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THE SPECTRAL MAPPING PROPERTY FOR p-MULTIPLIER OPERATORS ON COMPACT ABELIAN GROUPS

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Abstract

Let G be a compact abelian group and $1 < p < \infty$. It is known that the spectrum $\sigma(T_\psi)$, of a Fourier p-multiplier operator T_ψ acting in $L^p(G)$, may fail to coincide with its natural spectrum $\overline{\psi(\Gamma)}$ if $p \neq 2$; here Γ is the dual group to G and the bar denotes closure in $\mathbb{C}$. Criteria are presented, based on geometric, topological and/or algebraic properties of the compact set $\sigma(T_\psi)$, which are sufficient to ensure that the equality $\sigma(T_\psi) = \overline{\psi(\Gamma)}$ holds.

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1. Introduction

Let G be a compact abelian group with (discrete) dual group Γ. The *Fourier transform*

$$\hat{f}(\gamma) := \int_G f(g)\langle -g, \gamma\rangle \, dg, \quad \gamma \in \Gamma$$

is defined for all $f \in L^1(G)$. According to Hausdorff-Young's inequality, the Fourier transform map $f \mapsto \hat{f}$ is linear and continuous from $L^p(G)$ into $\ell^{p'}(\Gamma)$, where $1/p + 1/p' = 1$ and $1 \leq p \leq 2$. In the above formula for $\hat{f}$, replacing $f(g)dg$ with $d\mu(g)$ gives the definition of the *Fourier-Stieltjes transform* $\hat{\mu} : \Gamma \to \mathbb{C}$ for any finite regular Borel measure μ on G. An element T from $\mathscr{L}(L^p(G))$, the Banach algebra of all continuous linear operators from $L^p(G)$ into itself, is called a (*Fourier*) *p-multiplier operator* if it commutes with each *translation operator* τ_h, for $h \in G$, where $\tau_h f : g \mapsto f(g - h)$. Equivalently, there exists $\psi \in \ell^\infty(\Gamma)$, necessarily unique, such that $\widehat{Tf} = \psi \hat{f}$ for $f \in L^2 \cap L^p(G)$. The function $\psi : \Gamma \to \mathbb{C}$ is called a *p-multiplier* for G and the corresponding operator T is denoted by T_ψ; the p-dependence of T_ψ is suppressed since p will always be clearly identified. The space of all p-multipliers for G is denoted by $\mathscr{M}^p(G) \subseteq \ell^\infty(\Gamma)$ and the space of all p-multiplier operators by $O_p(G) \subseteq \mathscr{L}(L^p(G))$. The inequality

$$\|\psi\|_{\ell^\infty(\Gamma)} \leq \|T_\psi\|_{\mathscr{L}(L^p(G))}, \quad \psi \in \mathscr{M}^p(G)$$

is well known. If we equip $\mathscr{M}^p(G)$ with the norm $|||\psi|||_p := \|T_\psi\|_{\mathscr{L}(L^p(G))}$, then $\mathscr{M}^p(G)$ is a unital commutative semisimple Banach algebra for pointwise multiplication. For each $\psi \in \mathscr{M}^p(G)$, the functions $\mathrm{Re}(\psi)$, $\mathrm{Im}(\psi)$ and $\overline{\psi}$ (the complex conjugate of ψ) also belong to $\mathscr{M}^p(G)$ with $\psi \mapsto \overline{\psi}$ being an isometric involution on $\mathscr{M}^p(G)$. Since $\mathscr{M}^p(G)$ is isometrically isomorphic to $\mathscr{M}^{p'}(G)$ we will restrict attention to $1 < p \leq 2$. It is known that $\hat{\mu} \in \mathscr{M}^p(G)$ for every $1 \leq p < \infty$ and every finite regular Borel measure μ on G. As a general reference for p-multipliers, see [9].

The *spectrum* $\sigma(T)$, of an operator $T \in \mathscr{L}(L^p(G))$, is defined by

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } \mathscr{L}(L^p(G))\}.$$

For elements $T_\psi \in O_p(G)$, a basic fact is that

$$\overline{\psi(\Gamma)} \subseteq \sigma(T_\psi), \tag{1}$$

for every $1 \leq p < \infty$, [13, Lemma 2.1], where the bar denotes closure in $\mathbb{C}$. Fundamental work of Igari [7] and Zafran [14] established that (1) fails to be an equality in general, even for such a 'nice' group as $G = \mathbb{T}$ (the circle group) and for every $1 < p < 2$. Indeed, there even exist elements $\psi \in \mathscr{M}^p(\mathbb{T}) \cap c_0(\mathbb{Z})$ which fail to satisfy (1).

We will say that $\psi \in \mathscr{M}^p(G)$ satisfies the *spectral mapping property* if (1) is an equality. The class of *decomposable operators* (in an arbitrary Banach space) was introduced by Foiaş [5]; see also [2]. In [1], Albrecht made a detailed study of decomposability for the particular class of p-multiplier operators acting in lca groups; see also [4]. Using a functional calculus approach and local spectral theory he showed that the class of decomposable p-multiplier operators is rather extensive and, most importantly, that all such operators *must* satisfy the spectral mapping property, [1, Lemma 3.2].

The purpose of this note is to present criteria of a rather different nature, which ensure the spectral mapping property. The criteria are based directly on geometric, topological and/or algebraic properties of the spectrum itself. Given a compact set $K \subseteq \mathbb{C}$, let $\mathscr{I}(K)$ denote the set of all *isolated points* of K. We can now state the main result.

THEOREM 1.1. *Let G be a compact abelian group, $1 < p \le 2$ and $\psi \in \mathcal{M}^p(G)$.*

(i) *ψ satisfies the spectral mapping property if and only if $\overline{\psi}$ does.*

(ii) *If $\sigma(T_\psi) = \overline{\mathcal{I}(\sigma(T_\psi))}$, then ψ satisfies the spectral mapping property.*

(iii) *Suppose that $\sigma(T_\psi)$ is totally disconnected. Then T_ψ is decomposable and hence, satisfies the spectral mapping property. This is the case if either:*

(a) *$\sigma(T_\psi)$ is countable.*

(b) *$\sigma(T_\psi)$ is independent, as a subset of the abelian group $\mathbb{R}^2$.*

(c) *$\sigma(T_\psi)$ is a Kronecker set.*

(iv) *If $\overline{\psi(\Gamma)}$ is totally disconnected, then T_ψ satisfies the spectral mapping property if and only if T_ψ is decomposable.*

(v) *Let μ be a finite regular Borel measure on G such that $\overline{\hat{\mu}(\Gamma)}$ has capacity zero. Then $\hat{\mu}$ satisfies the spectral mapping property.*

(vi) *The following statements are equivalent:*

(a) *T_ψ fails the spectral mapping property.*

(b) *$\sigma(T_\psi) \setminus \overline{\psi(\Gamma)}$ is an uncountable set.*

(c) *$\overline{\sigma(T_\psi) \setminus \overline{\psi(\Gamma)}}$ is a non-empty perfect set.*

The proof is via a series of steps.

For $T \in \mathcal{L}(L^p(G))$, let $\sigma_{pt}(T)$, $\sigma_r(T)$ and $\sigma_c(T)$ denote the *point, residual* and *continuous spectra* of T, respectively, in which case the three sets are pairwise disjoint and have union equal to $\sigma(T)$, [3, page 580].

LEMMA 1.2. *Let $1 < p \le 2$ and $\psi \in \mathcal{M}^p(G)$. Then*

$$\sigma_c(T_{\overline{\psi}}) = \{\overline{\lambda} : \lambda \in \sigma_c(T_\psi)\} \tag{2}$$

and also

$$\sigma_{pt}(T_{\overline{\psi}}) = \{\overline{\lambda} : \lambda \in \sigma_{pt}(T_\psi)\} = \overline{\psi}(\Gamma). \tag{3}$$

In particular,

$$\sigma(T_{\overline{\psi}}) = \{\overline{\lambda} : \lambda \in \sigma(T_\psi)\}. \tag{4}$$

PROOF. For any $\varphi \in \mathcal{M}^p(G)$, it is routine to check that T_φ is injective if and only if $0 \notin \varphi(\Gamma)$. It follows easily that

$$\sigma_{pt}(T_\varphi) = \varphi(\Gamma), \quad \varphi \in \mathcal{M}^p(G). \tag{5}$$

Putting φ equal to ψ and $\overline{\psi}$ in (5), we can deduce (3).

Suppose that $\lambda \in \sigma_c(T_\psi)$. Then (5) implies that $\lambda \notin \psi(\Gamma)$ and hence, $\overline{\lambda} \notin \overline{\psi}(\Gamma)$. For each $\gamma \in \Gamma$, the function $f_\gamma : g \mapsto \langle g, \gamma\rangle \cdot (\overline{\psi(\gamma)} - \overline{\lambda})^{-1}$, for $g \in G$, belongs to $L^p(G)$ and satisfies $(T_{\overline{\psi}} - \overline{\lambda} I) f_\gamma = h_\gamma$, where $h_\gamma(g) := \langle g, \gamma\rangle$, for $g \in G$. This

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shows that all trigonometric polynomials belong to the range of $T_{\overline{\psi}} - \overline{\lambda} I$ and hence, that this range is dense in $L^p(G)$. Accordingly , $\overline{\lambda} \in \sigma_c(T_{\overline{\psi}})$ which shows that the right-hand side of (2) is contained in $\sigma_c(T_{\overline{\psi}})$. The reverse inclusion is established similarly.

Since $\sigma_r(T_\psi) = \emptyset = \sigma_r(T_{\overline{\psi}})$, [13, Lemma 2.4], (4) follows. □

Given any $\psi \in \mathscr{M}^p(G)$ it is routine to verify that

$$\{\overline{\lambda} : \lambda \in \overline{\psi(\Gamma)}\} = \{\mu : \mu \in \overline{\overline{\psi}(\Gamma)}\}.$$

This observation, together with Lemma 1.2, imply (i) of Theorem 1.1.

To establish Theorem 1.1 (ii) we require the fact that every isolated point of $\sigma(T_\psi)$ belongs to $\sigma_{pt}(T_\psi)$. But, since all elements of $\mathscr{M}^p(G)$ are *continuous* on the discrete space Γ, this follows from [13, Theorem 2.3]. In view of (5), we can conclude that

$$(6) \qquad \mathscr{I}(\sigma(T_\psi)) \subseteq \psi(\Gamma) \subseteq \sigma(T_\psi), \qquad \psi \in \mathscr{M}^p(G).$$

Under the particular hypothesis that $\overline{\mathscr{I}(\sigma(T_\psi))} = \sigma(T_\psi)$, we see from (6) that ψ *satisfies the spectral mapping property. This completes the proof of part (ii).*

REMARK. It is straightforward to exhibit compact sets $K \subseteq \mathbb{C}$ which satisfy $\overline{\mathscr{I}(K)} = K$, but K is not totally disconnected. So, (ii) of Theorem 1.1 does not follow from part (iii). Of course, there also exist totally disconnected, compact sets K for which $\overline{\mathscr{I}(K)} \neq K$ (for example, the Cantor set, where $\mathscr{I}(K) = \emptyset$).

Concerning the proof of Theorem 1.1 (iii), it is known that decomposability of T_ψ always follows from the total disconnectedness of $\sigma(T_\psi)$; see the proof of [10, Lemma 2.2] which also applies to arbitrary compact abelian groups G.

For the definition of a subset of $\mathbb{R}^2 \simeq \mathbb{C}$ being an *independent set* (an algebraic notion) or a *Kronecker set*, we refer to [11, Chapter 5]. Compact, independent subsets of $\mathbb{R}^2$ are always totally disconnected, [11, Theorem 5.2.9], and every Kronecker set is independent, [11, Theorem 5.1.4]. So, (b) and (c) of part (iii) are valid. To verify (a), let C be a connected subset of $\sigma(T_\psi)$. Since metric spaces are completely regular and C is countable, it follows that C is actually a singleton set, [8, page 129]. So, $\sigma(T_\psi)$ is totally disconnected whenever it is countable. This completes the proof of part (iii).

For part (iv), we have seen that decomposability of T_ψ always implies the spectral mapping property, even if $\overline{\psi(\Gamma)}$ is not totally disconnected. On the other hand, the spectral mapping property means that $\sigma(T_\psi) = \overline{\psi(\Gamma)}$ and so $\sigma(T_\psi)$ is totally disconnected whenever $\overline{\psi(\Gamma)}$ is totally disconnected. Then part (iii) yields the decomposability of T_ψ.

For the notion of *capacity*, which is relevant to (v), we refer to [12] and the references therein. When $p = 2$, it is known that $\sigma(T_{\hat{\mu}}) = \overline{\hat{\mu}(\Gamma)}$, [14, page 357]. This observation, together with [12, Corollary 2.2], establishes (v). For related criteria which also imply the spectral mapping property see [6, Section 4].

REMARK. If Γ is discrete and countable (that is, G is metrizable) and μ is a finite regular Borel measure on G with $\hat{\mu} \in c_0(\Gamma)$, then $\overline{\hat{\mu}(\Gamma)}$ is countable and hence, has capacity zero. It follows from (v) of Theorem 1.1 that $\hat{\mu}$ satisfies the spectral mapping property for all $1 < p < \infty$; see also [12, page 309]. The condition $\hat{\mu} \in c_0(\Gamma)$ is not necessary for this conclusion to hold, [12, Examples 2 and 3]. There also exist μ (even on $\mathbb{T}$) which satisfy the spectral mapping property for all $1 < p < \infty$ but, $\overline{\hat{\mu}(\Gamma)}$ is uncountable, [12, pages 310–312].

Finally, part (vi) of Theorem 1.1 follows from (6) and the following result (with the choice $J := \overline{\psi(\Gamma)}$ and $K := \sigma(T_\psi)$).

LEMMA 1.3. *Let $K \subseteq \mathbb{C}$ be non-empty and compact.*

(i) *$\mathscr{I}(K)$ is a countable set, possibly empty.*

(ii) *K is countable if and only if the set $\mathscr{A}(K)$ of all accumulation points of K is countable.*

(iii) *Let J be a closed subset of K with $\mathscr{I}(K) \subseteq J$. Then either $J = K$ or $\overline{K \setminus J}$ is a non-empty perfect set (that is, $\overline{K \setminus J} = \mathscr{A}(\overline{K \setminus J})$). If $J \neq K$, then $K \setminus J$ (hence, also $\overline{K \setminus J}$) is uncountable.*

PROOF. (i) K is a separable metric space and so has a countable base for its topology. Since each set $\{x\}$, for $x \in \mathscr{I}(K)$, belongs to this base, it follows that $\mathscr{I}(K)$ is countable.

(ii) Since K is the disjoint union of $\mathscr{I}(K)$ and $\mathscr{A}(K)$, (ii) follows from (i).

(iii) Suppose that $\overline{K \setminus J} \neq \emptyset$ and let $x \in \overline{K \setminus J}$.

If $x \notin K \setminus J$, then $x \in \mathscr{A}(K \setminus J) \subseteq \mathscr{A}(\overline{K \setminus J})$.

If $x \in K \setminus J$, then $K \setminus J$ being open in K ensures the existence of a ball B_x (centre x and positive radius) which is open in K and satisfies $B_x \subseteq K \setminus J$. Since $\mathscr{I}(K) \subseteq J$ it follows that $x \in \mathscr{A}(K)$. Choose any sequence $\{x_n\}_{n=1}^\infty$ in $K \setminus \{x\}$ which converges to x. Then all but finitely many of the x_n must belong to B_x. Remove these finitely many points leaves a sequence in $(K \setminus J) \setminus \{x\}$ which converges to x. Accordingly, $x \in \mathscr{A}(K \setminus J) \subseteq \mathscr{A}(\overline{K \setminus J})$.

This establishes that $\overline{K \setminus J}$ is a perfect set whenever $J \neq K$.

Suppose now that $J \neq K$. The set $K \setminus J$ is open in K and each singleton set $\{x\}$, for $x \in K \setminus J$, is nowhere dense in K (because $\mathscr{I}(K) \subseteq J$). So, if $K \setminus J$ is countable, then it is of first category in K. By Baire's Theorem $J = K \setminus (K \setminus J)$ would be dense

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in K, that is, $K = \overline{J}$ ($= J$) contrary to the assumption that $J \neq K$. Hence, $K \setminus J$ is uncountable. □

As a concluding remark, we point out that Theorem 1.1 (vi) is an extension of a result of Zafran, [13, Lemma 2.6], proved for $\psi \in \mathscr{M}^p(G) \cap c_0(\Gamma)$. Our result shows that the condition $\psi \in c_0(\Gamma)$ can be omitted.

References

[1] E. Albrecht, 'Decomposable systems of operators in harmonic analysis', in: *Toeplitz Memorial Conference in Operator Theory* (ed. I. Gohberg), Oper. Theory Adv. Appl. 4 (Birkhäuser, Basel, 1982) pp. 19–35.
[2] I. Colojoară and C. Foiaş, *Theory of generalized spectral operators* (Gordon and Breach, New York, 1968).
[3] N. Dunford and J. T. Schwartz, *Linear operators I: General theory* (Interscience, New York, 1964).
[4] J. Eschmeier, 'Operator decomposability and weakly continuous representations of locally compact abelian groups', *J. Operator Theory* **7** (1982), 201–208.
[5] C. Foiaş, 'Spectral maximal spaces and decomposable operators', *Arch. Math. (Basel)* **14** (1963), 341–349.
[6] C. C. Graham, K. E. Hare and D. L. Ritter, 'The size of L^p-improving measures', *J. Funct. Anal.* **84** (1989), 472–495.
[7] S. Igari, 'Functions of L^p multipliers', *Tôhoku Math. J.* **21** (1969), 304–320.
[8] K. Kuratowski, *Topology II* (PWN-Polish Scientific Publ., Warszawa, 1968).
[9] R. Larsen, *An introduction to the theory of multipliers* (Springer, Berlin, 1971).
[10] G. Mockenhaupt and W.J. Ricker, 'Fuglede's theorem, the bicommutant theorem and p-multiplier operators for the circle', *J. Operator Theory* **49** (2003), 295–310.
[11] W. Rudin, *Fourier analysis on groups* (McGraw-Hill, New York, 1973).
[12] P. Sarnak, 'Spectra of singular measures as multipliers on L^p', *J. Funct. Anal.* **37** (1980), 302–317.
[13] M. Zafran, 'On the spectra of multipliers', *Pacific J. Math.* **47** (1973), 609–626.
[14] ———, 'The spectra of multiplier transformations on the L_p-spaces', *Ann. of Math. (2)* **103** (1976), 355–374.

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