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DISCRETE FOCAL BOUNDARY-VALUE PROBLEMS

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Abstract In this paper we shall employ the nonlinear alternative of Leray–Schauder and known sign properties of a related Green's function to establish the existence results for the nth-order discrete focal boundary-value problem. Both the singular and non-singular cases will be discussed.

Keywords: Leray–Schauder alternative; focal problems; Green's function; singular; non-singular

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1. Introduction

This paper discusses the nth-order ($n \geqslant 2$) discrete focal boundary-value problem

$$\left.\begin{aligned}(-1)^{n-p}\Delta^n y(k-p) = f(k, y(k), y(k+1), \dots, y(k+n-p-1)), \quad k \in J_p,\\ \Delta^i y(0) = 0, \quad 0 \leqslant i \leqslant p-1,\\ \Delta^i y(T+1) = 0, \quad p \leqslant i \leqslant n-1,\end{aligned}\right\} \tag{1.1}$$

where $T \in \{1, 2, \dots\}$, $1 \leqslant p \leqslant n-1$, $J_p = \{p, p+1, \dots, T+p\}$, and $y : I_n = \{0, 1, \dots, T+n\} \to \mathbb{R}$. We will let $C(I_n)$ denote the class of maps w continuous on I_n (discrete topology) with norm $\|w\| = \max_{k \in I_n} |w(k)|$. By a solution to (1.1) we mean a $w \in C(I_n)$ such that w satisfies the difference equation in (1.1) for $k \in J_p$ and w satisfies the focal boundary data. The results presented in this paper are all new and supplement those recently discussed in [**1–4, 6, 7, 11, 13–15**]. In fact, this is the first time the singular discrete focal boundary-value problem has been discussed successfully. For this we shall employ the nonlinear alternative of Leray–Schauder and known sign properties of a related Green's function cleverly. The continuous analogue of the results established here, which improve several known existence criteria (see, for example, [**2, 8, 9**]), has appeared in [**5**].

For the remainder of this introduction we gather together some results that will be used in § 2 and in § 3. First, we recall the following well-known result from the literature [**1, 6, 10**].

Theorem 1.1. *The Green's function $G_1(k,j)$ of the boundary-value problem*

$$\Delta^n y = 0, \quad y(k_i) = 0, \quad 1 \leqslant i \leqslant n, \quad 0 = k_1 < k_2 < \cdots < k_n \mp T + n$$

exists and $G_1(k,j)Q(k) \geqslant 0$ for $(k,j) \in I_n \times I_0$, where

$$I_0 = \{0,1,\ldots,T\} \quad \text{and} \quad Q = \prod_{i=1}^{n}(k-k_i).$$

In [**1,6**] it was shown that if y satisfies

$$\left.\begin{aligned} \Delta^n y(k) &= \phi(k), && k \in I_0, \\ \Delta^i y(0) &= 0, && 0 \leqslant i \leqslant p-1, \\ \Delta^i y(T+1) &= 0, && p \leqslant i \leqslant n-1, \end{aligned}\right\} \tag{1.2}$$

then

$$y(k) = \sum_{j=0}^{T} G_2(k,j)\phi(j), \quad \text{for } k \in I_n, \tag{1.3}$$

where

$$G_2(k,j) = (-1)^{n-p} \sum_{i=0}^{j} \frac{(k-i-1)^{(p-1)}(j+n-p-1-i)^{(n-p-1)}}{(p-1)!(n-p-1)!},$$

if $j \in \{0,1,\ldots,k-1\}$, and

$$G_2(k,j) = (-1)^{n-p} \sum_{i=0}^{k-1} \frac{(k-i-1)^{(p-1)}(j+n-p-1-i)^{(n-p-1)}}{(p-1)!(n-p-1)!},$$

if $j \in \{k,k+1,\ldots,T\}$. Next consider

$$\left.\begin{aligned} \Delta^n y(k-p) &= \phi(k), && k \in J_p, \\ \Delta^i y(0) &= 0, && 0 \leqslant i \leqslant p-1, \\ \Delta^i y(T+1) &= 0, && p \leqslant i \leqslant n-1. \end{aligned}\right\} \tag{1.4}$$

Notice (1.4) is the same as

$$\left.\begin{aligned} \Delta^n y(k) &= \phi(k+p), && k \in I_0, \\ \Delta^i y(0) &= 0, && 0 \leqslant i \leqslant p-1, \\ \Delta^i y(T+1) &= 0, && p \leqslant i \leqslant n-1, \end{aligned}\right\} \tag{1.5}$$

and so

$$y(k) = \sum_{j=0}^{T} G_2(k,j)\phi(j+p), \quad \text{for } k \in I_n.$$

This is the same as

$$y(k) = \sum_{j=p}^{T+p} G(k,j)\phi(j), \qquad \text{for } k \in I_n, \tag{1.6}$$

where

$$G(k,j) = G_2(k, j-p), \quad \text{for } k \in I_n \text{ and } j \in J_p. \tag{1.7}$$

Next suppose $y : I_n \to \mathbb{R}$ satisfies

$$\left.\begin{aligned} (-1)^{n-p}\Delta^n y(k) \geqslant 0, &\quad k \in I_0, \\ \Delta^i y(0) = 0, &\quad 0 \leqslant i \leqslant p-1, \\ \Delta^i y(T+1) = 0, &\quad p \leqslant i \leqslant n-1. \end{aligned}\right\} \tag{1.8}$$

Now (1.3) implies

$$\Delta^i y(k) = \sum_{j=0}^{T} (-1)^{n-p}\Delta^i G_2(k,j)(-1)^{n-p}\Delta^n y(j),$$

and since [**6**]

$$(-1)^{n-p}\Delta^i G_2(k,j) \geqslant 0, \qquad (k,j) \in I_{n-i} \times I_0, \quad 0 \leqslant i \leqslant p-1$$

and

$$(-1)^{n-p+i}\Delta^{i+p} G_2(k,j) \geqslant 0, \qquad (k,j) \in I_{n-i-p} \times I_0, \quad 0 \leqslant i \leqslant n-p-1,$$

we have

$$\Delta^i y(k) \geqslant 0, \quad \text{for } k \in I_{n-i}, \ 0 \leqslant i \leqslant p \tag{1.9}$$

and

$$\Delta^{p+1} y(k) \leqslant 0, \quad \text{for } k \in I_{n-p-1}, \tag{1.10}$$

where $I_j = \{0, 1, \ldots, T+j\}$. As a result we have

$$\sup_{k \in I_{n-i}} \Delta^i y(k) = \Delta^i y(T+n-i), \quad 0 \leqslant i \leqslant p-1. \tag{1.11}$$

Fix $i \in \{0, 1, \ldots, p-1\}$ and let $\phi_i(k) = \Delta^i y(k)$. It is easy to see that $\phi_i(k)$ satisfies the following $p-i+1$ conditions

$$\left.\begin{aligned} \Delta^j \phi_i(0) = 0, &\quad j = 0, 1, \ldots, p-i-1, \\ \phi_i(T+n-i) = \Delta^i y(T+n-i); & \end{aligned}\right\} \tag{1.12}$$

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these are conjugate conditions [**6**]. In addition, (1.10) implies

$$\Delta^{p-i+1}\phi_i(k) = \Delta^{p+1}y(k) \leqslant 0, \quad \text{for } k \in I_{n-p-1}. \tag{1.13}$$

Now [**1**, **6**], $\phi_i(k)$ can be written as

$$\phi_i(k) = \frac{k^{(p-i)}}{(T+n-i)^{(p-i)}}\phi_i(T+n-i) + \sum_{j=0}^{T+n-p-1} G_3(k,j)\Delta^{p-i+1}\phi_i(j), \tag{1.14}$$

for $k \in I_{n-i}$, where G_3 is the Green's function for the problem

$$\left.\begin{aligned} \Delta^{p-i+1}\phi_i(k) &= 0, \quad k \in I_{n-p-1}, \\ \Delta^j\phi_i(0) &= 0, \quad j = 0,1,\dots,p-i-1, \\ \phi_i(T+n-i) &= 0. \end{aligned}\right\} \tag{1.15}$$

Theorem 1.1 implies that, for $k \in I_{n-i}$,

$$\operatorname{sgn} G_3(k,j) = \operatorname{sgn}(k^{(p-i)}(k-T-n+i)) = -$$

(here we use the convention $\operatorname{sgn} 0 = -$). This, together with (1.14), gives

$$\phi_i(k) \geqslant \frac{k^{(p-i)}}{(T+n-i)^{(p-i)}}\Delta^i y(T+n-i), \qquad \text{for } k \in I_{n-i} \text{ and } 0 \leqslant i \leqslant p-1,$$

i.e.

$$\Delta^i y(k) \geqslant \frac{k^{(p-i)}}{(T+n-i)^{(p-i)}} \sup_{j \in I_{n-i}} \Delta^i y(j), \qquad \text{for } k \in I_{n-i} \text{ and } 0 \leqslant i \leqslant p-1. \tag{1.16}$$

Next, suppose that $y : I_n \to \mathbb{R}$ satisfies

$$\left.\begin{aligned} (-1)^{n-p}\Delta^n y(k-p) &\geqslant 0, \quad k \in J_p, \\ \Delta^i y(0) &= 0, \quad 0 \leqslant i \leqslant p-1, \\ \Delta^i y(T+1) &= 0, \quad p \leqslant i \leqslant n-1. \end{aligned}\right\} \tag{1.17}$$

Now, since $(-1)^{n-p}\Delta^n y(k-p) \geqslant 0$ for $k \in J_p$ is the same as $(-1)^{n-p}\Delta^n y(k) \geqslant 0$ for $k \in I_0$, we have

$$\Delta^i y(k) \geqslant \frac{k^{(p-i)}}{(T+n-i)^{(p-i)}} \sup_{j \in I_{n-i}} \Delta^i y(j), \quad \text{for } k \in I_{n-i} \text{ and } 0 \leqslant i \leqslant p-1. \tag{1.18}$$

In particular,

$$y(k) \geqslant \frac{p^{(p)}}{(T+n)^{(p)}} \sup_{j \in I_n} y(j), \quad \text{for } k \in J_p. \tag{1.19}$$

Next we present a new existence principle for the discrete focal boundary-value problem

$$\left.\begin{aligned} (-1)^{n-p}\Delta^n y(k-p) = f(k, y(k), y(k+1), \dots, y(k+n-p-1)), \quad k \in J_p, \\ y(0) = a \\ \Delta^i y(0) = 0, \quad 1 \leqslant i \leqslant p-1, \\ \Delta^i y(T+1) = 0, \quad p \leqslant i \leqslant n-1. \end{aligned}\right\} \tag{1.20}$$

Theorem 1.2. *Suppose $f : J_p \times \mathbb{R}^{n-p} \to \mathbb{R}$ is continuous (i.e. continuous as a map from the topological space $J_p \times \mathbb{R}^{n-p}$ into the topological space $\mathbb{R}$ (of course, the topology on J_p will be the discrete topology)). Assume there is a constant $M > |a|$, independent of λ, with*

$$\|y\| = \max_{j \in I_n} |y(j)| \neq M,$$

for any solution $y \in C(I_n)$ to

$$\left.\begin{aligned} (-1)^{n-p}\Delta^n y(k-p) = \lambda f(k, y(k), y(k+1), \dots, y(k+n-p-1)), \quad k \in J_p, \\ y(0) = a, \\ \Delta^i y(0) = 0, \quad 1 \leqslant i \leqslant p-1, \\ \Delta^i y(T+1) = 0, \quad p \leqslant i \leqslant n-1, \end{aligned}\right\} \tag{1.21$_\lambda$}$$

for each $\lambda \in (0,1)$. Then (1.20) has a solution.

Proof. Solving $(1.21)_\lambda$ is equivalent to finding a $y \in C(I_n)$ that satisfies

$$y(k) = a + \lambda \sum_{j=p}^{T+p} (-1)^{n-p} G(k,j) f(j, y(j), y(j+1), \dots, y(j+n-p-1)), \quad \text{for } k \in I_n, \tag{1.22$_\lambda$}$$

where G is as in (1.7). Define the operator $S : C(I_n) \to C(I_n)$ by setting

$$Sy(k) = a + \sum_{j=p}^{T+p} (-1)^{n-p} G(k,j) f(j, y(j), y(j+1), \dots, y(j+n-p-1)).$$

Now $(1.22)_\lambda$ is equivalent to the fixed-point problem

$$y = (1-\lambda)a + \lambda Sy.$$

It is easy to see [**3, 6**] that $S : C(I_n) \to C(I_n)$ is continuous and completely continuous. Let

$$U = \{u \in C(I_n) : \|u\| < M\} \quad \text{and} \quad E = C(I_n).$$

The nonlinear alternative of Leray–Schauder [**12**] guarantees that S has a fixed point in $\overline{U}$, i.e. (1.20) has a solution. □

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2. Non-singular focal problems

In this section we establish existence of solutions to discrete focal non-singular boundary-value problems. For convenience, we discuss (1.1).

Theorem 2.1. *Suppose the following conditions are satisfied:*

$$f : J_p \times \mathbb{R}^{n-p} \to \mathbb{R} \quad \text{is continuous;} \tag{2.1}$$

$$\left.\begin{array}{l}\text{there exists a continuous, non-decreasing function } \psi : [0,\infty) \to \\ [0,\infty) \text{ with } \psi > 0 \text{ on } (0,\infty) \text{ and a function } q : J_p \to [0,\infty) \text{ with} \\ |f(k,u_1,\dots,u_{n-p})| \leqslant q(k)\psi(|u|) \text{ for all } u_i \in \mathbb{R},\ i = 1,2,\dots,n-p \\ \text{and } k \in J_p, \text{ where } |u| = \max\{|u_i| : i = 1,2,\dots,n-p\};\end{array}\right\} \tag{2.2}$$

and

$$\sup_{c\in(0,\infty)} \left(\frac{c}{\psi(c)}\right) > Q, \quad \text{where } Q = \max_{k\in I_n} \sum_{j=p}^{T+p} q(j)(-1)^{n-p}G(k,j). \tag{2.3}$$

Then (1.1) has a solution.

Proof. Let $M > 0$ satisfy

$$(M/\psi(M)) > Q. \tag{2.4}$$

Consider the family of problems

$$\left.\begin{array}{r}(-1)^{n-p}\Delta^n y(k-p) = \lambda f(k, y(k), y(k+1), \dots, y(k+n-p-1)), \quad k \in J_p, \\ \Delta^i y(0) = 0, \quad 0 \leqslant i \leqslant p-1, \\ \Delta^i y(T+1) = 0, \quad p \leqslant i \leqslant n-1,\end{array}\right\} \tag*{$(2.5)_\lambda$}$$

for $0 < \lambda < 1$. Let y be any solution of $(2.5)_\lambda$ for $0 < \lambda < 1$. Then

$$y(k) = \lambda \sum_{j=p}^{T+p} (-1)^{n-p} G(k,j) f(j, y(j), y(j+1), \dots, y(j+n-p-1)), \quad \text{for } k \in I_n. \tag{2.6}$$

Now, (2.6) together with (2.2) implies that for $k \in I_n$,

$$|y(k)| \leqslant \sum_{j=p}^{T+p} (-1)^{n-p} G(k,j) q(j) \psi(\|y\|) \leqslant Q\psi(\|y\|),$$

where $\|y\| = \sup_{k\in I_n} |y(k)|$. Consequently,

$$\frac{\|y\|}{\psi(\|y\|)} \leqslant Q. \tag{2.7}$$

Now, (2.4) together with (2.7) implies $\|y\| \neq M$. Thus, any solution y of $(2.5)_\lambda$ satisfies $\|y\| \neq M$. Now, Theorem 1.2 implies that (1.1) has a solution. □

Remark 2.2. It is easy to put conditions [**3**, **4**, **6**] on f to guarantee that (1.1) has a non-negative solution.

Remark 2.3. The ideas in this section can be trivially extended in order to establish existence results for the non-singular conjugate nth-order problem,

$$\left.\begin{aligned}(-1)^{n-p}\Delta^n y(k) = f(k, y(k), y(k+1), \ldots, y(k+n-1)), \quad k \in I_0,\\ \Delta^i y(0) = 0, \quad 0 \leqslant i \leqslant p-1,\\ \Delta^i y(T+n-i) = 0, \quad 0 \leqslant i \leqslant n-p-1,\end{aligned}\right\}$$

the non-singular focal nth-order problem,

$$\left.\begin{aligned}(-1)^{n-p}\Delta^n y(k) = f(k, y(k), y(k+1), \ldots, y(k+n-1)), \quad k \in I_0,\\ \Delta^i y(0) = 0, \quad 0 \leqslant i \leqslant p-1,\\ \Delta^i y(T+1) = 0, \quad p \leqslant i \leqslant n-1,\end{aligned}\right\}$$

and the non-singular (n,p) problem,

$$\left.\begin{aligned}\Delta^n y(k) = f(k, y(k), y(k+1), \ldots, y(k+n-1)), \quad k \in I_0,\\ \Delta^i y(0) = 0, \quad 0 \leqslant i \leqslant n-2,\\ \Delta^p y(T+n-p) = 0, \quad 0 \leqslant p \leqslant n-1 \quad (p \text{ fixed}).\end{aligned}\right\}$$

3. Singular focal problems

Next we discuss

$$\left.\begin{aligned}(-1)^{n-p}\Delta^n y(k-p) = f(k, y(k)), \quad k \in J_p,\\ \Delta^i y(0) = 0, \quad 0 \leqslant i \leqslant p-1,\\ \Delta^i y(T+1) = 0, \quad p \leqslant i \leqslant n-1,\end{aligned}\right\} \tag{3.1}$$

where $f(i, y)$ may be singular at $y = 0$.

Theorem 3.1. *Suppose the following conditions are satisfied:*

$$f : J_p \times (0, \infty) \to (0, \infty) \text{ is continuous;} \tag{3.2}$$

$$\left.\begin{aligned}&f(k,u) \leqslant g(u) + h(u) \text{ on } J_p \times (0,\infty) \text{ with } g > 0 \text{ continuous}\\ &\text{and non-increasing on } (0,\infty),\ h \geqslant 0 \text{ continuous on } [0,\infty)\\ &\text{and } (h/g) \text{ non-decreasing on } (0,\infty);\end{aligned}\right\} \tag{3.3}$$

$$\left.\begin{aligned}&\text{for each constant } H > 0, \text{ there exists a continuous function}\\ &\psi_H : J_p \to (0,\infty) \text{ with } f(k,u) \geqslant \psi_H(k) \text{ on } J_p \times (0, H];\end{aligned}\right\} \tag{3.4}$$

$$\left.\begin{aligned}&\text{there exists a constant } K_\theta > 0 \text{ with } g(\theta u) \leqslant K_\theta g(u)\\ &\text{for all } u \geqslant 0, \text{ where } \theta = [p^{(p)}/(T+n)^{(p)}];\end{aligned}\right\} \tag{3.5}$$

and

$$\sup_{c\in(0,\infty)}\left(\frac{c}{g(c)+h(c)}\right) > K_\theta Q, \tag{3.6}$$

where

$$Q=\sum_{j=p}^{T+p}(-1)^{n-p}G(T+n,j), \quad \text{and } G \text{ is as in (1.7)}. \tag{3.7}$$

Then (3.1) has a solution $y\in C(I_n)$ with $y(i)>0$ for $i\in J_p$.

Proof. Choose $M>0$ with

$$\frac{M}{QK_\theta[g(M)+h(M)]}>1. \tag{3.8}$$

Next choose $\epsilon>0$ and $\epsilon<M$ with

$$\frac{M}{QK_\theta[g(M)+h(M)]+\epsilon}>1. \tag{3.9}$$

Let $n_0\in\{1,2,\dots\}$ be chosen so that $(1/n_0)<\epsilon$ and let $N_0=\{n_0,n_0+1,\dots\}$. We show first that

$$\left.\begin{aligned} (-1)^{n-p}\Delta^n y(k-p)&=f^\star(k,y(k)), \quad k\in J_p,\\ y(0)&=(1/m),\\ \Delta^i y(0)&=0, \quad 1\leqslant i\leqslant p-1,\\ \Delta^i y(T+1)&=0, \quad p\leqslant i\leqslant n-1, \end{aligned}\right\} \tag*{$(3.10)^m$}$$

has a solution for each $m\in N_0$, where

$$f^\star(k,u)=\begin{cases} f(k,u), & u\geqslant(1/m),\\ f(k,(1/m)), & u<(1/m).\end{cases}$$

To show that $(3.10)^m$ has a solution for each $m\in N_0$, we will apply Theorem 1.2. Consider the family of problems

$$\left.\begin{aligned} (-1)^{n-p}\Delta^n y(k-p)&=\lambda f^\star(k,y(k)), \quad k\in J_p,\\ y(0)&=(1/m),\\ \Delta^i y(0)&=0, \quad 1\leqslant i\leqslant p-1,\\ \Delta^i y(T+1)&=0, \quad p\leqslant i\leqslant n-1, \end{aligned}\right\} \tag*{$(3.11)^m_\lambda$}$$

for $0<\lambda<1$. Let $y\in C(I_n)$ be any solution of $(3.11)^m_\lambda$. Then

$$y(k)=(1/m)+\lambda\sum_{j=p}^{T+p}(-1)^{n-p}G(k,j)f^\star(j,y(j)), \quad \text{for } k\in I_n, \tag{3.12}$$

and so $y(k) \geqslant (1/m)$ for $k \in I_n$. Also, as in §1 (see (1.11)), we know that $\|y\| = \sup_{j\in I_n} y(j) = y(T+n)$. We next *claim* that

$$\|y\| = y(T+n) \neq M \quad \text{(here } M \text{ is as in (3.8))}. \tag{3.13}$$

We have immediately, from (3.12), (3.3), (1.19) and (3.5), that

$$\begin{aligned} y(T+n) &\leqslant \frac{1}{m} + \left\{1 + \frac{h(y(T+n))}{g(y(T+n))}\right\} \sum_{j=p}^{T+p} (-1)^{n-p} G(T+n,j) g(y(j)) \\ &\leqslant \epsilon + \left\{1 + \frac{h(y(T+n))}{g(y(T+n))}\right\} \sum_{j=p}^{T+p} (-1)^{n-p} G(T+n,j) g(\theta y(T+n)) \\ &\leqslant \epsilon + [g(y(T+n)) + h(y(T+n))] K_\theta Q. \end{aligned}$$

Consequently,

$$\frac{y(T+n)}{\epsilon + [g(y(T+n)) + h(y(T+n))] K_\theta Q} \leqslant 1. \tag{3.14}$$

Now (3.9) and (3.14) imply $y(T+n) \neq M$, and so (3.13) is true. Consequently, Theorem 1.2 guarantees that $(3.10)^m$ has a solution $y_m \in C(I_n)$ with $(1/m) \leqslant y_m(i) \leqslant M$ for $i \in I_n$. Next we obtain a sharper lower bound on y_m. Notice that y_m satisfies

$$y_m(i) = \frac{1}{m} + \sum_{j=p}^{T+p} (-1)^{n-p} G(i,j) f(j, y_m(j)), \quad \text{for } i \in I_n. \tag{3.15}$$

Also, (3.4) guarantees the existence of a continuous function $\psi_M : J_p \to (0,\infty)$ with $f(i,u) \geqslant \psi_M(i)$ for $(i,u) \in J_p \times (0,M]$. This, together with (3.15), yields

$$y_m(i) \geqslant \sum_{j=p}^{T+p} (-1)^{n-p} G(i,j) \psi_M(j) \equiv \Phi_M(i), \quad \text{for } i \in J_p. \tag{3.16}$$

Clearly,

$$\{y_m\}_{m\in N_0} \text{ is a bounded family on } I_n. \tag{3.17}$$

The Arzela–Ascoli Theorem [**3**] guarantees the existence of a subsequence N of N_0 and a function $y \in C(I_n)$ with $y_n \to y$ in $C(I_n)$ as $n \to \infty$ through N. Also

$$y(0) = \cdots = y(p-1) = 0 \quad \text{and} \quad \Delta^i y(T+1) = 0, \quad p \leqslant i \leqslant n-1.$$

Fix $i \in J_p$, then $y_m, m \in N$ satisfies (3.15). Also,

$$\Phi_M = \min_{i\in J_p} \Phi_M(i) \leqslant y_m(j) \leqslant M, \quad \text{for } j \in J_p \text{ and } m \in N. \tag{3.18}$$

Let $m \to \infty$ through N in (3.15) to obtain

$$y(i) = \sum_{j=p}^{T+p} (-1)^{n-p} G(i,j) f(j, y(j)), \quad \text{for } i \in J_p.$$

Also, notice that (3.18) implies $y(j) \geqslant \Phi_M > 0$ for $j \in J_p$. □

Example 3.2. Consider the focal discrete boundary-value problem

$$\left.\begin{aligned} (-1)^{n-p}\Delta^n y(k-p) &= \mu([y(k)]^{-\alpha} + Ae^{y(k)}), \quad \text{for } k \in J_p, \\ \Delta^i y(0) &= 0, \quad 0 \leqslant i \leqslant p-1, \\ \Delta^i y(T+1) &= 0, \quad p \leqslant i \leqslant n-1, \end{aligned}\right\} \tag{3.19}$$

with $\alpha > 0$, $\beta \geqslant 0$, $A \geqslant 0$ and $\mu > 0$. If

$$\mu < \frac{\theta^\alpha}{Q} \sup_{c \in (0,\infty)} \left(\frac{c^{\alpha+1}}{1 + Ac^\alpha e^c} \right), \tag{3.20}$$

where

$$\theta = \frac{p^{(p)}}{(T+n)^{(p)}} \quad \text{and} \quad Q = \sum_{j=p}^{T+p} (-1)^{n-p} G(T+n, j),$$

then (3.19) has a solution $y \in C(I_n)$ with $y(i) > 0$ for $i \in J_p$.

The result follows immediately from Theorem 3.1 with $g(u) = \mu u^{-\alpha}$ and $h(u) = \mu A e^u$.

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