aD + \sqrt{C}}{D} q_n\right)\left(p_c + aq_c - \frac{aD + \sqrt{C}}{D} q_c\right)^m;$$

$$\therefore \quad p_{mc+n} - \frac{\sqrt{C}}{D} q_{mc+n} = \left(p_n - \frac{\sqrt{C}}{D} q_n\right)\left(p_c - \frac{\sqrt{C}}{D} q_c\right)^m \quad . \quad (13).$$

Example $$\frac{\sqrt{80}}{5} = 1 + \frac{1}{\dot{1}+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{\dot{2}+} \cdots\cdots$$

$c = 4$, and the first four convergents are $\frac{1}{1}, \frac{2}{1}, \frac{7}{4}, \frac{9}{5}$.

Taking $n = 3$ and $m = 2$ in (13) we have

$$p_{11} - \frac{\sqrt{80}}{5} q_{11} = \left(7 - \frac{\sqrt{80}}{5} \times 4\right)\left(9 - \frac{\sqrt{80}}{5} \times 5\right)^2 = 2279 - \frac{1274\sqrt{80}}{5};$$

$$\therefore \quad p_{11} = 2279 \text{ and } q_{11} = 1274.$$

Again, since $$\frac{aD + \sqrt{C}}{D} = \dot{2a} + \frac{1}{a_1+}\ \frac{1}{a_2+} \cdots\cdots \frac{1}{\dot{a}_{mc-1}+} \cdots\cdots$$

$$\therefore \text{ by (11)} \quad -\frac{aD - \sqrt{C}}{D} = \frac{1}{\dot{a}_{mc-1}+}\ \frac{1}{a_{mc-2}+} \cdots \frac{1}{a_2+}\ \frac{1}{a_1+}\ \frac{1}{\dot{2a}+} \cdots (14)$$

$$\therefore \quad \frac{\sqrt{C}}{D} = a + \frac{1}{\dot{a}_{mc-1}+}\ \frac{1}{a_{mc-2}+} \cdots \frac{1}{a_2+}\ \frac{1}{a_1+}\ \frac{1}{\dot{2a}+} \cdots (15).$$

Let $\frac{p_r}{q_r}$ and $\frac{P'_r}{Q'_r}$ be the rth convergents to the continued fractions (15) and (14).

Then by (12) we have

$$P'_{mc-n-1} + \frac{aD + \sqrt{C}}{D} Q'_{mc-n-1}$$

$$= (-1)^{mc+n}\left(P_n - \frac{aD + \sqrt{C}}{D} Q_n\right)\left(P_c - \frac{aD + \sqrt{C}}{D} Q_c\right)^{-m}.$$

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Now $\dfrac{p_{r+1}}{q_{r+1}} = a + \dfrac{1}{a_{mc-1}+} \dfrac{1}{a_{mc-2}+} \cdots\cdots \dfrac{1}{a_{mc-r}} = a + \dfrac{P'_r}{Q'_r} = \dfrac{P'_r + aQ'_r}{Q'_r}$.

The numerator is prime to the denominator.

$$\therefore \quad Q'_r = q_{r+1} \text{ and } P'_r + aQ'_r = p_{r+1};$$

$$\therefore \quad P'_r = p_{r+1} - aq_{r+1};$$

$$\therefore \quad p_{mc-n} - aq_{mc-n} + \frac{aD + \sqrt{C}}{D} q_{mc-n}$$

$$= (-1)^{mc+n}\left(p_n + aq_n - \frac{aD + \sqrt{C}}{D} q_n\right)\left(p_c + aq_c - \frac{aD + \sqrt{C}}{D} q_c\right)^{-m};$$

$$\therefore \quad p_{mc-n} + \frac{\sqrt{C}}{D} q_{mc-n} = (-1)^{mc+n}\left(p_n - \frac{\sqrt{C}}{D} q_n\right)\left(p_c - \frac{\sqrt{C}}{D} q_c\right)^{-m} \qquad (16).$$

Example $\qquad \dfrac{\sqrt{80}}{5} = 1 + \dfrac{1}{\dot{1}+} \dfrac{1}{3+} \dfrac{1}{1+} \dfrac{1}{\dot{2}+} \cdots\cdots .$

Taking $n = 3$, $m = 2$ in (16) we find

$$p_5 + \frac{\sqrt{80}}{5} q_5 = (-1)^{11}\left(7 - \frac{\sqrt{80}}{5} \times 4\right)\left(9 - \frac{\sqrt{80}}{5} \times 5\right)^{-2} = 25 + \frac{14\sqrt{80}}{5};$$

$$\therefore \quad p_5 = 25 \text{ and } q_5 = 14.$$

The formulae (13) and (16) could be obtained independently, and on account of their importance in the solution of indeterminate equations of the second degree, it might be advisable to prove them directly.

5. To find integral solutions of $x^2 - Cy^2 = (-1)^n D_n$, where D_n is the $(n+1)$th divisor in the development of $\sqrt{C}$ as a continued fraction.

(i) when c, the number of quotients in the cycle of $\sqrt{C}$ is even.

Taking $D = 1$ in (13) we have

$$p_{mc+n} - \sqrt{C}.q_{mc+n} = (p_n - \sqrt{C}.q_n)(p_c - \sqrt{C}.q_c)^m.$$

Now p_{mc+n}, q_{mc+n}, p_n, q_n, p_c, q_c are all rational, whereas $\sqrt{C}$ is irrational, hence we may change the sign of $\sqrt{C}$.

$$\therefore \quad p_{mc+n} + \sqrt{C}.q_{mc+n} = (p_n + \sqrt{C}.q_n)(p_c + \sqrt{C}.q_c)^m.$$

On multiplying we obtain

$$p_{mc+n}^2 - Cq_{mc+n}^2 = (p_n^2 - Cq_n^2)(p_c^2 - Cq_c^2)^m$$
$$= (-1)^n D_n\{(-1)^c D_c\}^m$$
$$= (-1)^n D_n,$$

since c is even, and $D_c = 1$.

Similarly from (16) we get

$$p_{mc-n}^2 - Cq_{mc-n}^2 = (-1)^n D_n.$$

$\therefore$ integral solutions of $x^2 - Cy^2 = (-1)^n D_n$

are $\left.\begin{matrix} x = \pm p_{mc+n} \\ y = \pm q_{mc+n} \end{matrix}\right\}$ and $\left.\begin{matrix} x = \pm p_{mc-n} \\ y = \mp q_{mc-n} \end{matrix}\right\}$;

$$\therefore \quad x - y\sqrt{C} = \pm(p_{mc+n} - \sqrt{C}.q_{mc+n}) \text{ or } \pm(p_{mc-n} + \sqrt{C}.q_{mc-n})$$
$$= \pm(p_n - \sqrt{C}.q_n)(p_c - \sqrt{C}.q_c)^{\pm m} \qquad \text{by (13) and (16).}$$

$\therefore$ integral solutions of $x^2 - Cy^2 = (-1)^n D_n$ are furnished by

$$x - y\sqrt{C} = \pm(p_n - q_n\sqrt{C})(p_c - q_c\sqrt{C})^m \quad - \quad - \quad (17)$$

where m is zero, or any integer positive or negative.

Example: Find positive integral solutions of $x^2 - 7y^2 = -3$.

We have $\sqrt{7} = 2 + \frac{1}{\dot{1}+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{\dot{4}+} \ldots\ldots$

$c = 4$, and the first four convergents are $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}$.

When $n = 1$, $\quad p_n^2 - 7q_n^2 = 2^2 - 7 \times 1^2 = -3$;

$\therefore$ by (17) we have

$$x - y\sqrt{7} = \pm(2 - \sqrt{7})(8 - 3\sqrt{7})^m.$$

Taking $\quad m = 0, \ -1, \ +1, \ -2, \ +2 \ldots\ldots$

we find $\left.\begin{matrix} x = 2 \\ y = 1 \end{matrix}\right\} \left.\begin{matrix} 5 \\ 2 \end{matrix}\right\} \left.\begin{matrix} 37 \\ 14 \end{matrix}\right\} \left.\begin{matrix} 82 \\ 31 \end{matrix}\right\} \left.\begin{matrix} 590 \\ 223 \end{matrix}\right\} \ldots\ldots$

(ii) When c, the number of partial quotients in the cycle of $\sqrt{C}$ is odd.

$$(-1)^c = -1, \text{ but if } m \text{ is even } \{(-1)^c\}^m = +1;$$

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and proceeding in the same way as above we see that integral solutions of $x^2 - Cy^2 = (-1)^n D_n$ are given by

$$x - y\sqrt{C} = \pm(p_n - q_n\sqrt{C})(p_c - q_c\sqrt{C})^{2m} \quad . \quad . \quad (18)$$

where m is zero, or any integer positive or negative.

Example: Find positive integral solutions of $x^2 - 13y^2 = +3$.

We have $$\sqrt{13} = 3 + \frac{1}{\dot{1}+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{\dot{6}+}\ \cdots\cdots$$

$c = 5$, and the first five convergents are $\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}$;

when $n = 2$, $\quad p_n^2 - 13q_n^2 = 4^2 - 13 \times 1^2 = +3$;

$\therefore$ by (18) we have

$$x - y\sqrt{13} = \pm(4 - \sqrt{13})(18 - 5\sqrt{13})^{2m}.$$

Taking $\quad m = 0, \quad -1, \quad +1 \ldots\ldots$

we find $\quad \left.\begin{matrix} x = 4 \\ y = 1 \end{matrix}\right\} \left.\begin{matrix} 256 \\ 71 \end{matrix}\right\} \left.\begin{matrix} 4936 \\ 1369 \end{matrix}\right\} \ldots\ldots$

6. Two lemmas on continued fractions.

Lemma 1. If x lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$,

where $\frac{p_n}{q_n}$ is the preceding convergent to $\frac{p_{n+1}}{q_{n+1}}$ when converted into a continued fraction, then $\frac{p_n}{q_n}$ is a convergent to the continued fraction which represents x.

For, let $$\frac{p_{n+1}}{q_{n+1}} = a_1 + \frac{1}{a_2+}\ \cdots\cdots\cdots \frac{1}{a_{n+1}};$$

then by supposition x lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$;

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$$\therefore \quad \frac{p_n}{q_n} \sim x < \frac{p_n}{q_n} \sim \frac{p_{n+1}}{q_{n+1}}$$

$$< \frac{p_n}{q_n} \sim \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}}$$

$$< \frac{p_n q_{n-1} \sim p_{n-1} q_n}{q_n(a_{n+1}q_n + q_{n-1})}$$

$$< \frac{1}{q_n(a_{n+1}q_n + q_{n-1})}$$

$$< \frac{1}{q_n(q_n + q_{n-1})}, \qquad \text{since } a_{n+1} \nless 1.$$

$\therefore \frac{p_n}{q_n}$ is a convergent to the continued fraction which represents x.*

Lemma 2. If $x = \frac{Mx' + M'}{Nx' + N'}$,

where x' is one of the complete quotients in the development of x as a continued fraction, and M, N, M′, N′ integers all of the same sign, such that $MN' - M'N = \pm 1$

and $M > M'$, $N > N'$ in absolute magnitude,

then $\frac{M'}{N'}$ and $\frac{M}{N}$ are consecutive convergents to the continued fraction which represents x.

We may suppose M, N, M′, N′ to be all positive; for if they were all negative, they could be made all positive by changing the signs in the numerator and the denominator.

Then from the above conditions it can be proved, as in Serret, p. 36, that $\frac{M'}{N'}$ is the preceding convergent to $\frac{M}{N}$ when converted into a continued fraction.

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* See *Chrystal's Algebra*, Vol. 2, Chap. XXXII., § 9, Cor. 4.

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$$\text{Now} \qquad \frac{M}{N} \sim x = \frac{M}{N} \sim \frac{Mx' + M'}{Nx' + N'}$$

$$= \frac{MN' \sim M'N}{N(Nx' + N')}$$

$$= \frac{1}{N(Nx' + N')};$$

$$\therefore \quad \frac{M}{N} \sim x < \frac{1}{N(N + N')}, \qquad \text{since } x' > 1.$$

$\therefore \frac{M}{N}$ is a convergent to x, the preceding convergent being $\frac{M'}{N'}$.*

7. To connect the convergents of the continued fraction representing $\frac{E - \sqrt{C}}{D}$ with those of the continued fraction representing $\frac{E + \sqrt{C}}{D}$.

$$\text{Let } \frac{E + \sqrt{C}}{D} = a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_k +} \frac{1}{\dot{a}_1 +} \frac{1}{a_2 +} \cdots\cdots \frac{1}{\dot{a}_c +} \cdots\cdots$$

$$= a_1 + \frac{1}{a_2 +} \cdots\cdots \frac{1}{a_k +} \frac{1}{\dfrac{E' + \sqrt{C}}{D'}};$$

$$\therefore \quad \frac{E + \sqrt{C}}{D} = \frac{\dfrac{E' + \sqrt{C}}{D'} p_k + p_{k-1}}{\dfrac{E' + \sqrt{C}}{D'} q_k + q_{k-1}} \quad - \quad - \quad - \quad - \quad - \quad (19)$$

and, changing the sign of $\sqrt{C}$ we have

$$\frac{E - \sqrt{C}}{D} = \frac{\dfrac{E' - \sqrt{C}}{D'} p_k + p_{k-1}}{\dfrac{E' - \sqrt{C}}{D'} q_k + q_{k-1}} = \frac{-\dfrac{E' - \sqrt{C}}{D'} p_k - p_{k-1}}{-\dfrac{E' - \sqrt{C}}{D'} q_k - q_{k-1}} \quad - \quad (20)$$

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* See *Chrystal's Algebra*, Vol. 2, Chap. XXXII., §9, Cor. 4.

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Now let

$$-\frac{E'-\sqrt{C}}{D'}=\frac{1}{a_{mc}+}\frac{1}{a_{mc-1}+}\cdots\cdots\frac{1}{a_{mc-(r-1)}+}\frac{1}{\dfrac{E''+\sqrt{C}}{D''}}$$

$$=\frac{\dfrac{E''+\sqrt{C}}{D''}P'_r+P'_{r-1}}{\dfrac{E''+\sqrt{C}}{D''}Q'_r+Q'_{r-1}}, \qquad \text{(where } r>1\text{)}.$$

Substitute in (20) and reduce

$$\therefore \frac{E-\sqrt{C}}{D}=\frac{\dfrac{E''+\sqrt{C}}{D''}(p_kP'_r-p_{k-1}Q'_r)+(p_kP'_{r-1}-p_{k-1}Q'_{r-1})}{\dfrac{E''+\sqrt{C}}{D''}(q_kP'_r-q_{k-1}Q'_r)+(q_kP'_{r-1}-q_{k-1}Q'_{r-1})}$$

$$=\frac{\dfrac{E''+\sqrt{C}}{D''}M+M'}{\dfrac{E''+\sqrt{C}}{D''}N+N'} \quad \text{(say)} \quad . \quad . \quad . \quad . \quad (21).$$

$\dfrac{E''+\sqrt{C}}{D''}$ is one of the complete quotients of $-\dfrac{E'-\sqrt{C}}{D'}$

or of $\dfrac{E-\sqrt{C}}{D}$, since $\dfrac{E-\sqrt{C}}{D}$ and $-\dfrac{E'-\sqrt{C}}{D'}$

terminate by the same quotients. (Serret, p. 49.)

The conditions that M, N, M′, N′, may be all of the same sign, are that $\dfrac{p_{k-1}}{p_k}$ and $\dfrac{q_{k-1}}{q_k}$ shall not lie between $\dfrac{P'_{r-1}}{Q'_{r-1}}$ and $\dfrac{P'_r}{Q'_r}$ and these conditions are fulfilled.

For if $\dfrac{p_{k-1}}{p_k}$ and $\dfrac{q_{k-1}}{q_k}$ lay between $\dfrac{P'_{r-1}}{Q'_{r-1}}$ and $\dfrac{P'_r}{Q'_r}$,

then by Lemma 1, $\dfrac{P'_{r-1}}{Q'_{r-1}}$ would be a convergent to $\dfrac{p_{k-1}}{p_k}$ and $\dfrac{q_{k-1}}{q_k}$.

Now $$\frac{p_k}{q_k}=a_1+\frac{1}{a_2+}\cdots\cdots\cdots\frac{1}{a_{k-1}+}\frac{1}{a_k};$$

$$\therefore \frac{p_{k-1}}{p_k}=\frac{1}{a_k+}\frac{1}{a_{k-1}+}\cdots\cdots\cdots\frac{1}{a_2+}\frac{1}{a_1}$$

$$\frac{q_{k-1}}{q_k}=\frac{1}{a_k+}\frac{1}{a_{k-1}+}\cdots\cdots\cdots\cdots\frac{1}{a_2}$$

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$$\text{and } \frac{P'_{r-1}}{Q'_{r-1}} = \frac{1}{a_c +}\,\frac{1}{a_{c-1} +}\cdots\cdots\cdots\cdots\frac{1}{a_{c-(r-2)}};$$

hence we would have $a_k = a_c$, $a_{k-1} = a_{c-1}, \ldots\ldots a_{k-(r-2)} = a_{c-(r-2)}$.

But $$a_k \mp a_c;$$

for, if this were so, the period of $\frac{E+\sqrt{C}}{D}$ ought to begin one step earlier.

$\therefore\quad \frac{p_{k-1}}{p_k}$ and $\frac{q_{k-1}}{q_k}$ do not lie between $\frac{P'_{r-1}}{Q'_{r-1}}$ and $\frac{P'_r}{Q'_r}$;

$\therefore\quad$ M, N, M′, N′ are all of the same sign.

$$\begin{aligned}\text{Again, } MN' - M'N &= (p_k P'_r - p_{k-1}Q'_r)(q_k P'_{r-1} - q_{k-1}Q'_{r-1}) \\ &\quad - (p_k P'_{r-1} - p_{k-1}Q'_r)(q_k P'_r - q_{k-1}Q'_r) \\ &= -(p_k q_{k-1} - p_{k-1}q_k)(P'_r Q'_{r-1} - P'_{r-1}Q'_r) \\ &= -(-1)^k(-1)^r \\ &= \pm 1;\end{aligned}$$

thus M is prime to N.

If now $\frac{M}{M'} > 1$ and $\frac{N}{N'} > 1$, then by Lemma 2

$\frac{M'}{N'}$ and $\frac{M}{N}$ will be consecutive convergents to $\frac{E-\sqrt{C}}{D}$.

But if $\frac{M}{M'} < 1$ and $\frac{N}{N'} < 1$, continue the development of $-\frac{E'-\sqrt{C}}{D'}$ to one more quotient, so that $\frac{E''+\sqrt{C}}{D''} = a_{mc-r} + \cfrac{1}{\frac{E'''+\sqrt{C}}{D'''}}$;

substituting in (21) we obtain

$$\frac{E-\sqrt{C}}{D} = \frac{\frac{E'''+\sqrt{C}}{D'''}(Ma_{mc-r} + M') + M}{\frac{E'''+\sqrt{C}}{D'''}(Na_{mc-r} + N') + N},$$

and now $\frac{Ma_{mc-r} + M'}{M} > 1$, $\frac{Na_{mc-r} + N'}{N} > 1$;

$\therefore\quad$ as above it follows that $\frac{M}{N}$ and $\frac{Ma_{mc-r} + M'}{Na_{mc-r} + N'}$

are consecutive convergents to $\frac{E-\sqrt{C}}{D}$;

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$$\therefore \quad \frac{M}{N} \text{ or } \frac{p_k P'_r - p_{k-1} Q'_r}{q_k P'_r - q_{k-1} Q'_r} \text{ is a convergent to } \frac{E - \sqrt{C}}{D}.$$

The general idea of the above is due to Serret, pp. 70–71,* but the details have been considerably modified.

Let $\frac{p'}{q'}$ denote the convergent $\frac{p_k P'_r - p_{k-1} Q'_r}{q_k P'_r - q_{k-1} Q'_r}$.

Since the numerator is prime to the denominator,

$$\therefore \quad \left.\begin{aligned} p' &= p_k P'_r - p_{k-1} Q'_r \\ q' &= q_k P'_r - q_{k-1} Q'_r \end{aligned}\right\} \quad - \quad - \quad - \quad (22).$$

Then by eliminating p_{k-1} and q_{k-1} from (19) by means of (22) in the same way as in § 1 we deduce that

$$p' - \frac{E + \sqrt{C}}{D} q' = \left(p_k - \frac{E + \sqrt{C}}{D} q_k\right)\left(P'_r + \frac{E' + \sqrt{C}}{D'} Q'_r\right).$$

Now let $r = mc - s - 1$, then if $mc - s - 1 > 1$ or $m > \frac{s+2}{c}$,

$$\begin{aligned} p' &- \frac{E + \sqrt{C}}{D} q' \\ &= \left(p_k - \frac{E + \sqrt{C}}{D} q_k\right)\left(P'_{mc-s-1} + \frac{E' + \sqrt{C}}{D'} Q'_{mc-s-1}\right) \\ &= \left(p_k - \frac{E + \sqrt{C}}{D} q_k\right)\left(P_s - \frac{E' + \sqrt{C}}{D'} Q_s\right)\left(P_c - \frac{E' + \sqrt{C}}{D'} Q_c\right)^{-m} \quad \text{by (12)} \\ &= \left(p_{k+s} - \frac{E + \sqrt{C}}{D} q_{k+s}\right)\left(P_c - \frac{E' + \sqrt{C}}{D'} Q_c\right)^{-m} \quad - \quad - \quad - \quad \text{by (5).} \end{aligned}$$

Put $k + s = n$ so that $n \nless k$.

$$\therefore \quad p' - \frac{E + \sqrt{C}}{D} q' = \left(p_n - \frac{E + \sqrt{C}}{D} q_n\right)\left(P_c - \frac{E' + \sqrt{C}}{D'} Q_c\right)^{-m} \qquad (23)$$

provided $n \nless k$ and $m > \frac{n - k + 2}{c}$.

Comparing (23) with (7) we see that the power of $P_c - \frac{E' + \sqrt{C}}{D'} Q_c$ is $-m$ instead of m.

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* See also Legendre's Theorie des Nombres, 3me éd. t. i., §§ 59-74.

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Serret's § 29, in which infinite limits are used, and the latter part of § 30 have been dispensed with.

Example: $\dfrac{123+\sqrt{37}}{28} = 4 + \dfrac{1}{1+}\;\dfrac{1}{1+}\;\dfrac{1}{1+}\;\dfrac{1}{\dot{1}+}\;\dfrac{1}{3+}\;\dfrac{1}{\dot{2}+}\;\cdots\cdots\cdots$

$k=4$, $c=3$, and the first four convergents are $\dfrac{4}{1}, \dfrac{5}{1}, \dfrac{9}{2}, \dfrac{14}{3}$;

and $\dfrac{3+\sqrt{37}}{7} = \dot{1} + \dfrac{1}{3+}\;\dfrac{1}{\dot{2}+}\;\cdots\cdots\cdots$

the first three convergents being $\dfrac{1}{1}, \dfrac{4}{3}, \dfrac{9}{7}$.

The requi ite conditions are $n \not< 4$ and $m > \dfrac{n-2}{3}$.

Taking $n=4$, $m=1$, these conditions are satisfied.

$\therefore$ by (23)

$$p' - \frac{123+\sqrt{37}}{28}q' = \left(14 - \frac{123+\sqrt{37}}{28}\times 3\right)\left(9 - \frac{3+\sqrt{37}}{7}\times 7\right)^{-1}$$

$$= \frac{-27-5\sqrt{37}}{28}$$

$$\therefore \quad q' = 5$$

$$\text{and} \quad p' = \frac{123}{28}\times 5 - \frac{27}{28} = 21.$$

Similarly if $\left.\begin{matrix} n=4 \\ m=2 \end{matrix}\right\}$ then $\dfrac{p'}{q'} = \dfrac{238}{57}$

and if $\left.\begin{matrix} n=4 \\ m=3 \end{matrix}\right\}$, $\dfrac{p'}{q'} = \dfrac{2877}{689}$.

Thus $\dfrac{21}{5}, \dfrac{238}{57}, \dfrac{2877}{689}$ are convergents to

$$\frac{123-\sqrt{37}}{28} = 4 + \frac{1}{5+}\;\frac{1}{\dot{1}+}\;\frac{1}{2+}\;\frac{1}{\dot{3}+}\;\cdots\cdots\cdots$$

namely the 2nd, 5th, and 8th convergents.

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