Canad. Math. Bull. Vol. **23** (3), 1980

A FUNCTION WHICH TRANSFORMS CERTAIN GRAPHS INTO STRAIGHT LINES FOR SIMULTANEOUS SOLUTION

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A function M is defined which maps the plane onto a square region in such a way that the planar graphs ln, exp, X, $-X$, $1/X$, and all compositions formed from them are transformed into straight lines. One can then solve for their intersections. It also provides a natural definition for the repeated composition of ln with itself t times, where t can be a non-integer.

An *infinite exponential* (inf exp) means a sequence of the form $a_1, a_1^{a_2}, a_1^{a_2^{a_3}}, \ldots$, and is sometimes [see 1, p. 150] denoted by $E(a_1, a_2, \ldots)$. A *base e inf exp* is one in which each a_n is e or e^{-1}. Since $E(e, e, e, \ldots) \to \infty$ we know $E(e^{-1}, e, e, e, \ldots) \to 0$. This allows any base e inf exp of the form $E(a_1, \ldots, a_n, e^{-1}, e, e, e, \ldots)$ to be replaced by a *base e finite exponential* $E(a_1, \ldots, a_n, 0)$. In the finite case it makes no difference whether a_n is e or e^{-1}; so every base e fin exp $E(a_1, \ldots, a_{n-1}, a_n, 0)$ has two base e inf exp representations $E(a_1, \ldots, a_{n-1}, e, e^{-1}, e, e, e, \ldots)$ and $E(a_1, \ldots, a_{n-1}, e^{-1}, e^{-1}, e, e, e, \ldots)$ which differ only in the nth element.

THEOREM 1. *If $x > 0$ and x is not a base e fin exp, then x has exactly one base e inf exp representation.*

We define $N(x) = e^{-x}$, $E^0(x) = x$, $E^1(x) = \exp(x)$, $E^2(x) = \exp\circ\exp(x)$, $E^3(x) = \exp\circ\exp\circ\exp(x), \ldots$, and shorten $N\circ E^n$ to NE^n.

Proof. If $0 < x < 1$ there is an x_1 between 0 and 1 and a nonnegative integer n_1 (also denoted by $n1$ or $n, 1$) such that $x = NE^{n1}(x_1)$. Recursion gives a sequence $x_1, x_2, \ldots$, each between 0 and 1, and a sequence of nonnegative integers $n_1, n_2, \ldots$ such that

$$x = NE^{n1}(x_1) = NE^{n1}\circ NE^{n2}(x_2) = NE^{n1}\circ NE^{n2}\circ Ne^{n3}(x_3) = \cdots .$$

Subject Classifications: Primary; 40A99 sequences, series, summability—convergence and divergence of infinite limiting processes—miscellaneous. Secondary; 41A30 Approximations and expansions—approximations by other special function classes.

Key words and phrases: Infinite exponential representation, Base e infinite exponential, Repeated-composition function, The m function, The M function, Planar graphs' intersection points, Ordered commutative group of $\ln^t$ graphs.

Received by the editors July 25, 1975 and, in revised form, December 1, 1977 and January 3, 1979.

Now x is interior to each of the intervals $[0, 1]$, $[NE^{n1}(1), NE^{n1}(0)]$, $[NE^{n1}\circ NE^{n2}(0), NE^{n1}\circ NE^{n2}(1)]$, $[NE^{n1}\circ NE^{n2}\circ NE^{n3}(1), NE^{n1}\circ NE^{n2}\circ NE^{n3}(0)], \ldots$ because $0<x_i<1$, and the composition function $NE^{n1}\circ NE^{n2}\circ\cdots\circ NE^{ni}$ is increasing/decreasing if the number of iterations of N is even/odd. Moreover, this sequence of intervals is nested because of the fact that $[NE^n(1), NE^n(0)]\subseteq [0, 1]$ for each n, and the just-mentioned increasing/decreasing property.

These interval lengths $\to 0$. As proof we show it for the even-numbered intervals. If i is even, the ith interval has length $|[NE^{n1}\circ NE^{n2}\circ\cdots\circ NE^{ni}(1)-NE^{n1}\circ NE^{n2}\circ\cdots\circ NE^{ni}(0)]/[1-0]|$ which, by the mean value theorem, is $|[NE^{n1}\circ NE^{n2}\circ\cdots\circ NE^{ni}]'(\xi_i)|$ for some ξ_i in $[0, 1]$. Grouping the composition by two's and using the chain rule gives

$$\begin{aligned}&|[NE^{n1}\circ NE^{n2}]'(NE^{n3}\circ\cdots\circ NE^{ni}(\xi_i))\\ &\qquad\cdot[NE^{n3}\circ NE^{n4}]'(NE^{n5}\circ\cdots\circ NE^{ni}(\xi_i))\cdot\cdots\\ &\qquad\cdots\cdot[NE^{n,i-1}\circ NE^{ni}]'(\xi_i)|\\ &=|[NE^{n1}\circ NE^{n2}]'(\xi_2)|\cdot|[NE^{n3}\circ NE^{n4}]'(\xi_4)|\cdot\cdots\cdot|[NE^{n,i-1}\circ NE^{ni}]'(\xi_i)|,\end{aligned}$$

where $\xi_2, \xi_4, \ldots, \xi_i$ are all >0. This product is $\leq(4/e^2)^{i/2}$ because the $i/2$ factors can each be shown to be $\leq 4/e^2$ as follows.

$$\begin{aligned}|[NE^n]'(\xi)|&=|[-NE^n\cdot E^n\cdot E^{n-1}\cdot\cdots\cdot E^1](\xi)|\\ &=[(E^n\cdot E^n/E^{n+1})\cdot(E^{n-1}\cdot E^{n-1}/E^n)\cdot\cdots\cdot(E^1\cdot E^1/E^2)\cdot 1/E^1](\xi)\\ &=[X^2/E^1](E^n(\xi))\cdot[X^2/E^1](E^{n-1}(\xi))\cdot\cdots\cdot[X^2/E^1](E(\xi))\cdot N(\xi)\\ &\leq(4/e^2)\cdot(4/e^2)\cdot\cdots\cdot(4/e^2)\cdot 1=(4/e^2)^n\end{aligned}$$

because $4/e^2$ is the maximum x^2/e^x when $x\geq 0$. Therefore

$$\begin{aligned}|[NE^{n,k-1}\circ NE^{n,k}]'(\xi_k)|&=|[NE^{n,k-1}]'(NE^{n,k}(\xi_k))|\cdot|[NE^{n,k}]'(\xi_k)|\\ &\leq(4/e^2)^{n,k-1}\cdot(4/e^2)^{n,k}\leq 4/e^2\end{aligned}$$

provided that $n_{k-1}+n_k>0$.

In case $n_{k-1}+n_k=0$, then $|[NE^0\circ NE^0]'(\xi_k)|=|[N\circ N]'(\xi_k)|\leq 1/e<4/e^2$, because $1/e$ is the maximum of $[N\circ N]'$ when $x\geq 0$. Since each of the $i/2$ factors is $\leq 4/e^2$, the ith interval has length $\leq(4/e^2)^{i/2}$, and this approaches 0 as $i\to\infty$.

Since the nested intervals close down on x, their end points 0, 1, $NE^{n1}(0)$, $NE^{n1}(1)$, $NE^{n1}\circ NE^{n2}(0), \ldots$ form a subsequence of a unique base e inf exp converging to x.

If $x>1$ then $x=E^n(x^*)$ for an x^* in $(0, 1)$ and $x=E^n$ (inf exp for x^*).

THEOREM 2. *Every base e inf exp converges except* $E(e, e, e, \ldots)$.

Proof. If $E(a_1, a_2, \ldots)$ is of the form $E(a_1, \ldots, a_n, e^{-1}, e, e, e, \ldots)$, the limit is $E(a_1, \ldots, a_{n-1}, 0)$. If it is not of this form, certain terms of $E(a_1, a_2, \ldots)$ are

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end points in a unique sequence of nested intervals closing down on one number, as was seen above.

DEFINITION 1. *Let m denote the function such that, for each real number x and base e inf exp representation $E(a_0, a_1, a_2, \ldots)$ for e^x,*

$$m(x) = \sum_{j=0}^{\infty} \left[2^{-j} \prod_{i=0}^{j} \ln(a_i)\right].$$

For example, if e^x is $E(e, e, e^{-1}, e, e^{-1}, e, e^{-1}, \ldots)$ then $m(x) = 1 + 2^{-1} - 2^{-2} - 2^{-3} + 2^{-4} + 2^{-5} - 2^{-6} - \ldots$, or 6/5. If x is a fin exp, e^x has two representations, $E(a_0, \ldots, a_k, e, e^{-1}, e, e, e, \ldots)$ and $E(a_0, \ldots, a_k, e^{-1}, e^{-1}, e, e, e, \ldots)$, but this causes no ambiguity because $\sum_{j=k+1}^{\infty} [2^{-j} \prod_{i=0}^{j} \ln(a_i)]$ is zero in both cases.

THEOREM 3. *m has domain $(-\infty, \infty)$, range $(-2, 2)$, and is increasing and continuous.*

Proof. That m has domain $(-\infty, \infty)$ is clear from Theorem 1 and Def. 1. m has range $(-2, 2)$ because this is the set of all sums of series of the form $S_0 + S_1/2 + S_2/4 + S_3/8 + \cdots$ where each S_i is ± 1 and not all S_i are alike. Each such series determines an $a_0, a_1, a_2, \ldots$ (let $\prod_{i=0}^{j} \ln a_i = S_j$) for a base e inf exp which, by Theorem 2, converges. ± 2 is not in m's range because $a_0, a_1, a_2, a_3, \ldots$ would be $e^{\pm 1}, e, e, e, \ldots$ and x would be the divergent $\pm E(e, e, e, \ldots)$.

Now we show that m is increasing. Suppose $a < b$ and $e^a = E(a_0, a_1, \ldots)$ and $e^b = E(b_0, b_1, \ldots)$. Then $a_i \neq b_i$ for some i, otherwise $E(a_0, a_1, \ldots)$ would have two limits: e^a and e^b. Let k denote the lowest such i. Now $E(a_0, \ldots, a_{k-1}, t)$ is a function of t which is increasing/decreasing if the number of iterations of e^{-1} is even/odd. Let p denote this number of iterations. When p is even or zero we have $a_k = e^{-1}$, $b_k = e$, and $E(a_0, \ldots, a_{k-1}, e^{-1}, a_{k+1}, \ldots) = e^a \leq E(a_0, \ldots, a_{k-1}, 1) \leq e^b = E(a_0, \ldots, a_{k-1}, e, b_{k+1}, \ldots)$, because the alternative is that $a_k = e$, $b_k = e^{-1}$, $e^a \geq e^b$, and $a \not< b$. Therefore

$$m(a) = \left[\sum_{j=0}^{k-1} \left(2^{-j} \prod_{i=0}^{j} \ln a_i\right)\right] - 2^{-k} + \sum_{j=k+1}^{\infty} \left(2^{-j} \prod_{i=0}^{j} \ln a_i\right)$$

$$< \left[\sum_{j=0}^{k-1} \left(2^{-j} \prod_{i=0}^{j} \ln a_i\right)\right] + 2^{-k} + \sum_{j=k+1}^{\infty} \left(2^{-j} \prod_{i=0}^{j} \ln b_i\right) = m(b).$$

Equality is impossible because we could not have the equations $\sum_{j=k+1}^{\infty} (2^{-j} \prod_{i=0}^{j} \ln a_i) = 2^{-k}$ and $\sum_{j=k+1}^{\infty} (2^{-j} \prod_{i=0}^{j} \ln b_i) = -2^{-k}$ both true; it would mean that $(a_{k+1}, a_{k+2}, \ldots) = (e^{-1}, e, e, e, \ldots)$ and $(b_{k+1}, b_{k+2}, \ldots) = (e^{-1}, e, e, e, \ldots)$ are both true and $e^a = e^b$. If p is odd then $a_k = e$ and $b_k = e^{-1}$, and $m(a) < m(b)$ can be shown by modifying the above argument to suit the odd case. m is continuous because it increases and maps $R \times R$ onto $(-2, 2)$.

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THEOREM 4. *If* $x \in \{\text{fin exp}\}$ *then* $m'(x)=\infty$; *this set is dense in* R.

Proof. If x is a fin exp, there is a function $E(a_1, \ldots, a_k, t)$ such that $x = E(a_1, \ldots, a_k, 0)$. Denote this function by $A_k(t)$. For $m'(x)$ to exist it is necessary that

$$\begin{aligned} m'(x) &= \operatorname*{limit}_{n\to\infty} [m(A_k \circ NE^n(0)) - m(A_k(0))]/[A_k \circ NE^n(0) - A_k(0)] \\ &= \operatorname*{limit}_{n\to\infty} \pm 2^{-k-n}/[A_k \circ NE^n(0) - A_k(0)] \\ &= \operatorname*{limit}_{n\to\infty} [\pm e^{(-k-n)\ln 2}/NE^n(0)] \cdot [NE^n(0) - 0]/[A_k \circ NE^n(0) - A_k(0)] \\ &= \operatorname*{limit}_{n\to\infty} \pm e^{-k} \cdot \exp(E^n(0) - n \cdot \ln 2) \cdot 1/A_k'(0) = \pm e^{-k} \exp(\infty)/A_k'(0) = \infty. \end{aligned}$$

Lastly, {fin exp} is dense in R because every $x \in R$ is the limit of a sequence of fin exps: its inf exp representation.

THEOREM 5.
(i) $m(-x) = -m(x)$,
(ii) $m(1/x) = -m(x) \pm 2$ *if* x *is positive/negative*,
(iii) $m(E^n(x)) = 2^{-n}m(x) + 2 - 2^{-n+1}$, *and*
(iv) $m(\ln^n(x)) = 2^n m(x) + 2 - 2^{n+1}$.

Proof. (i) If $e^x = E(a_0, a_1, a_2, \ldots)$ then $e^{-x} = E(a_0^{-1}, a_1, a_2, \ldots)$ and $m(-x)$ is obtained by reversing all signs in the series for $m(x)$.

(ii) $e^{1/x} = E(a_0, a_1^{-1}, a_2, \ldots)$. Adding the equations

$$m(x) = \ln a_0 + \textstyle\sum_{j=1}^{\infty} [2^{-j} \prod_{i=0}^{j} \ln a_i]$$

and

$$m(1/x) = \ln a_0 - \textstyle\sum_{j=1}^{\infty} [2^{-j} \prod_{i=0}^{j} \ln a_i]$$

gives $m(x) + m(1/x) = 2 \cdot \ln a_0$. But $\ln a_0 = \pm 1$ if $x > 0/x < 0$.

(iii) $\exp(E^n(x)) = E(e, e, \ldots, e, a_0, a_1, a_2, \ldots)$, and

$$\begin{aligned} m(E^n(x)) &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \sum_{j=0}^{\infty} \left[2^{-j-n} \prod_{i=0}^{j} \ln a_i\right], \\ &= [1 - 2^{-n}]/[1 - \tfrac{1}{2}] + 2^{-n} \sum_{j=0}^{\infty} \left[2^{-j} \prod_{i=0}^{j} \ln a_i\right] = 2 - 2^{-n+1} + 2^{-n} m(x). \end{aligned}$$

(iv) We let $E^n(x) = y$ and $x = \ln^n y$ and substitute into (iii).

DEFINITION 2. *Let* M *denote the function such that* $M(x, y) = (m(x), m(y))$ *for each ordered real-number pair* (x, y).

Then M is a $1-1$ mapping of $R \times R$ onto $(-2, 2) \times (-2, 2)$. For each planar graph G let G_M denote its M image in $(-2, 2) \times (-2, 2)$. Then $M(x, G(x))$ is both $(m(x), G_M(m(x)))$ and $(m(x), m(G(x)))$; so $G_M(m(x)) = m(G(x))$. The

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planar graph $\ln^n$ has an M image which is that part of the line $Y = 2^n X + 2 - 2^{n+1}$ lying in $(-2, 2) \times (-2, 2)$, because $M(w, \ln^n w) = (m(w), 2^n m(w) + 2 - 2^{n+1})$ by Theorem 5-iv. Similarly, X, $-X$, $1/X$, and E^n have M images on the lines $Y = X$, $Y = -X$, $Y = -X \pm 2$ if $x > 0/x < 0$, and $Y = 2^{-n} X + 2 - 2^{-n+1}$, respectively.

THEOREM 6. *If $G_1, G_2, \ldots, G_p$ is any finite sequence of functions selected, with repetition allowed, from the set $\{X, -X, 1/X, E^1, \ln, E^2, \ln^2, E^3, \ln^3, \ldots\}$, then the composition function $G_1 \circ G_2 \circ \cdots \circ G_p$, if it exists, has an M image which is either a straight line or a finite set of linear intervals in the space $(-2, 2) \times (-2, 2)$.*

Proof. If F and G are any two planar graphs whose M images are straight lines, say $a_f X + b_f$ and $a_g X + b_g$, and if $F \circ G$ exists, then

$$\begin{aligned} M(x, F \circ G(x)) &= (m(x), m(F \circ G(x))) = (m(x), F_M(m(G(x)))) \\ &= (m(x), F_M \circ G_M(m(x))) = (m(x), [a_f X + b_f] \circ [a_g X + b_g](m(x))) \\ &= (m(x), a_f a_g m(x) + a_f b_g + b_f). \end{aligned}$$

So $[F \circ G]_M$ lies on the line $a_f a_g X + a_f b_g + b_f$. By induction, any finite multiple-composition $F \circ G \circ H \circ \cdots$ of members of $\{X, -X, 1/X, E^1, \ln, E^2, \ln^2, \ldots\}$ has an M image which is linear, or linear intervals.

Using $\ln^{-n}$ and $\ln^0$ for E^n and X, Theorem 5 suggests a generalization of $\ln^n$.

DEFINITION 3. *If $t \in R$ let $\ln^t$ mean the planar graph such that $m(\ln^t(x)) = 2^t m(x) + 2 - 2^{t+1}$ for each $x \in R$ such that $m(x) > 2 - 2^{-t+2}$.*

Then $\ln^t$ means the graph whose M image is that subinterval of the straight line with slope 2^t and y-intercept $2 - 2^{t+1}$ which lies within the space $(-2, 2) \times (-2, 2)$.

EXAMPLE. Evaluating $\ln^{3/2}(e)$, $m(\ln^{3/2}(e)) = 2^{3/2} m(e) + 2 - 2^{1+3/2} = (\sqrt{8})(\frac{3}{2}) + 2 - \sqrt{32} = 0.5858$. Therefore $\ln^{3/2}(e) = m^{-1}(0.5858)$.

Tables of the m function are available from the author.

THEOREM 7. *If t and u are real numbers then $\ln^t \circ \ln^u = \ln^{t+u}$.*

Proof. If $x \in R$ and $\ln^t \circ \ln^u(x)$ exists, then

$$\begin{aligned} m(\ln^t \circ \ln^u(x)) &= [\ln^t]_M(m(\ln^u(x))) = [\ln^t]_M \circ [\ln^u]_M(m(x)) \\ &= [2^t X + 2 - 2^{t+1}] \circ [2^u X + 2 - 2^{u+1}](m(x)) \\ &= 2^{t+u} m(x) + 2 - 2^{t+u+1} = m(\ln^{t+u}(x)). \end{aligned}$$

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