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BOOK REVIEWS

HITCHIN, N. J., SEGAL, G. B. AND WARD, R. S. *Integrable systems: twistors, loop groups and Riemann surfaces* (Oxford Graduate Texts in Mathematics, vol. 4, Clarendon Press, 1999), ix+136 pp., 0 19 850421 7, £25.

The notion of an 'integrable system' arose from the study of classical mechanics. There one encounters systems of differential equations with conserved quantities such as momentum or angular momentum. Occasionally, as in the classical equations of motion of a rigid body about its centre of mass, the motion of the system can be found explicitly, in this case in terms of elliptic functions. More generally, a completely integrable system of ordinary differential equations with $2n$ degrees of freedom must admit n independent conserved quantities $H_1, \dots, H_n$ in involution—roughly speaking, this means that they can be set freely and independently of each other—and then the motion is completely described by the uniform change of n angular variables, $\phi_j = \omega_j t$, say, where the ω_j are constant angular velocities. More geometrically, the conserved quantities specify an n-dimensional torus and the motion of the system is equivalent to a straight-line motion on it. Thus $(H_1, \dots, H_n, \phi_1, \dots, \phi_n)$ give a new system of coordinates on the phase-space, relative to which the dynamics of the system are particularly simple. These coordinates are known as action-angle variables, and the transformation to them is in general quite non-trivial. This is reflected in the fact that explicit formulae for the solutions will involve elliptic functions or more generally abelian functions associated with Riemann surfaces of higher genus.

Integrable systems of ordinary differential equations may be rare (the generic ODE will not be integrable), but they are mathematically remarkably rich, and often of particular interest in classical mechanics. The same is true of nonlinear *partial* differential equations. The most famous example is the Korteweg–de Vries (KdV) equation for water waves in a shallow channel. The complete integrability of this equation 'explains' its solitary-wave 'soliton' solutions (the first recorded observation of which was by Scott Russell, on the Edinburgh–Glasgow canal near Edinburgh, in 1834, long before the KdV equation was written down). It was only in the 1960s that the KdV and related equations began to be understood in a systematic way; in particular, it was realized that the scattering transform was the analogue of the transformation to action-angle variables in this case.

The literature is now vast and includes a mountain of examples.* The problem of approaching the literature is compounded by the different mathematical approaches to the subject; today one can find important work on the KdV equation, for example, by applied mathematicians, mathematical physicists, analysts and differential and algebraic geometers. I need scarcely add that it is very difficult to compare the results claimed by these very different communities!

This book is an introduction to integrable systems that makes no pretence at completeness, either as a list of examples or as an account of all the methods and results that are available. The general aim of the book is to introduce the reader, at a level accessible to graduate students, to three distinct but related facets, each of which illustrates fundamental features of the subject.

* In the Introduction to this book, Hitchin lists approximately 50 integrable systems, from just two of the standard books on the subject!

The Introduction and first chapter are by Nigel Hitchin; there is a chapter by Graeme Segal and another by Richard Ward. The authors have focused on aspects of the subject to which they have made important contributions, and their accounts are accordingly authoritative, as well as being very well written. Each chapter serves its purpose extremely well: anyone who has read and understood this book will be well-placed to grapple with the landmark papers in integrable systems listed in bibliographies at the end of each chapter.

Hitchin's chapter on Riemann surfaces and integrable systems describes the relation between algebraic geometry and integrable systems with a finite number of degrees of freedom. This excellent account leads rapidly from integrable systems in Lax form to algebraic curves and the 'linearization' of the system on the Jacobian torus of this curve. Contact is then made with Hamiltonian methods and the chapter ends with a brief description of an integrable system intimately related to the moduli space of stable vector bundles over a compact Riemann surface. Despite its sophisticated endpoint, the author begins by assuming almost no algebraic geometry; Riemann surfaces, line-bundles and sheaves are developed from scratch, so that this chapter is an essentially self-contained account not only of the algebraic theory of completely integrable Hamiltonian systems, but also the algebraic geometry that is used to study them.

Segal's chapter deals with more analytic aspects of integrable systems, and focuses on the scattering transform for the Korteweg–de Vries equation. This is a cornerstone of the theory, since it is the infinite-dimensional analogue of the transformation to action-angle variables. The reader will find here as well a beautiful account of the infinite-dimensional geometry associated with the KdV equation: the Grassmannian of Hilbert space, loop groups, and the relation between infinite determinants and conserved quantities. Much of this appears far from the picture painted by Hitchin, but the relation with this is explained at the end.

Ward's chapter describes a less classical approach to integrable systems: their relations with twistor theory and the geometry of the self-dual Yang–Mills equation. In this relatively short chapter the basic geometric set-up is described, along with some of the successes of the twistor approach. More could perhaps have been said here, for this relatively new approach has yielded a unifying geometric framework which sheds light on much that has gone before. However, a recent monograph by Mason and Woodhouse deals with this at length and Ward's chapter may be seen as a swift introduction to it.

In conclusion, this book is an excellent introduction to some of the most important theoretical aspects of integrable systems and a very valuable addition to the literature. The authors have made tremendous efforts to avoid technicalities and alienating jargon, and the book is remarkably self-contained for its length.

The well-motivated graduate student will be able to turn to it for guidance before tackling the more advanced monographs or research papers in the subject. I would expect that it will also be of help to researchers in the area who perhaps know one approach to the subject well, but want to learn more about others, or, for example, to algebraic geometers who are interested in applications of their subject.

The one note of caution that has to be added is that a different three authors would have written a different book, emphasizing different aspects of the subject. That is inevitable when such a broadly based subject is introduced in a book of less than 150 pages. The topics that are treated here, however, are clearly in the mathematical mainstream, and seem certain to underpin the subject as it is developed further. The mathematical diversity that is contained here exemplifies the fact that the most interesting parts of mathematics cannot be compartmentalized as pure or applied, analysis or algebra. Integrable systems yield up their secrets only to those who appreciate the importance of the different available approaches to the subject, and remind us all that we can often only see the big picture clearly if we allow it to be illuminated from every side.

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