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On a class of finite soluble groups

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Let the group T be the direct product of groups S_i $(i = 1, \ldots, r)$ where for a given group A_i, S_i is the direct product of n_i factors $A_i \times A_i \times \ldots \times A_i$. Let B be a group that has a faithful permutation representation Γ_i of degree n_i $(i = 1, \ldots, r)$. Consider G, the split extension of T by B defined by letting B act on T as follows.

Each S_i is normal in G. If $\left(a_1, \ldots, a_{n_i}\right) \in S_i$ and $b \in B$ then $\left(a_1, \ldots, a_{n_i}\right)^b = \left(a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_{n_i}}\right)$ where $\Gamma_i(b) = \begin{pmatrix} \alpha_1\alpha_2\ldots\alpha_{n_i} \\ 1\ 2\ldots\ n_i \end{pmatrix}$. It is proved that if T is an M-group and all subgroups of B are M-groups, then G is an M-group. This is a generalisation of a result of Gary M. Seitz, *Math. Z.* 110 (1969), 101-122, who proved the particular case where $r = 1$ and Γ_1 is the regular representation of B.

A finite group G is an M-group if each irreducible complex character of G is induced from a linear character of a subgroup of G. Some of the difficulties involved in studying M-groups are indicated by a result of Dade which states that any finite soluble group can be embedded

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in an M-group [3]. Seitz sharpened Dade's result by proving that a finite soluble group can be embedded in an M-group with the same derived length. This statement is a corollary of the following result of Seitz [4]:

Let A be an M-group and suppose B is a group all of whose subgroups are M-groups. Then A wr B, the wreath product of A with B is an M-group.

The aim of this paper is to generalise this result of Seitz.

We shall use the following notation:

aut G is the group of automorphisms of the group G;

ker χ is the kernel of the character χ;

an $\tilde{M}$-group is a group all of whose subgroups are M-groups;

if H is a subgroup of the group G and χ a character of H then χ^G is the character of G induced from χ;

$N_G(\chi)$ is the stabilizer of χ in G;

$\psi \downarrow H$ is the restriction of a character ψ of a group G to its subgroup H;

$a^b = b^{-1}ab$.

LEMMA 1 [2]. *Let $H \triangleleft G$, χ an irreducible character of H which has an extension $\hat{\chi}$ to $T = N_G(\chi)$. Then $\chi^G = \sum_{\omega} \omega(1)(\omega\hat{\chi})^G$ where the sum runs over the irreducible characters of T/H. Each character $(\omega\hat{\chi})^G$ is irreducible and $(\omega_1\hat{\chi})^G = (\omega_2\hat{\chi})^G$ implies $\omega_1 = \omega_2$.*

LEMMA 2 [1]. *Suppose $H \triangleleft G$. If H is abelian and complemented in G, then each irreducible character of H extends to its stabilizer.*

LEMMA 3. *Let $G = A \cdot B$ $(A \triangleleft G, A \cap B = 1)$. Then each linear character of A extends to its stabilizer.*

Proof. Let $T = N_G(\chi)$ be the stabilizer of the linear character χ of A. Then $T = A \cdot B_0$ $(B_0 = B \cap T)$. If $\ker \chi = K$ $(K \triangleleft T)$ then $T/K = (A/K) \cdot (B_0 K/K)$ (semidirect product) and A/K is cyclic. Now use

Lemma 2.

It is clear that not every extension of an M-group by an $\tilde{M}$-group is an M-group. We formulate one sufficient condition for such an extension to be an M-group.

THEOREM 1. *Let* $G = A \cdot B$ $(A \triangleleft G,\ A \cap B = 1)$ *where* A *is an* M*-group and* B *is an* $\tilde{M}$*-group. If for each irreducible character* $\chi = \phi^A$ *of the group* A *where* ϕ *is a linear character of a group* $H \subseteq A$ *,* $N_G(\chi) \cap B \subseteq N_G(\phi)$*, then* G *is an* M*-group.*

Proof. Let $N_G(\chi) = T$ and $B_0 = T \cap B$; then $T = A \cdot B_0$. In view of $B_0 \subseteq N_G(\phi)$ the stabilizer of the character ϕ in the group $S = HB_0$ $(H \triangleleft S,\ H \cap B_0 = 1)$ is S . Thus the linear character ϕ of the group H can be extended to a linear character $\hat{\phi}$ of the group S (see Lemma 3). Now $\hat{\phi}^T \downarrow A = (\hat{\phi} \downarrow HB_0 \cap A)^A = (\hat{\phi} \downarrow H)^A = \phi^A = \chi$. (We have used the subgroup theorem [1] and the fact that $(HB_0)A = T$.) Thus the character χ has an extension to the irreducible monomial character $\hat{\phi}^T$ of the group $T = N_G(\chi)$. Now using Lemma 1 we have $\chi^G = \sum_{\omega} \omega(1)\left(\omega\hat{\phi}^T\right)^G$. Since B is an $\tilde{M}$-group, the group $T/A \cong B_0 \subseteq B$ is an M-group, and therefore if ω is any irreducible character of T/A then $\omega = \psi^T$ where ψ is a linear character of the group R $(A \subseteq R \subseteq T)$. Further

$$\left(\omega\hat{\phi}^T\right)^G = \left(\psi^T \cdot \hat{\phi}^T\right)^G = \left[\left(\psi\left(\hat{\phi}^T \downarrow R\right)\right)^T\right]^G = \left[\psi\left(\hat{\phi}^T \downarrow R\right)\right]^G .$$

But $\hat{\phi}^T \downarrow R = \left(\hat{\phi} \downarrow HB_0 \cap R\right)^R$ and hence

$$\left(\omega\hat{\phi}^T\right)^G = \left[\psi\left(\hat{\phi} \downarrow HB_0 \cap R\right)^R\right]^G = \left[\left[(\psi\hat{\phi}) \downarrow HB_0 \cap R\right]^R\right]^G = \left[(\psi\hat{\phi}) \downarrow HB_0 \cap R\right]^G ,$$

which proves the theorem.

Now let $T = A \times A \times \ldots \times A$ (n factors) and assume that a group B admits a faithful representation Γ by permutations of degree n . Let $a = (a_1, a_2, \ldots, a_n)$ $(a_i \in A)$ be any element of T and b any element

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of B with $\Gamma(b) = \begin{pmatrix} \alpha_1\alpha_2\ldots\alpha_n \\ 1\ 2\ldots\ n \end{pmatrix}$. Then

(1) $$\phi_b(a) = \left(a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_n}\right)$$

is an automorphism of the group T and the mapping $\psi : B \to \text{aut } T$ where, for each $b \in B$, $\psi(b) = \phi_b \in \text{aut } T$, is a homomorphism of the group B into the group aut T . We shall call the automorphism ϕ_b , defined by (1), the automorphism corresponding to Γ .

Consider the group $G = T \cdot B$ $(T \triangleleft G, T \cap B = 1)$ where $a^b = \phi_b(a)$ $(a \in T, b \in B)$. Then this group is isomorphic to the wreath product of the group T with the permutation group $\Gamma(B)$. When Γ is the regular representation of the group B we have the standard wreath product T wr B . The following theorem considers a more general type of group.

THEOREM 2. *Let* $T = S_1 \times S_2 \times \ldots \times S_r$ *where* $S_i = A_i \times A_i \times \ldots \times A_i$ $\left(n_i\right.$ *factors;* $\left.i = 1, 2, \ldots, r\right)$ *and let* B *be a group which admits a faithful representation* Γ_i *by permutations of degree* n_i $(i = 1, 2, \ldots, r)$. *Let* G *be the split extension of* T *by* B , *where each* S_i *is invariant under* B , *and the action of* $b \in B$ *on* S_i *is given by the automorphism determined by* $\Gamma_i(b)$ *according to the rule given in* (1). *Then if* T *is an M-group and* B *is an* $\tilde{M}$*-group then* G *is an M-group.*

Proof. First of all consider the case when $r = 1$. Then $G = TB$ $(T \triangleleft G, T \cap B = 1)$, $T = A \times A \times \ldots \times A$ (n factors). If $b \in B$ is any element of B and $\Gamma(b) = \begin{pmatrix} \alpha_1\alpha_2\ldots\alpha_n \\ 1\ 2\ldots\ n \end{pmatrix}$ then

(2) $$a^b = \left(a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_n}\right)$$

where $a = \left(a_1, a_2, \ldots, a_n\right)$ is any element of T $\left(a_i \in A\right)$. Let $\chi = \chi_1\chi_2 \ldots \chi_n$ be any irreducible character of T . Thus $\chi(a) = \chi_1\left(a_1\right)\chi_2\left(a_2\right) \ldots \chi_n\left(a_n\right)$ and χ_i is an irreducible character of A

$(i = 1, 2, \ldots, n)$. Now since A is an M-group, $\chi_i = \psi_i^A$, where ψ_i is a linear character of some $H_i \subseteq A$ $(i = 1, 2, \ldots, n)$. Now $\psi = \psi_1\psi_2 \ldots \psi_n$ is a linear character of the group $H = H_1 \times H_2 \times \ldots \times H_n \subseteq T$ and $\chi = \psi^T$. In view of (2), $\chi(a^b) = \chi_1(a_{\alpha_1})\chi_2(a_{\alpha_2}) \ldots \chi_n(a_{\alpha_n})$, and $\chi(a^b) = \chi(a)$ implies $\chi_i = \chi_{\alpha_i}$ $(i = 1, 2, \ldots, n)$. Thus we can assume that $H_i = H_{\alpha_i}$ and $\psi_i = \psi_{\alpha_i}$ $(i = 1, 2, \ldots, n)$. Now $N_G(\chi) = TB_0$ $(B_0 = N_G(\chi) \cap T)$, and for each $h = (h_1, h_2, \ldots, h_n) \in H$ $(h_i \in H_i;\ i = 1, 2, \ldots, n)$, $b \in B_0$, we have $h^b = (h_{\alpha_1}, \ldots, h_{\alpha_n})$, where $b \to \begin{pmatrix}\alpha_1\alpha_2\ldots\alpha_n\\ 1\ 2\ldots\ n\end{pmatrix}$ and so $b^{-1}Hb \subseteq H$. Moreover, if $\psi(h) = \psi_1(h_1)\psi_2(h_2) \ldots \psi_n(h_n)$ then $\psi(h^b) = \psi_1(h_{\alpha_1}) \ldots \psi_n(h_{\alpha_n}) = \psi_{\alpha_1}(h_{\alpha_1}) \ldots \psi_{\alpha_n}(h_{\alpha_n}) = \psi$. Hence

(3) $$N_G(\chi) \cap B \subseteq N(\psi) \cap B .$$

This proves Theorem 2 for the case $r = 1$ (by applying Theorem 1). Now consider the general case. Let $\chi = \chi_1\chi_2 \ldots \chi_r$ be any irreducible character of T , where χ_i is an irreducible character of S_i $(i = 1, 2, \ldots, r)$. Further, for some linear character ψ_i of the group $H_i \subseteq S_i$, $\chi_i = \psi_i^{S_i}$ $(i = 1, 2, \ldots, r)$. As above $\psi = \psi_1\psi_2 \ldots \psi_r$ is a linear character of $H = H_1 \times H_2 \times \ldots \times H_r$ and $\chi = \psi^T$. For each group $G_i = S_iB$ $(S_i \triangleleft G_i;\ S_i \cap B = 1,\ i = 1, \ldots, r)$, we use the formula (3) to obtain $N_{G_i}(\chi_i) \cap B \subseteq N_{G_i}(\psi_i) \cap B$. Hence

(4) $$\bigcap_{i=1}^{r} \left[N_{G_i}(\chi_i) \cap B\right] \subseteq \bigcap_{i=1}^{r} \left[N_{G_i}(\psi_i) \cap B\right] .$$

Since $B_i \triangleleft G$ $(i = 1, 2, \ldots, r)$,

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$$N_G(\chi) \cap B = \bigcap_{i=1}^{s} [N_G(\chi_i) \cap B] = \bigcap_{i=1}^{s} \left[N_{G_i}(\chi_i) \cap B\right] .$$

Now using the formula (4) we have

$$N_G(\chi) \cap B \subseteq \bigcap_{i=1}^{r} \left[N_{G_i}(\psi_i) \cap B\right] \subseteq N_G(\psi) \cap B .$$

In view of Theorem 1 this completes the proof.

REMARK. The result of Seitz [4] is a particular case of Theorem 2 for $r = 1$, and Γ_1 the regular representation of the group B .

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