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Isoresonant Complex-valued Potentials and Symmetries

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Abstract. Let X be a connected Riemannian manifold such that the resolvent of the free Laplacian $(\Delta - z)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}^+$, has a meromorphic continuation through $\mathbb{R}^+$. The poles of this continuation are called resonances. When X has some symmetries, we construct complex-valued potentials, V, such that the resolvent of $\Delta + V$, which has also a meromorphic continuation, has the same resonances with multiplicities as the free Laplacian.

1 Introduction and Statement of the Results

Let (X, g) be a connected Riemannian manifold with dimension $n \geq 2$. On X we have the free non-negative Laplacian, Δ, acting on functions with domain $H^2(X)$ whose spectrum is included in $\mathbb{R}^+$. So for $z \in \mathbb{C} \setminus \mathbb{R}^+$, the resolvent $R_0(z) := (\Delta - z)^{-1}$ of the Laplacian is a bounded operator from $L^2(X)$ to $H^2(X)$. We will assume that the resolvent has a meromorphic continuation through $\mathbb{R}^+$ in a domain of $\mathbb{C}$, D^+. For example, this holds for Euclidean spaces, asymptotically hyperbolic manifolds and manifolds with asymptotically cylindrical ends.

We call a pole of R_0 in D^+ a *resonance* of Δ, and we write $\operatorname{Res}(\Delta)$ for the set of these poles. This definition includes eigenvalues in the set of resonances. If $z_0 \in \operatorname{Res}(\Delta)$, then, in a neighbourhood of z_0 in D^+, we have a finite Laurent expansion:

$$R_0(z) = \sum_{i=1}^{p} (z - z_0)^{-i} S_i + H(z),$$

where S_i has a finite rank, H is holomorphic and p is the *order* of the resonance. We call the *multiplicity* of z_0 the dimension of the resonant space which is the range of S_1. See [Agm98].

If we perturb the Laplacian with a potential V and if V is sufficiently decreasing at infinity on X, for example compactly supported, then the resolvent of $\Delta + V$, $(\Delta + V - z)^{-1}$, can also be continued meromorphically to D^+. Then we can introduce the resonances of $(\Delta + V)$, and we write their set $\operatorname{Res}(\Delta + V)$.

For such a V, sufficiently decreasing, we have the equality $\sigma_{\text{ess}}(\Delta + V) = \sigma_{\text{ess}}(\Delta)$ for the essential spectrum, because V is then relatively compact with respect to Δ. So we can wonder how these potentials modify resonances. We reach the main question of this work: Do there exist potentials V such that $\operatorname{Res}(\Delta + V) = \operatorname{Res}(\Delta)$?

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We will construct such potentials and call them *isoresonant*. In terms of the inverse problem, we can't detect their presence only with the observation of the set of resonances.

Our potentials will be complex-valued, and this is crucial. For example, it is known that in $\mathbb{R}^n$, for $n \geq 2$ and even or $n = 3$, nontrivial, real valued, smooth and compactly supported potentials create an infinite number of resonances. See [Mel95], [SBZ95], [Chr99], [SB99].

We have been inspired by the work of Christiansen in [Chr06] and [Chr08]. She constructs in Euclidean spaces $\mathbb{R}^n$ ($n \geq 2$) isoresonant complex potentials, *i.e.*, in this case: $\operatorname{Res}(\Delta + V) = \operatorname{Res}(\Delta) = \varnothing$. She uses an action of $\mathbb{S}^1$ on $\mathbb{R}^n$. We generalize this construction to manifolds that have an isometric action of $\mathbb{S}^1$, and we use other symmetries, such as $(\mathbb{S}^1)^m$ and $SO(n)$. On these manifolds, the free Laplacian already has some resonances, so there is more work to prove the isoresonance of the potentials. By comparison, in the Euclidean space it is sufficient to prove $\operatorname{Res}(\Delta+V) \subset \operatorname{Res}(\Delta)$, because $\operatorname{Res}(\Delta) = \varnothing$.

We are going to describe the method for the construction and the statements of the results. We assume that (X, g) has an isometric action of $\mathbb{S}^1$. This action induces a unitary representation of $\mathbb{S}^1$ on $L^2(X)$:

$$
\begin{aligned}
S^1 &\to U(L^2(X)),\\
e^{i\theta} &\to f \to \left(x \to f(e^{-i\theta}.x)\right).
\end{aligned}
$$

Then we can decompose $L^2(X)$ according to isotypical components:

$$
L^2(X) = \overline{\bigoplus_{j\in\mathbb{Z}}^{\perp} L^2_j(X)},
$$

with, for all $j \in \mathbb{Z}$,

$$
L^2_j(X) := \{f \in L^2(X) : \forall\theta \in [0, 2\pi], \forall x \in X, f(e^{-i\theta}.x) = e^{ij\theta} f(x)\},
$$

is the space of $\mathbb{S}^1$ homogeneous functions of weight j.

We take for our isoresonant potentials sums of $\mathbb{S}^1$ homogeneous functions with weights of the same sign. Such functions create a shift on the isotypical components of $L^2(X)$: if $V \in L^\infty(X) \cap L^2_m(X)$ and $f \in L^2_j(X)$, then $Vf \in L^2_{j+m}(X)$. On the contrary, the Laplacian stabilizes these isotypical components. Thanks to this shift we will prove the inclusion $\operatorname{Res}(\Delta+V) \subset \operatorname{Res}(\Delta)$, first for truncated V, and then for all V, thanks to a characterization of resonances as zeros of regularized determinants.

On the way, we have to estimate, for all compact K, the lower bound of the spectrum of the Dirichlet Laplacian acting on $\mathbb{S}^1$ homogeneous functions of weight j supported in K (we denote this space $L^2_j(K)$). This is an interesting result on its own:

Proposition 1.1 *Let K be a compact manifold with boundary, having an action of $\mathbb{S}^1$ and a metric g such that $\mathbb{S}^1$ acts by isometries on (K, g) and g has a product form in a neighborhood of the boundary of K. Then there exist strictly positive constants, $C_1(K)$ and $C_2(K)$, such that, for all $j \in \mathbb{Z}$, we have:*

$$
C_1 j^2 \leq \operatorname{Min}\operatorname{Spec} \Delta_{L^2_j(K)} \leq C_2(1 + j^2).
$$

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For the other inclusion, $\operatorname{Res}(\Delta) \subset \operatorname{Res}(\Delta + V)$, we use Agmon's perturbation theory of resonances, developed in [Agm98]. Thanks to this theory we can view resonances as eigenvalues of auxiliary operators and so we can use Kato's theory in order to study their perturbations.

Finally we get the following result, given here in restricted cases for simplicity:

Theorem 1.2 *On the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $\mathbb{H}^n$, let be the potential*

$$V = \sum_{m=1}^{M} V_m,$$

where $V_m \in L^\infty(X)$ is compactly supported and $\mathbb{S}^1$ homogeneous with weight m.

Then in $\mathbb{C}$ we have $\operatorname{Res}(\Delta + V) = \operatorname{Res}(\Delta)$ with the same multiplicities.

See the theorem 3.1 for the general case with a more general manifold, an infinite sum for V, and V not compactly supported.

Remark 1.3 Instead of the free Laplacian we can perturb $\Delta + V_0$ with V_0 a real, compactly supported and $\mathbb{S}^1$ invariant potential and the result becomes $\operatorname{Res}(\Delta + V_0 + V) = \operatorname{Res}(\Delta + V_0)$ with the same multiplicities. We can imagine the perturbation of other operators which respect the decomposition of $L^2(X)$ according to the isotypical components.

The construction of isoresonant potentials using the action of $(\mathbb{S}^1)^m$ is essentially the same as in the case $\mathbb{S}^1$ so we don't describe it in this article, but it can be found in [Aut08]. On the contrary, if we look at the action of $\mathrm{SO}(n)$ $(n \geq 3)$, as this group is not commutative, we don't have any simple description of the isotypical components. Then we add an hypothesis and assume that we can write

$$L^2(X) = \bigoplus_{k\in\mathbb{N}} L^2(\mathbb{R}^+) \otimes H^k,$$

where $H^k = \operatorname{Ker}\big(\Delta_{\mathbb{S}^{n-1}} - k(k+n-2)\big)$, $k \in \mathbb{N}$, is the eigenspace of the Laplacian on the sphere $\mathbb{S}^{n-1}$. As in the case $\mathbb{S}^1$, we are going to construct some V which induces a shift in this decomposition of $L^2(X)$. This time V is a sum of highest weight vectors of the representations H^k of the complexification of the Lie algebra $\mathfrak{so}_n$. Moreover, for the action of $\mathrm{SO}(n)$ we don't need to use the first proposition of this introduction, which simplifies the proof of the isoresonance. Here we have been inspired by the construction of isospectral potentials by Guillemin and Uribe in [GU83]. In that article, the authors treat the case of compact manifolds and isospectral potentials on them.

The isoresonant potentials don't modify the set of resonances of the free Laplacian and their multiplicity. We can wonder if, with more information, we would be able to detect them. In this way, we prove that on $\mathbb{H}^2$ there exist some potentials among the family of isoresonant potentials which modify the order of the resonances. On $\mathbb{H}^2$ resonances of the free Laplacian are, up to a change of spectral parameter, the negative integers with order one. Taking for the hyperbolic plane the model $\mathbb{R}^+ \times \mathbb{S}^1$ with coordinates (r, θ) and metric $g = \mathrm{d}r^2 + \mathrm{sh}(r)^2\mathrm{d}\theta^2$, we have

Proposition 1.4 *On the hyperbolic plane $\mathbb{H}^2$, let k be a strictly positive integer. There exists a potential $V \in \mathcal{F} := \{V_m(r)e^{\mathrm{i}m\theta} : m \in \mathbb{Z} \setminus \{0\}, V_m \in L_c^\infty(\mathbb{R}^+)\}$ such that $-k$ is a resonance of $\Delta + V$ with order strictly bigger than one.*

In the last part of this article, we construct isoresonant potentials using the $\mathbb{S}^1$ action on another example, the catenoid, *i.e.*, $\left(\mathbb{R} \times \mathbb{S}^1, \mathrm{d}r^2 + (r^2 + a^2)\mathrm{d}\alpha^2\right)$ with $(r, e^{i\alpha}) \in \mathbb{R} \times \mathbb{S}^1$ and $a \in \mathbb{R}$. We treat this example separately because we can't use Agmon's theory for defining the continuation of the resolvent and for perturbations of resonances; instead we have to use a complex scaling method, following [WZ00].

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2 Framework and Conditions

We take a cover $f \colon \Sigma \to \Omega$, where Ω is an open set of $\mathbb{C}$, and an unbounded domain $D \subset \Sigma$ such that $f(D) \subset \mathbb{C} \setminus \mathbb{R}^+$. We note $R_0(\lambda) := \left(\Delta - f(\lambda)\right)^{-1}$ which is first defined holomorphic in D with values in $\mathcal{L}\left(L^2(X)\right)$ (the space of bounded operators from $L^2(X)$ to itself). Let two Banach spaces B_0 and B_1 be such that

$$B_0 \overset{J_0}{\hookrightarrow} L^2(X) \overset{J}{\hookrightarrow} B_1,$$

where J_0 and J are continuous injections, $J_0(B_0)$ is dense in $L^2(X)$ and $J\left(L^2(X)\right)$ is dense in B_1. We note, for $\lambda \in D$,

$$\widetilde{R}_0(\lambda) = JR_0(\lambda)J_0.$$

$\widetilde{R}_0$ is holomorphic on D with values in $\mathcal{L}(B_0, B_1)$.

Our first assumption is the following condition.

Condition A $\widetilde{R}_0$ has a meromorphic continuation with finite rank poles (we will say *finite-meromorphic*) from D to D^+ a domain of Σ.

In order that Condition A not hold trivially, we assume that $f(D^+)$ intersects the essential spectrum of Δ.

Let us give some examples:

- X is $\mathbb{R}^n$ with the Euclidean metric,
  - If n is odd, then we take $\Sigma = \mathbb{C}$ and $R_0(\lambda) := (\Delta - \lambda^2)^{-1}$ is first defined in $D = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > 0\}$ with values in $\mathcal{L}\left(L^2(\mathbb{R}^n)\right)$ and has, for all $N > 0$, a holomorphic continuation in $D_N^+ = \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < N\}$ with values in $\mathcal{L}\left(e^{-N\langle z\rangle}L^2(\mathbb{R}^n), e^{N\langle z\rangle}L^2(\mathbb{R}^n)\right)$, where $\langle z\rangle = (1 + |z|^2)^{\frac{1}{2}}$. See [Mel95] and [SBZ95].
  - If n is even, then we take for Σ the logarithmic cover of $\mathbb{C} \setminus \{0\}$, and $R_0(\lambda) := (\Delta - e^{2\lambda})^{-1}$ is first defined in $D = \{\lambda \in \mathbb{C}; 0 < \operatorname{Im}\lambda < \pi\}$ with values in $\mathcal{L}\left(L^2(\mathbb{R}^n)\right)$ and has, for all $N > 0$, a holomorphic continuation in $D_N^+ = \{\lambda \in \mathbb{C} : |\operatorname{Im}(e^\lambda)| < N\}$ with values in $\mathcal{L}\left(e^{-N\langle z\rangle}L^2(X), e^{N\langle z\rangle}L^2(X)\right)$. See [Mel95].

- X is an asymptotically hyperbolic manifold. We begin with the definition of such a manifold. Let $\overline{X} = X \cup \partial\overline{X}$, a smooth compact manifold of dimension n with boundary $\partial\overline{X}$ and ρ_0 a boundary-defining function that is a smooth function on $\overline{X}$ such that

$$\rho_0 \geq 0, \quad \partial\overline{X} = \{m \in \overline{X}; \rho_0(m) = 0\}, \quad \mathrm{d}\rho_0|_{\partial\overline{X}} \neq 0.$$

We say that a smooth metric g on X is asymptotically hyperbolic if $\rho_0^2 g$ continues as a smooth metric on $\overline{X}$ and $|\mathrm{d}\rho_0|_{\rho_0^2 g} = 1$ on $\partial\overline{X}$. Thanks to this condition, the sectional curvature of g tends to -1 at the boundary, and there exists a function ρ defining the boundary, a collar neighborhood of the boundary, $U_\rho := [0,\varepsilon) \times \partial\overline{X}$, and a family $h(\rho)$, $\rho \in [0,\varepsilon)$, of smooth metrics on $\partial\overline{X}$ such that

$$g = \frac{\mathrm{d}\rho^2 + h(\rho)}{\rho^2} \quad \text{on } U_\rho. \tag{2.1}$$

For example the hyperbolic space $\mathbb{H}^n$ and its convex co-compact quotients are asymptotically hyperbolic.
We take $\Sigma = \mathbb{C}$ and $R_0(\lambda) := \big(\Delta - \lambda(n-1-\lambda)\big)^{-1}$ is first defined and meromorphic in $D = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \frac{n-1}{2}\}$ with values in $\mathcal{L}(L^2(X))$ and λ is one of its poles if and only if $\lambda(n-1-\lambda) \in \sigma_d(\Delta)$, and it is of finite rank.
Mazzeo and Melrose [MM87] and after Guillarmou [Gui05] have proved that R_0 has a finite-meromorphic continuation in $\mathbb{C} \setminus (\frac{n}{2} - \mathbb{N})$ and in all $\mathbb{C}$ if and only if the metric g is even. The metric g is *even* if the family $h(\rho)$ defined in (2.1) has a Taylor's series in $\rho = 0$ only with even powers of ρ (it does not depend on the choice of ρ). More precisely, for all $N \geq 0$, R_0 has a finite-meromorphic continuation on $D_N^+ := \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > \frac{n-1}{2} - N\}$ if g is even, and otherwise on $D_N^+ \setminus (\frac{n}{2} - \mathbb{N})$, with values in $\mathcal{L}\big(\rho^N L^2(X), \rho^{-N} L^2(X)\big)$.
- X is a Riemannian manifold with asymptotically cylindrical ends. As in the previous case, let $\overline{X} = X \cup \partial\overline{X}$ be a smooth compact manifold of dimension n with boundary. We say that a smooth metric g on X is a metric with asymptotically cylindrical ends if there exists a function ρ defining the boundary, a collar neighborhood of the boundary, $U_\rho := [0,\varepsilon) \times \partial\overline{X}$, and a family $h(\rho)$, $\rho \in [0,\varepsilon)$, of smooth metrics on $\partial\overline{X}$ such that

$$g = \frac{\mathrm{d}\rho^2}{\rho^2} + h(\rho) \quad \text{on } U_\rho. \tag{2.2}$$

Let $\Delta_{\partial\overline{X}}$ be the Laplacian on the compact manifold $\partial\overline{X}$ and $0 = \sigma_1 < \sigma_2 < \cdots$ its spectrum. For Δ, the Laplacian on X, $[\sigma_j, \sigma_{j+1})$ is continuous spectrum for all $j > 0$ with a multiplicity equal to the sum of the multiplicities of $\{\sigma_1, \dots, \sigma_j\}$ as eigenvalues of $\Delta_{\partial\overline{X}}$ and there may be embedded eigenvalues with finite multiplicity in $[0, +\infty)$.
Melrose, in [Mel93], proves the continuation of the resolvent of the free Laplacian on the Riemannian surface Σ which is the surface such that all the functions $r_j(\lambda) := (\lambda - \sigma_j)^{\frac{1}{2}}$ are holomorphic on it. This surface is ramified at points

$\lambda = \sigma_j$. $R_0(\lambda) = (\Delta - \lambda)^{-1}$ is first defined in $D = \{\lambda \in \Sigma : \forall j \operatorname{Im}(r_j(\lambda)) > 0\}$ with values in $\mathcal{L}(L^2(X))$ and, for all $N \geq 0$, it has a finite-meromorphic continuation in a domain D_N^+ with values in $\mathcal{L}(\rho^N L^2(X), \rho^{-N} L^2(X))$.

- X is a rank-one symmetric space of the noncompact type. In this case, Hilgert and Pasquale prove, in [HP09], the finite-meromorphic continuation of the resolvent of the free Laplacian.

In order to treat all these examples with a single notation, we reformulate Condition A, with $N > 0$, and $\rho = e^{-\langle z \rangle}$ for the Euclidean case and a boundary-defining function on $\overline{X}$ for the other examples, as follows.

Condition $\mathbf{A}_{N,\rho}$ $\widetilde{R}_0$ has a finite-meromorphic continuation from D to D_N^+ an unbounded domain of Σ, with values in $\mathcal{L}(\rho^N L^2(X), \rho^{-N} L^2(X))$.

Agmon [Agm98] shows that notions of resonances, multiplicity and order do not depend of the weight ρ^N chosen.

In order to have the finite-meromorphic continuation to D_N^+ of the resolvent of $\Delta + V$, $(\Delta + V - z)^{-1}$, $z \in \mathbb{C} \setminus \operatorname{Spec}(\Delta + V)$, we introduce a condition for V:

Condition $\mathbf{B}_{N,\rho}$ $\widetilde{R}_V(\lambda) := J(\Delta + V - f(\lambda))^{-1} J_0$ with values in $\mathcal{L}(\rho^N L^2(X), \rho^{-N} L^2(X))$ has a finite-meromorphic continuation from D to D_N^+, and $\rho^{-2N} V$ is bounded on X.

Remark 2.1 With the hypothesis $\rho^{-2N} V$ bounded, we will be able to apply Agmon's perturbation theory of resonances [Agm98].

Remark 2.2 If V is compactly supported or if V is smooth on $\overline{X}$ and vanishes to all orders in ρ at the boundary, then V verifies Condition $\mathrm{B}_{N,\rho}$ for all N. In these cases, the resolvent of $\Delta + V$ has a finite-meromorphic continuation in all Σ.

3 Circular Symmetries

3.1 Statement of the result and examples

We can now give the main result of this part:

Theorem 3.1 *Let (X, g) be a connected Riemannian manifold with an action of $\mathbb{S}^1$ by isometries verifying the Condition $A_{N,\rho}$ for some $N > 0$ and some $\mathbb{S}^1$ invariant function ρ.*

Let V be the potential

$$V = \sum_{m=1}^{+\infty} V_m,$$

where $V_m \in L^\infty(X)$ is $\mathbb{S}^1$ homogeneous of weight m, with $\sum_{m=1}^{+\infty} \|V_m\|_\infty < +\infty$. If V verifies Condition $B_{N,\rho}$ and for all $\lambda \in D_N^+ \setminus \operatorname{Res}(\Delta)$, $\rho^{-(N+1)} V \widetilde{R}_0(\lambda) \rho^N$ is in a Schatten class $\mathcal{S}_q$, $q \in \mathbb{N} \setminus \{0\}$, then on D_N^+, $\operatorname{Res}(\Delta + V) = \operatorname{Res}(\Delta)$ with the same multiplicities.

We will recall the definition of the Schatten classes in Definition 3.8 below.

Remark 3.2 The last assumption, $\rho^{-(N+1)}V\widetilde{R}_0(\lambda)\rho^N \in \mathcal{S}_q$, is technical and will allow us to use regularized determinants. If V is compactly supported then it holds with $q > \frac{\dim X}{2}$ and for any N. In the Euclidean space $\mathbb{R}^n$, if V is bounded and super-exponentially decaying we still have the assumption for all N. On an asymptotically hyperbolic manifold X, if V is smooth on $\overline{X}$ and vanishes to all orders in ρ at the boundary then V verifies the assumption for all N.

Let us describe the $\mathbb{S}^1$ action and the potentials for the examples of Section 2 verifying Condition A:

- Let $\mathbb{R}^n = (\mathbb{R}^2)^k \times \mathbb{R}^{n-2k}$ be the Euclidean space with the following $\mathbb{S}^1$ action

$$\bigoplus_{i=1}^{k} R(p_i\theta) \oplus \mathrm{Id}_{\mathbb{R}^{n-2k}},$$

  where $\theta \in [0, 2\pi)$, $(p_1, \dots, p_k) \in (\mathbb{Z} \setminus \{0\})^k$, and $R(\phi)$ is the rotation of angle ϕ on $\mathbb{R}^2$. The components $\mathbb{S}^1$ homogeneous of weight m of the isoresonant potentials of the theorem have the following form:

$$V_m(r_1e^{i\alpha_1}, \dots, r_ke^{i\alpha_k}, z) = \sum_{\substack{(\ell_1,\dots,\ell_k)\in\mathbb{Z}^k \\ \sum_{i=1}^k \ell_i p_i = -m}} W_{m,\ell_1,\dots,\ell_k}(\bar{x})e^{i\ell_1\alpha_1}\cdots e^{i\ell_k\alpha_k}$$

  where $(r_1e^{i\alpha_1}, \dots, r_ke^{i\alpha_k}, z) \in (\mathbb{R}^2)^k \times \mathbb{R}^{n-2k}$, $\bar{x} \in \mathbb{R}^n/\mathbb{S}^1$ and the sum converges in infinite norm.
- For the hyperbolic space $\mathbb{H}^n$, we can take the Poincaré model, *i.e.*, the unit ball of $\mathbb{R}^n$ centered at the origin with the metric $4(1 - \|x\|^2)^{-2}g_{\mathrm{euclid}}$. The action of $\mathbb{S}^1$ on $\mathbb{R}^n$ described in the previous point induces an isometric action of $\mathbb{S}^1$ on $\mathbb{H}^n$. The isoresonant potentials have the same form as in the Euclidean case. We recall [GZ95a] that if n is odd $\mathrm{Res}(\Delta) = \varnothing$ and if n is even $\mathrm{Res}(\Delta) = -\mathbb{N}$ and the multiplicity of the integer $-k$ is the multiplicity of $k(k + n - 1)$ as eigenvalue of the Laplacian on the Euclidean sphere $\mathbb{S}^n$.
- Let us consider $\mathbb{H}^n$ with the model $\mathbb{R}^+_* \times \mathbb{R}^{n-1}$ with the corresponding coordinates (x, y). We take for X the hyperbolic cylinder $\mathbb{H}^n/\langle\gamma\rangle$ where γ is the isometry $w \to e^\ell w$. $\mathbb{S}^1$ acts on X isometrically by $e^{i\theta}.[x, y] = [e^{\frac{\ell\theta}{2\pi}}x, e^{\frac{\ell\theta}{2\pi}}y]$. We can see X as $\mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ with the coordinates (r, θ, ω), the metric $dr^2 + \mathrm{ch}^2\, r d\theta^2 + \mathrm{sh}^2\, r d\omega^2$, and the $\mathbb{S}^1$ action is the trivial action on the factor $\mathbb{S}^1$. The components $\mathbb{S}^1$ homogeneous of weight m of the isoresonant potentials have the form:

$$V_m(r, \theta, \omega) = W_m(r, \omega)e^{-\mathrm{i}m\theta}.$$

  Here, according to [GZ95a], we have $\mathrm{Res}(\Delta) = -\mathbb{N} + i\mathbb{Z}2\pi/\ell$ [GZ95a].

We recall that the isometric $\mathbb{S}^1$ action induces a decomposition of $L^2(X)$ according to isotypical subspaces:

$$L^2(X) = \bigoplus_{j\in\mathbb{Z}} L^2_j(X).$$

Let $P_j \colon L^2(X) \to L^2_j(X)$ be the corresponding orthogonal projection.

The main idea of the proof is that if $V_m \in L^\infty(X)$ is $\mathbb{S}^1$ homogeneous of weight m then it induces by multiplication a shift on these isotypical representations:

$$V_m \colon L^2_j(X) \to L^2_{j+m}(X).$$

3.2 A Preliminary Lemma

Lemma 3.3 *Let (X, g) be a connected Riemannian manifold with a $\mathbb{S}^1$ action by isometries. For all compact $K \subset X$ there exists a compact manifold with boundary $\widetilde{K}$ which is diffeomorphic to a compact of X containing K, and which has an isometric $\mathbb{S}^1$ action, with a smooth metric $\widetilde{g}$ such that $\widetilde{g}_{|K} = g_{|K}$, and $\widetilde{g}$ is a product metric $d\delta^2 + \widetilde{h}_{\partial\widetilde{K}}$ in a neighbourhood of the boundary $\partial\widetilde{K}$ of $\widetilde{K}$, with δ a $\mathbb{S}^1$ invariant function defining $\partial\widetilde{K}$ and $\widetilde{h}$ is independent of δ.*

Before giving the proof of this general result, let us verify this result for our previous examples:

- For the Euclidean space, we can remark that every compact K can be included in a ball $B(0, R)$ and we can take for $\widetilde{K}$ a bigger ball $B(0, \widetilde{R})$ with $\widetilde{R} > R$ with the following metric in polar coordinates:

$$\widetilde{g} = dr^2 + f(r)d\omega^2, \quad (r, \omega) \in \mathbb{R}^+ \times \mathbb{S}^{n-1},$$

  where $d\omega^2$ is the metric on the $(n-1)$-sphere in $\mathbb{R}^n$ and f is smooth on $[0, \widetilde{R}]$, constant near $\widetilde{R}$ and $f(r) = r^2$ on $[0, R]$.
- For the hyperbolic space $\mathbb{H}^n$, if K is included in a ball of radius R and centered at the origin, then we take for $\widetilde{K}$ a ball of radius $\widetilde{R} > R$ and, in polar coordinates, $\widetilde{g} = dr^2 + f(r)d\omega^2$ with this time $f(r) = \operatorname{sh}^2 r$ on $[0, R]$.
- For the hyperbolic cylinder $\mathbb{H}^n/\langle\gamma\rangle$, every compact is included in a $K = [0, R] \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$, so we can take $\widetilde{K} = [0, \widetilde{R}] \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ with the metric $\widetilde{g} = \mathrm{d}r^2 + h_1(r)\mathrm{d}\theta^2 + h_2(r)\mathrm{d}\omega^2$ where h_1 and h_2 are smooth on $[0, \widetilde{R}]$, constant near $\widetilde{R}$ and on $[0, R]$, $h_1(r) = \operatorname{ch}^2 r$, $h_2(r) = \operatorname{sh}^2 r$.

Proof Let ϕ be an exhaustion function on X and ψ defined by:

$$\forall m \in X, \psi(m) = \int_{\mathbb{S}^1} \phi(u.m)\,\mathrm{d}u.$$

For all $A \in \mathbb{R}$, by definition of ϕ, there exists a compact K_A such that $\phi \geq A$ outside K_A. With $L_A = \mathbb{S}^1.K_A$, $\psi \geq A$ outside L_A, and so ψ is also an exhaustion function on X. In addition ψ is $\mathbb{S}^1$ invariant.

Take K a compact of X. Thanks to the Sard's theorem, there exists a sequence $(v_i)_{i\in\mathbb{N}}$ of regular values of ψ with $\lim_{i\to+\infty} v_i = +\infty$. So there exists a v_i for which $K \subset \psi^{-1}(]-\infty, v_i[)$. We note that $L = \psi^{-1}(]-\infty, v_i[)$; it is an $\mathbb{S}^1$ invariant compact of X with a smooth boundary.

Let ρ be an $\mathbb{S}^1$ invariant boundary-defining function of $X \setminus L$. We have a collar neighborhood, in $X \setminus L$, of the boundary $\partial L = \psi^{-1}(\{v_i\})$, $[0, 2\varepsilon) \times \partial L$, where the metric g of X can be written in the form $\mathrm{d}\rho^2 + h(\rho)$, where $h(\rho)$ is a family of smooth metrics on ∂L. Let $g_\varepsilon = \mathrm{d}\rho^2 + h(\varepsilon)$ and $\chi \in \mathcal{C}^\infty(X)$ which only depend on ρ and such that:

$$\begin{cases} \chi = 1 & \text{on } L, \\ \chi = 0 & \text{if } \rho \geq \varepsilon. \end{cases}$$

Then consider the compact $\widetilde{K} = \big(L \sqcup ([0, 2\varepsilon) \times \partial L)\big) / \partial L$ with the metric $\widetilde{g} = \chi g + (1-\chi) g_\varepsilon$. Then $(\widetilde{K}, \widetilde{g})$ has an isometric $\mathbb{S}^1$ action and its metric near its boundary is g_ε which can be write in the form $\mathrm{d}\delta^2 + \widetilde{h}$ taking $\delta = 2\varepsilon - \rho$ and $\widetilde{h} = h(\varepsilon)$. ■

3.3 Spectral Lower Bound for the Laplacian on Homogeneous Functions

In the following we will need a spectral lower bound for the Laplacian on $\mathbb{S}^1$ homogeneous and compactly supported functions. As we have not read this result anywhere before, we also give an upper bound for the first eigenvalue. A more precise discussion about this result can be found in [Aut08].

Proposition 3.4 *Let K be a compact manifold with boundary with an $\mathbb{S}^1$ action and equipped with a metric g such that $\mathbb{S}^1$ acts by isometries on (K, g). We assume g has a product form $d\delta^2 + h_{\partial K}$ in a neighbourhood of ∂K with δ a $\mathbb{S}^1$ invariant function defining ∂K, and h is independent of δ. Then there are constants $C_1 = C_1(K) > 0$ and $C_2 = C_2(K) > 0$ such that for all $j \in \mathbb{Z}$, we have*

$$C_1 j^2 \leq \operatorname{Min} \operatorname{Spec} \Delta_{L^2_j(K)} \leq C_2 \langle j \rangle^2,$$

where $\Delta_{L^2_j(K)}$ is the Friedrichs selfadjoint extension in $L^2_j(K)$ of the Laplacian defined on $C^\infty_c(K) \cap L^2_j(K)$ and $\langle j \rangle := (1 + |j|^2)^{\frac{1}{2}}$.

Remark 3.5 In the case where K is a disk centered at 0 in $\mathbb{R}^2$, we can apply this lemma by including K in a bigger disk $\widetilde{K}$ as explained before, but we can also prove the lower bound directly because it is just an estimate of the first zero of Bessel functions.

Proof We begin with the lower bound. Consider two copies of K. We can identify their regular boundaries and get a compact closed manifold M. More precisely, there exists a collar neighbourhood W of ∂K diffeomorphic to $[0, \epsilon) \times \partial K$ by the diffeomorphism:

$$\begin{aligned} \psi \colon [0, \epsilon) \times \partial K &\to W \\ (t, y) &\to \psi_t(y), \end{aligned}$$

with ψ_t the gradient flow of δ for the metric g. So, on the topological space $M = (K \sqcup K)/\partial K$, we can construct a differential atlas beginning with $\partial K \subset M$ which is

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included in a open set $[W] = (W \sqcup W)/\partial K$ diffeomorphic to $(-\epsilon,\epsilon)\times\partial K$ by

$$(-\epsilon,\epsilon)\times\partial K \to [W]$$
$$(t,y) \to \begin{cases} \psi_t(y), & \text{if } t \geq 0,\\ \psi_{-t}(y), & \text{if } t \leq 0.\end{cases}$$

On $[W]$ the $\mathbb{S}^1$ action is, via the previous diffeomorphism, the action on ∂K. The other charts are those in the interior of K.

As the metric g has a product form in a neighbourhood of ∂K, it can be continued by symmetry on δ. We get a smooth metric on M, and we still have an isometrical action of $\mathbb{S}^1$ on M. Let Y be the corresponding vector field, which we will consider as a differential operator of order one (so $Y.f(m) = -i\partial_\theta(f(e^{-i\theta}.m))_{|\theta=0}$). Another pseudo-differential operator of order one on M is $P := \sqrt{\Delta_M + 1}$, where Δ_M is the Laplacian on (M,g). P and Y commute because $\mathbb{S}^1$ acts by isometries on M. We consider $Q := P^2 + Y^2$, whose principal symbol is $q(x,\xi) = |\xi|^2 + (\xi(Y))^2$, $(x,\xi)\in T^*M$; so Q is elliptic.

Let $\Lambda \subset \mathbb{R}^2$ be the joint spectrum of (P,Y), it is constituted by the points $(\lambda_k^P, \lambda_k^Y)$ such that $P\phi_k = \lambda_k^P\phi_k$ and $Y\phi_k = \lambda_k^Y\phi_k$ where (ϕ_k) is a orthonormal basis of $L^2(M)$. We note that the spectrum of Y is equal to $\mathbb{Z}$ and we are looking for the minimum of the first coordinates of points of Λ whose second coordinate is j. We call this minimum λ_1^j.

Let $p(x,\xi) = (|\xi|, \xi(Y))$ be the joint principal symbol of P and Y. Then p is a homogeneous function of degree one. Let Γ be the linear cone $\Gamma = \mathbb{R}^+ p(S(T^*M))$, where $S(T^*M)$ is the unit sphere bundle of T^*M. Moreover, we have $p(S(T^*M)) = \{(1,\xi(Y)), |\xi| = 1\}$. Now, as P and Y commute and Q is elliptic, we can apply [CdV79, Theorem 0.6]: if C is a cone of $\mathbb{R}^2$ such that $C\cap\Gamma = \{0\}$, then $C\cap\Lambda$ is finite. Taking, for example, $C_K = \mathbb{R}\{(a, |Y|_K), |a| \leq \frac{1}{2}\}$ where $|Y|_K = \sup_{m\in K}|Y(m)|$, it means that there exists a constant $c := 1/2|Y|_K$ and $J \in \mathbb{N}$ such that, for all $j\in\mathbb{Z}$, $|j| \geq J$, we have $\lambda_1^j \geq c|j|$, and we can take a smaller c and have $\lambda_1^j \geq c|j|$ for all $j\in\mathbb{Z}$. As $P = \sqrt{\Delta_M + 1}$, there is another constant c, such that the minimum of the spectrum of the Laplacian on homogeneous functions of weight j on M is superior than cj^2.

Moreover, the spectrum of $\Delta_{L_j^2(K)}$ with Dirichlet conditions is included in the spectrum of Δ_M acting on $L_j^2(M)$. Indeed, let I be the involution that exchanges the two copies of K in M. Then the eigenfunctions of Δ_M that are odd for I vanish on the image of ∂K in M. So they correspond to eigenfunctions of the Dirichlet Laplacian on K.

In order to prove the upper bound, we remark that

$$\operatorname{Min\,Spec}\Delta_{L_j^2(K)} = \inf_{\phi\in L_j^2(K)\setminus\{0\}} \frac{\langle\Delta\phi,\phi\rangle}{\|\phi\|^2}.$$

So it's sufficient to construct, for j large, one $\phi_j \in L_j^2(K)\setminus\{0\}$ such that $\langle\Delta\phi_j,\phi_j\rangle \leq Cj^2\|\phi_j\|^2$, with C independent of j. Let $\widehat{K}$ be the set of principal orbits of the action of $\mathbb{S}^1$ in K. The principal orbits are those for which the stability groups are the identity. $\widehat{K}$ is an open, connected and dense subset of X. We consider the principal $\mathbb{S}^1$ fibration $\widehat{K} \to \widehat{K}/\mathbb{S}^1$ and we take U_j, a $\mathbb{S}^1$ invariant open set of $\widehat{K}$, where this fibration is trivial. So U_j is diffeomorphic to $(U_j/\mathbb{S}^1) \times \mathbb{S}^1$ and we can take U_j small enough to be sure that $U_j/\mathbb{S}^1$ is a coordinate patch and we denote the corresponding coordinates by $(\mathbf{y}, \theta) := (y_1, \dots, y_N, \theta)$, where $N = \dim U_j/\mathbb{S}^1$. In those coordinates the metric has the form

$$g_{|U_{\phi_j}} = \sum_{k,\ell=1}^{N} a_{k,\ell}(\mathbf{y},\theta)\mathrm{d}y_k\mathrm{d}y_\ell + b(\mathbf{y},\theta)\mathrm{d}\theta^2 + \sum_{k=1}^{N} c_k(\mathbf{y},\theta)\mathrm{d}y_k\mathrm{d}\theta,$$

with $a_{k,\ell}$, b and c_k smooth on U_j, and the Laplacian becomes

$$\Delta = \sum_{k,\ell=1}^{N} A_{k,\ell}(\mathbf{y},\theta)\partial_{y_k}\partial_{y_\ell} + B(\mathbf{y},\theta)\partial_\theta^2 + \sum_{k=1}^{N} C_k(\mathbf{y},\theta)\partial_\theta\partial_{y_k}$$
$$+ \sum_{k=1}^{N} D_k(\mathbf{y},\theta)\partial_{y_k} + E(\mathbf{y},\theta)\partial_\theta,$$

where $A_{k,\ell}, B, C_k, D_k$ and E are smooth on U_j.

We take $\phi_j(\mathbf{y},\theta) = \psi(\mathbf{y})e^{-ij\theta}$ with ψ smooth and compactly supported in $U_j/\mathbb{S}^1$. So we have $\phi_j \in L_j^2(K)$ and

$$\Delta\phi_j = \Big(\sum_{k,\ell=1}^{N} A_{k,\ell}(\mathbf{y},\theta)\partial_{y_k}\partial_{y_\ell}\psi - j^2 B(\mathbf{y},\theta)\psi - ij\sum_{k=1}^{N} C_k(\mathbf{y},\theta)\partial_{y_k}\psi$$
$$+ \sum_{k=1}^{N} D_k(\mathbf{y},\theta)\partial_{y_k}\psi - ijE(\mathbf{y},\theta)\psi\Big) e^{ij\theta},$$

and

$$\begin{aligned}\langle\Delta\phi_j,\phi_j\rangle = &-j^2\int_{\operatorname{supp}\phi_j} B(\mathbf{y},\theta)|\psi|^2\,\mathrm{dvol}(g)\\ &- ij\int_{\operatorname{supp}\phi_j}\sum_{k=1}^{N} C_k(\mathbf{y},\theta)(\partial_{y_k}\psi)\overline{\psi} + E(\mathbf{y},\theta)|\psi|^2\,\mathrm{dvol}(g)\\ &+\int_{\operatorname{supp}\phi_j}\sum_{k,\ell=1}^{N} A_{k,\ell}(\mathbf{y},\theta)(\partial_{y_k}\partial_{y_\ell}\psi)\overline{\psi} + \sum_{k=1}^{N} D_k(\mathbf{y},\theta)(\partial_{y_k}\psi)\overline{\psi}\,\mathrm{dvol}(g),\end{aligned}$$

so

$$\langle\Delta\phi_j,\phi_j\rangle = |\langle\Delta\phi_j,\phi_j\rangle| \le \Lambda_1(\psi)j^2\|\psi\|^2_{L^2(K,g)} + \Lambda_2(\psi)|j| + \Lambda_3(\psi),$$

where $\Lambda_1, \Lambda_2, \Lambda_3$ are positive constants which only depend on ψ.

By fixing ψ as $\|\phi_j\| = \|\psi\|$, we get a constant C_2 such that, for all $j \in \mathbb{Z}$,

$$\langle\Delta\phi_j,\phi_j\rangle \le C_2\langle j\rangle^2\|\phi_j\|^2. \qquad \blacksquare$$

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With Proposition 3.4 and Lemma 3.3 we will prove the following:

Lemma 3.6 *Let $\lambda \in D_N^+ \setminus \mathrm{Res}(\Delta)$ and $\chi \in C_c^\infty(X)$ be $\mathbb{S}^1$-invariant. Then there is a constant $C = C(\lambda, \chi) > 0$ such that, for all $j \in \mathbb{Z}$,*

$$\|\chi \widetilde{R}_0(\lambda) P_j \chi\| \leq \frac{C}{1 + j^2}.$$

Proof Since $\mathbb{S}^1$ acts by isometries on X, we have, for all j, $\Delta P_j = P_j \Delta$ and $P_j \widetilde{R}_0 = \widetilde{R}_0 P_j$. The fact that χ is $\mathbb{S}^1$ invariant also gives $\chi P_j = P_j \chi$.

We have

$$\chi(\Delta - f(\lambda))\widetilde{R}_0(\lambda)\chi = \chi^2,$$

so

$$(\Delta - f(\lambda))\chi \widetilde{R}_0(\lambda) P_j \chi = \chi^2 P_j + [\Delta, \chi]\widetilde{R}_0(\lambda)\chi P_j,$$

then

$$\|(\Delta - f(\lambda))\chi \widetilde{R}_0(\lambda) P_j \chi\| \leq \|\chi^2 P_j\| + \|[\Delta, \chi]\widetilde{R}_0(\lambda)\chi P_j\| \leq C(\lambda, \chi). \tag{3.1}$$

Let $(\widetilde{K}, \widetilde{g})$ the compact containing $K := \operatorname{supp} \chi$ given by Lemma 3.3. If $v \in L^2(X)$ then $u = \chi P_j \widetilde{R}_0(\lambda)\chi v$ is in $L^2_j(K, g)$ and, as $\operatorname{supp} u \subset K$ and $\widetilde{g}_{|K} = g_{|K}$, we also have $u \in L^2_j(\widetilde{K}, \widetilde{g})$. In addition, u is, at the same time, in the domain of the Dirichlet Laplacian on $\widetilde{K}$, of the Dirichlet Laplacian on K and of the Laplacian on X, and we have

$$\Delta_{(\widetilde{K}, \widetilde{g})} u = \Delta_{(K,g)} u = \Delta_{(X,g)} u.$$

Let (ϕ_k) (depending on $\widetilde{K}$ and j) an orthonormal basis of $L^2_j(\widetilde{K}, \widetilde{g})$ constituted by eigenfunctions of $\Delta_{(\widetilde{K}, \widetilde{g})}$. We denote $\mu_k(j, \widetilde{K})$ the eigenvalue corresponding to ϕ_k. If we expand u following this basis: $u = \sum_k u_k \phi_k$, we have

$$(\Delta_{(\widetilde{K}, \widetilde{g})} - f(\lambda))u = \sum_k (\mu_k(j, \widetilde{K}) - f(\lambda)) u_k \phi_k$$

so that

$$\begin{aligned} \|(\Delta - f(\lambda))u\|^2 &= \sum_k |\mu_k(j, \widetilde{K}) - f(\lambda)|^2 \, |u_k|^2 \\ &\geq \sum_k \big(\mu_k(j, \widetilde{K}) - \mathrm{Re}(f(\lambda))\big)^2 |u_k|^2. \end{aligned}$$

Thus, using Proposition 3.4, there exists a constant $C = C(\widetilde{K}) > 0$ such that

$$\forall k \in \mathbb{N}, \forall j \in \mathbb{Z} \quad \mu_k(j, \widetilde{K}) \geq C j^2.$$

We take J in order to have $CJ^2 > \mathrm{Re}(f(\lambda))$, then for all $|j| \geq J$, we have

$$\|(\Delta - f(\lambda))u\|^2 \geq \big(Cj^2 - \mathrm{Re}(f(\lambda))\big)^2 \sum_k |u_k|^2 = \big(Cj^2 - \mathrm{Re}(f(\lambda))\big)^2 \|u\|^2.$$

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Using this in the inequality (3.1) we get that for all $|j| \geq J$

$$\|\chi\widetilde{R}_0(\lambda)P_j\chi\| \leq \frac{C(\lambda,\chi)}{Cj^2 - \mathrm{Re}(f(\lambda))}.$$

So there exists another constant $C > 0$ such that, for all $|j| \geq J$, $\|\chi\widetilde{R}_0(\lambda)P_j\chi\| \leq Cj^{-2}$, and we can take a greater C to have, for all $j \in \mathbb{Z}$,

$$\|\chi\widetilde{R}_0(\lambda)P_j\chi\| \leq \frac{C}{1+j^2}. \qquad \blacksquare$$

3.4 Localization of Resonances

We begin the proof of Theorem 3.1 by the inclusion $\mathrm{Res}(\Delta+V) \subset \mathrm{Res}(\Delta)$. First, we consider truncations in space of partial sums of V.

3.4.1 Localization of Resonances for the Truncated Partial Sums of V

We consider $S_M := \sum_{m=1}^{M} V_m$ where V_m is the component $\mathbb{S}^1$ homogeneous of weight m of V. Let $\chi \in C_c^\infty(X)$ be invariant under the action of $\mathbb{S}^1$. In this part, our purpose is to show that $\mathrm{Res}(\Delta+\chi S_M) \subset \mathrm{Res}(\Delta)$, on D_N^+.

For $\lambda \in D_N^+ \setminus \mathrm{Res}(\Delta)$, we have

$$\big(\Delta + \chi S_M - f(\lambda)\big)\widetilde{R}_0(\lambda)\rho^N = \rho^N\big(I + \rho^{-N}\chi S_M\widetilde{R}_0(\lambda)\rho^N\big).$$

In addition, we have

$$\rho^{-N}\chi S_M\widetilde{R}_0(\lambda)\rho^N = \chi\rho^{-2N}S_M\rho^N\widetilde{R}_0(\lambda)\rho^N.$$

So, thanks to the condition $A_{N,\rho}$, $\rho^{-N}\chi S_M\widetilde{R}_0(\lambda)\rho^N$ is a holomorphic family of compact operators in $D_N^+ \setminus \mathrm{Res}(\Delta)$ such that

$$\|\rho^{-N}\chi S_M\widetilde{R}_0(\lambda)\rho^N\| < 1$$

for $|\lambda|$ sufficiently large in D_N^+.

Then by the analytic Fredholm theory we get that $\big(I + \rho^{-N}\chi S_M\widetilde{R}_0(\lambda)\rho^N\big)^{-1}$ is meromorphic on $D_N^+ \setminus \mathrm{Res}(\Delta)$ and we have the often called Lipmann–Schwinger equation which establishes the link between the resolvent of $\Delta + \chi S_M$ and that of the free Laplacian:

$$\text{(LS)} \qquad \rho^N\widetilde{R}_{\chi S_M}(\lambda)\rho^N = \rho^N\widetilde{R}_0(\lambda)\rho^N\big(I + \rho^{-N}\chi S_M\widetilde{R}_0(\lambda)\rho^N\big)^{-1}.$$

So if λ_0 is a pole of $\widetilde{R}_{\chi S_M}$ in $D_N^+ \setminus \mathrm{Res}(\Delta)$, then λ_0 is a pole of $\big(I + \rho^{-N}\chi S_M\widetilde{R}_0(\lambda)\rho^N\big)^{-1}$, and still by Fredholm theory, there is a nontrivial $u \in L^2(X)$ such that

$$\big(I + \rho^{-N}\chi S_M\widetilde{R}_0(\lambda)\rho^N\big)u = 0.$$

We remark with the last equality that $\operatorname{supp} u \subset \operatorname{supp} \chi$. Let $\chi_2 \in C_c^\infty(X)$ invariant under the action of $\mathbb{S}^1$ and such that $\chi_2 = 1$ on the support of χ. If we denote $u_j := P_j u \in L^2_j(X)$, we have

$$u_j = P_j\big(-\rho^{-N}\chi S_M \widetilde{R}_0(\lambda)\rho^N u\big) = P_j\big(-\rho^{-N}\chi S_M \chi_2 \widetilde{R}_0(\lambda)\chi_2\rho^N u\big),$$

and by linearity,

$$u_j = -\sum_{m=1}^{M} P_j\big(\rho^{-N}\chi V_m \chi_2 \widetilde{R}_0(\lambda)\chi_2\rho^N u\big).$$

However, each V_m induces a shift on the isotypical representations:

$$V_m \colon L^2_j(X) \to L^2_{j+m}(X),$$

so we have

$$u_j = -\sum_{m=1}^{M} V_m P_{j-m}\big(\rho^{-N}\chi\chi_2 \widetilde{R}_0(\lambda)\chi_2\rho^N u\big),$$

$$u_j = -\sum_{m=1}^{M} V_m \rho^{-N}\chi\chi_2 \widetilde{R}_0(\lambda)\chi_2\rho^N P_{j-m}(u),$$

where we have also used that the projections P_{j-m} commute with $\widetilde{R}_0$, ρ, χ and χ_2.

By hypothesis, for all m, $\|V_m\|_\infty \le \sum_{m'=1}^{+\infty}\|V_{m'}\|_\infty < +\infty$. Then, applying Lemma 3.6 to $\chi_2\widetilde{R}_0\chi_2 P_{j-m}$, we get a constant C such that, for all $j \in \mathbb{Z}$,

$$\|u_j\| \le \sum_{m=1}^{M} \frac{C}{1+(j-m)^2}\|u_{j-m}\|,$$

so, for all $j \in \mathbb{Z}$,

$$\|u_j\| \le \epsilon_j \sum_{m=1}^{M}\|u_{j-m}\|,$$

where $\epsilon_j \to 0$ for $|j| \to +\infty$.

Thus we can use the following lemma:

Lemma 3.7 *Let $(a_j)_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ non-negative. If there is $M \in \mathbb{N}$ and, for all $j \in \mathbb{Z}$, $a_j \le \epsilon_j \sum_{m=1}^{M} a_{j-m}$ with $\epsilon_j \to 0$ for $|j| \to +\infty$, then $a_j = 0$ for all j.*

Proof Let $J' \le 0$ such that, for all $j \le J'$, $\epsilon_j \le \frac{1}{M}$. Then, for all $j \le J'$, we have $a_j \le \frac{1}{M}\sum_{m=1}^{M} a_{j-m}$ and if we sum all these inequalities we get, denoting $S = \sum_{j\le J'} a_j$,

$$\begin{aligned} S &\le \frac{1}{M}\big((S-a_{J'}) + (S-a_{J'}-a_{J'-1}) + \cdots + (S-a_{J'}-\cdots-a_{J'-M+1})\big)\\ &= \frac{1}{M}\big(MS - Ma_{J'} - (M-1)a_{J'-1} - \cdots - a_{J'-M+1}\big), \end{aligned}$$

from which we deduce

$$0 \leq -Ma_{J'} - (M-1)a_{J'-1} - \cdots - a_{J'-M+1},$$

and thus

$$a_{J'} = a_{J'-1} = \cdots = a_{J'-M+1} = 0.$$

Moreover, as $\epsilon_j \to 0$ for $|j| \to +\infty$, there exists a constant C such that, for all $j \in \mathbb{Z}$, $a_j \leq C \sum_{m=1}^{M} a_{j-m}$, so we have

$$\forall j \geq J' - M + 1, \quad a_j = 0,$$

and we can tighten J' to $-\infty$ and finally we have $a_j = 0$ for all $j \in \mathbb{Z}$. ∎

We apply Lemma 3.7 to the sequence $\{\|u_j\|^2\}_j$. We get that $\|u_j\| = 0$ for all j and thus $u \equiv 0$. This is in contradiction with the existence of a pole of $\widetilde{R}_{\chi S_m}$ in $D_N^+ \setminus \operatorname{Res}(\Delta)$.

Finally, for all M and all $\chi \in C_c^\infty(X)$, invariant under the action of $\mathbb{S}^1$, $\Delta + \chi S_M$ has no resonance in $D_N^+ \setminus \operatorname{Res}(\Delta)$, which can be expressed by

$$\operatorname{Res}(\Delta + \chi S_M) \subset \operatorname{Res}(\Delta), \quad \text{in } D_N^+.$$

3.4.2 Localization of Resonances for the Potential V

Let us recall some results and notations about regularized determinant that will be needed in the following [Yaf92].

Definition 3.8 Let $\mathcal{H}$ be an Hilbert space. If $A\colon \mathcal{H} \to \mathcal{H}$ is a compact operator, we define its *singular values* $(s_n(A))_{n\in\mathbb{N}}$ as the eigenvalues of the selfadjoint operator $(A^*A)^{1/2}$. For $1 \leq p < +\infty$, $\mathcal{S}_\mathbf{p}$, is the two-sided ideal of $\mathcal{L}(\mathcal{H})$ formed by operators A for which the sum

$$\|A\|_p^p = \sum_{n=0}^{\infty} s_n^p(A)$$

is finite.

Definition 3.9 For $A \in \mathcal{S}_p$ we define the *regularized determinant*, $\det_p$, by

$$\det_p(I+A) = \prod_{n=1}^{\infty}(1+\lambda_n(A)) \exp\Big(\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \lambda_n^k(A)\Big),$$

where the $(\lambda_n(A))_{n\in\mathbb{N}}$ are the eigenvalues of A.

We list some properties of this determinant.

Proposition 3.10
(1) $A \to \det_p(I+A)$ *is continuous on* $(\mathcal{S}_p, \|\cdot\|_p)$.

(2) *If* $z \to A(z)$ *is holomorphic in some domain of* $\mathbb{C}$, *with values in* $\mathcal{S}_p$, *then* $z \to \det_p(I + A(z))$ *is also holomorphic in the same domain.*
(3) *For* $A \in \mathcal{S}_p$, $I + A$ *is invertible if and only if* $\det_p(I + A) \neq 0$.

We have assumed that there exists q such that $\rho^{-(N+1)} V \widetilde{R}_0(\lambda)\rho^N$ is in a Schatten class $\mathcal{S}_q$ for all $\lambda \in D_N^+ \setminus \operatorname{Res}(\Delta)$, so $\rho^{-N} V \widetilde{R}_0(\lambda)\rho^N$ is in $\mathcal{S}_q$ too.

Let us first prove a preliminary fact. For all $\chi \in C_c^\infty(X)$ there exists p such that $\chi\rho^{-N}\widetilde{R}_0(\lambda)\rho^N \in \mathcal{S}_p$ for all $\lambda \in D_N^+ \setminus \operatorname{Res}(\Delta)$. To see this, take a compact K with a smooth boundary and containing $\operatorname{supp}\chi$. Let Δ_K be the Dirichlet Laplacian on K, and $(\mu_k)_{k\in\mathbb{N}}$ the eigenvalues of $(\Delta_K + 1)^{-1}$. Then Weyl's formula gives, when k tends to $+\infty$,

$$\mu_k \sim \frac{(2\pi)^2}{(\omega_n \operatorname{Vol}(K))^{\frac{2}{n}}} k^{-\frac{2}{n}},$$

where $n = \dim X$ and ω_n is the volume of the unity ball in $\mathbb{R}^n$. Thus for $p > \frac{n}{2}$, $(\Delta_K + 1)^{-1} \in \mathcal{S}_p$. Moreover, for all $\lambda \in D_N^+ \setminus \operatorname{Res}(\Delta)$, $(\Delta_K + 1)\chi\rho^{-N}\widetilde{R}_0(\lambda)\rho^N$ is a bounded operator in $L^2(X)$, and, as $\mathcal{S}_p$ is a two-sided ideal of $\mathcal{L}(L^2(X))$, we have

$$\chi\rho^{-N}\widetilde{R}_0(\lambda)\rho^N = (\Delta_K + 1)^{-1}(\Delta_K + 1)\chi\rho^{-N}\widetilde{R}_0(\lambda)\rho^N \in \mathcal{S}_p.$$

As $\mathcal{S}_{p_1} \subset \mathcal{S}_{p_2}$, for $p_1 \leq p_2$, we can take the maximum of p and q and we still note it q, and get that $\chi\rho^{-N}\widetilde{R}_0(\lambda)\rho^N$ and $\rho^{-N} V\widetilde{R}_0(\lambda)\rho^N$ are both in $\mathcal{S}_q$.

The Lipmann–Schwinger equation, (LS), with V instead of χS_M give

$$\rho^N \widetilde{R}_V(\lambda)\rho^N = \rho^N\widetilde{R}_0(\lambda)\rho^N \left(I + \rho^{-N} V \widetilde{R}_0(\lambda)\rho^N\right)^{-1}.$$

So thanks to the third point of Proposition 3.10 we have

$$\lambda \in \operatorname{Res}(\Delta + V) \cap D_N^+ \setminus \operatorname{Res}(\Delta) \iff \det_q\left(I + \rho^{-N} V\widetilde{R}_0(\lambda)\rho^N\right) = 0.$$

On $D_N^+ \setminus \operatorname{Res}(\Delta)$, we define

$$F(V, \lambda) := \det_q\left(I + \rho^{-N} V \widetilde{R}_0(\lambda)\rho^N\right).$$

If there exists $\lambda_0 \in \operatorname{Res}(\Delta + V) \setminus \operatorname{Res}(\Delta)$, then

$$F(V, \lambda_0) = 0.$$

Let Γ be a simple loop around λ_0 such that λ_0 is the only zero of $F(V, \cdot)$ in the domain U delimited by Γ, and such that $\overline{U} \subset D_N^+ \setminus \operatorname{Res}(\Delta)$. It is possible because, thanks to the second point of Proposition 3.10, F is holomorphic in λ and so its zeros are isolated.

Let χ_r a smooth family of compactly supported and $\mathbb{S}^1$-invariant functions such that,

$$\lim_{r\to+\infty} \|(\chi_r - 1)\rho\|_\infty = 0.$$

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As we have assumed that $V\rho^{-(N+1)}\widetilde{R}_0(\lambda)\rho^N \in \mathcal{S}_q$ we can write, for all $\lambda \in \Gamma$,

$$\|\chi_r V\rho^{-N}\widetilde{R}_0(\lambda)\rho^N - V\rho^{-N}\widetilde{R}_0(\lambda)\rho^N\|_q \leq \|(\chi_r - 1)\rho\|_\infty \|V\rho^{-(N+1)}\widetilde{R}_0(\lambda)\rho^N\|_q.$$

So when r tends to $+\infty$, $\chi_r V\rho^{-N}\widetilde{R}_0(\lambda)\rho^N$ tends to $V\rho^{-N}\widetilde{R}_0(\lambda)\rho^N$ in $\mathcal{S}_q$ uniformly on Γ. With the first point of Proposition 3.10, $F(\chi_r V, \lambda) \to F(V, \lambda)$ uniformly on Γ. From that, there exists r_0 such that for all $r > r_0$ and for all $\lambda \in \Gamma$ we have

$$|F(\chi_r V, \lambda) - F(V, \lambda)| < |F(V, \lambda)|.$$

Thus, by Rouché's theorem, $F(\chi_r V, \cdot)$ has the same number of zeros, in U, as $F(V, \cdot)$.

In the same way, fixing $r > r_0$, using $\chi_r\rho^{-N}\widetilde{R}_0(\lambda)\rho^N \in \mathcal{S}_q$ we can write

$$\|\chi_r S_M\rho^{-N}\widetilde{R}_0(\lambda)\rho^N - \chi_r V\rho^{-N}\widetilde{R}_0(\lambda)\rho^N\|_q \leq \|S_M - V\|_\infty \|\chi_r\rho^{-N}\widetilde{R}_0(\lambda)\rho^N\|_q.$$

Using the fact that by hypothesis, $\|S_M - V\|_\infty$ tends to 0 when M tends to ∞, we have $F(\chi_r S_M, \lambda) \to F(\chi_r V, \lambda)$ uniformly on Γ and we can use Rouché's theorem again.

In conclusion, there exist r and M such that $F(\chi_r S_M, \cdot)$ has the same number of zeros, in U, as $F(V, \cdot)$. It means that $\Delta + \chi_r S_M$ has a resonance in the domain $U \subset D_N^+ \setminus \mathrm{Res}(\Delta)$ which is in contradiction with the previous part. Therefore, on D_N^+,

$$\mathrm{Res}(\Delta + V) \subset \mathrm{Res}(\Delta).$$

Remark 3.11 In [Chr08], Christiansen proves the inclusion, $\mathrm{Res}(\Delta+V) \subset \mathrm{Res}(\Delta)$, without using the shift created by the potential V on the isotypical components of $L^2(X)$ but with regularized determinant and an hypothesis of analycity: $W(z) := \sum_{m=1}^{\infty} z^m V_m$ should be holomorphic in a domain of $\mathbb{C}$ containing the closed disc of center 0 and radius one.

3.5 Persistence of Resonances

In order to achieve the proof of Theorem 3.1, we have to show that the points in $\mathrm{Res}(\Delta) \cap D_N^+$ are also resonances of $\Delta + V$ with the same multiplicity. To make this, we will use Agmon's perturbation theory of resonances [Agm98].

Let $\lambda_0 \in D_N^+$ be a resonance of Δ with multiplicity m. Let $U \subset D_N^+$ with smooth boundary Γ such that $\overline{U} \cap \mathrm{Res}(\Delta) = \{\lambda_0\}$. If V satisfies the hypothesis of Theorem 3.1 then for all $t \geq 0$, tV satisfies these hypothesis too. So we can apply the result of the previous part: for all $t \geq 0$, $\mathrm{Res}(\Delta + tV) \subset \mathrm{Res}(\Delta)$ and thus

$$\mathrm{Res}(\Delta + tV) \cap U \subset \{\lambda_0\}.$$

Let $E := \{t_0 \geq 0 : \forall t \in [0, t_0], \mathrm{Res}(\Delta + tV) \cap U = \{\lambda_0\}$ with multiplicity $m\}$; we are going to prove by connectivity that E is in fact equal to $[0, +\infty[$. First it is not empty because $0 \in E$ by definition of λ_0.

We take $t_0 \in E$, and we want to establish the existence of $\delta > 0$ such that $]t_0 - \delta, t_0 + \delta[\subset E$. Following the theory of Agmon [Agm98], we begin with the definition of the Banach space

$$B_\Gamma = \Big\{ f \in \rho^{-N} L^2(X) : f = g + \int_\Gamma \widetilde{R}_{t_0 V}(\xi)\Phi(\xi)\, d\xi, \\ g \in \rho^N L^2(X), \Phi \in C(\Gamma, \rho^N L^2(X)) \Big\},$$

where $C(\Gamma, \rho^N L^2(X))$ is the space of continuous functions on Γ with values in $\rho^N L^2(X)$. On the space B_Γ we take the norm

$$\|f\|_{B_\Gamma} = \inf_{g,\Phi}(\|g\|_{\rho^N L^2(X)} + \|\Phi\|_{C(\Gamma,\rho^N L^2(X))}),$$

where the infimum is taken among all the $g \in \rho^N L^2(X)$ and the $\Phi \in C(\Gamma, \rho^N L^2(X))$ such that $f = g + \int_\Gamma \widetilde{R}_{t_0 V}(\xi)\Phi(\xi) d\xi$. On this Banach space, still following Agmon, we can define the operator $(\Delta + t_0 V)^\Gamma \colon \mathcal{D}((\Delta + t_0 V)^\Gamma) \to B_\Gamma$. It is a restriction of $\Delta + t_0 V$ in the sense that

$$(\Delta + t_0 V)^\Gamma u = (\overline{\Delta + t_0 V})u, \quad u \in \mathcal{D}((\Delta + t_0 V)^\Gamma),$$

where $\overline{\Delta + t_0 V}$ is the closure of the operator $\Delta + t_0 V$ viewed as an operator densely defined in $\rho^{-N} L^2(X)$. Agmon proves that $(\Delta + t_0 V)^\Gamma$ has a discrete spectrum in U that is exactly the set of the poles of $\widetilde{R}_{t_0 V}$, *i.e.*, the resonances of $\Delta + t_0 V$, with the same multiplicities.

Next, with the condition $B_{N,\rho}$, the family tV verifies all the hypothesis in order to apply the "perturbation" part of the paper [Agm98]. We perturb $\Delta + t_0 V$ by tV. So there exists $\delta > 0$ such that, for all $t \in]-\delta, \delta[$, we can define in B_Γ the operator $(\Delta + t_0 V + tV)^\Gamma$. Moreover, for all $t \in]-\delta, \delta[$, $(\Delta + t_0 V + tV)^\Gamma$ has a discrete spectrum in U which is exactly the set of the poles of $\widetilde{R}_{t_0 V + tV}$ with the same multiplicities.

Now our problem becomes a problem of eigenvalues. Using Kato's perturbation theory of eigenvalues [Kat66], we know that, perhaps taking a smaller δ, the eigenvalues of $(\Delta + t_0 V + tV)^\Gamma$ in U are continuous for all $t \in]-\delta, \delta[$. As these eigenvalues are also resonances of $\Delta + t_0 V + tV$, λ_0 is the unique possibility. So λ_0 is the unique eigenvalue in U of $(\Delta + t_0 V + tV)^\Gamma$ for all $t \in]-\delta, \delta[$ with constant multiplicity. Therefore, thanks to the parallel established before, λ_0 is the unique resonance in U of $\Delta + t_0 V + tV$ for all $t \in]-\delta, \delta[$ with constant multiplicity. It signifies that $]t_0 - \delta, t_0 + \delta[\subset E$ and so E is an open set.

We can prove that E is also a closed set doing the same proof with the complementary set of E. If t_0 is not in E, then λ_0 is a resonance of $\Delta + t_0 V$ with a multiplicity not equal to m (it can be 0). Perturbing this operator by tV and using Agmon's correspondence, we can prove that λ_0 is a resonance of $\Delta + tV$ with a multiplicity not equal to m for all t in a neighbourhood of t_0.

In conclusion, $E = [0, +\infty[$ and we can take $t_0 = 1$ to obtain, in U, $\operatorname{Res}(\Delta + V) = \operatorname{Res}(\Delta)$ with the same multiplicity. To finish, we have to do the same work in the neighbourhood of any resonance of the free Laplacian. This completes the proof of Theorem 3.1.

3.6 An Example where the Order of Resonances Grows

The isoresonant potentials introduced in Theorem 3.1 cannot be detected by only observing the set of resonances and their multiplicities. We can wonder if their existence can be seen through the order of the resonances. We are going to prove, in an example, that there exist potentials verifying Theorem 3.1 which change the order of resonances.

We consider the hyperbolic plane $\mathbb{H}^2$ with the model $\mathbb{R}^+ \times \mathbb{S}^1$, the coordinates (r, θ) and the metric $g = \mathrm{d}r^2 + \mathrm{sh}(r)^2 \mathrm{d}\theta^2$. We have already said that the resonances of the free Laplacian are all the negative integers and the multiplicity of $-k$, $k \in \mathbb{N}$, is $2k + 1$ (see [GZ95a]). Moreover the order of these resonances is one. We denote $\mathcal{F} := \left\{ V_m(r)e^{\mathrm{im}\,\theta} : m \in \mathbb{Z} \setminus \{0\}, V_m \in L^\infty_c(\mathbb{R}^+) \right\}$, each of which is a family of isoresonant potentials by Theorem 3.1.

Proposition 3.12 *Let $\mathbb{H}^2$ be the hyperbolic plane and k a strictly positive integer. There exists a potential $V \in \mathcal{F} := \left\{ V_m(r)e^{\mathrm{im}\,\theta} : m \in \mathbb{Z} \setminus \{0\}, V_m \in L^\infty_c(\mathbb{R}^+) \right\}$ such that $-k$ is a resonance of $\Delta + V$ with an order strictly greater than one.*

Proof Let $k \in \mathbb{N} \setminus \{0\}$, we suppose, *ad absurdum*, for all $V \in \mathcal{F}$, $-k$ is a resonance of order one of $\Delta + V$.

For all $V \in \mathcal{F}$, the resolvent $\left(\Delta + V - \lambda(1 - \lambda)\right)^{-1}$ has a meromorphic continuation $\widetilde{R}_V$ on $D_N^+ = \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda > \frac{1}{2} - N\}$ as an operator from $B_0 := \rho^N L^2(\mathbb{H}^2)$ to $B_1 := \rho^{-N} L^2(\mathbb{H}^2)$ where ρ is a boundary defining function of a compactification of $\mathbb{H}^2$. We take N sufficiently large to have $-k \in D_N^+$. B_0 and B_1 are dual, thanks to the nondegenerate symmetric form

$$\langle u, v \rangle = \int_{\mathbb{H}^n} uv \,\mathrm{dvol}(g).$$

We remark that, for all $t \in \mathbb{R}$ and all $V \in \mathcal{F}$, we have $tV \in \mathcal{F}$. With our hypothesis, for λ in a neighbourhood of $-k$ we have

$$\widetilde{R}_{tV}(\lambda) = (\lambda + k)^{-1} S(t) + H(t, \lambda),$$

where $H(t, \cdot)$ is holomorphic with values in $\mathcal{L}(B_0, B_1)$ and $S(t) \in \mathcal{L}(B_0, B_1)$ has a finite rank.

We apply the Agmon's perturbation theory of resonances. Consider a domain $U \subset D_N^+$ with smooth boundary Γ such that $\overline{U} \cap \mathrm{Res}(\Delta) = \{-k\}$. We have the corresponding Banach space,

$$B_\Gamma = \left\{ f \in B_1 : f = g + \int_\Gamma \widetilde{R}_0(\xi)\Phi(\xi)\,\mathrm{d}\xi, g \in B_0, \Phi \in C(\Gamma, B_0) \right\},$$

with $B_0 \subset B_\Gamma \subset B_1$.

Then there exists $\delta > 0$ such that, for all $V \in \mathcal{F}$ and all $t \in]-\delta, \delta[$, we can define the operators $(\Delta + tV)^\Gamma$ in B_Γ and their resolvents R^Γ_{tV}. Thanks to [Agm98], we know that $(\Delta + tV)^\Gamma$ has a discrete spectrum in U which correspond to the resonances of $\Delta + tV$ in U with the same multiplicities and orders. So for λ near $-k$ in U we have

$$R^\Gamma_{tV}(\lambda) = (\lambda + k)^{-1} S^\Gamma(t) + H^\Gamma(t, \lambda), \tag{3.2}$$

with $H^\Gamma(t, \cdot)$ holomorphic with values in $\mathcal{L}(B_\Gamma)$ and $S^\Gamma(t) \in \mathcal{L}(B_\Gamma)$ of finite rank. Still following [Agm98] we know that $S(t)$ and $S^\Gamma(t)$ have the same range and they coincide on B_0.

Let $V \in \mathcal{F}$, for all $t \in \,]-\delta, \delta[$ and $\phi \in B_\Gamma$ we define $\psi(t) := S^\Gamma(t)\phi$. From (3.2) we obtain for all $t \in \,]-\delta, \delta[$,

$$\big((\Delta + tV)^\Gamma + k(k+1)\big)\,\psi(t) = 0.$$

$\psi(t)$ is differentiable in t like $S^\Gamma(t)$ (because

$$S^\Gamma(t) = \frac{1}{2\pi i} \int_\Gamma \big(\Delta^\Gamma + tV - \lambda(1-\lambda)\big)^{-1} d\lambda),$$

so we can differentiate the last equality at $t = 0$ and get

$$V\psi(0) + \big(\Delta^\Gamma + k(k+1)\big)\,\psi'(0) = 0.$$

Compose this new equality with $S^\Gamma(0)$, using

$$S^\Gamma(0)\big(\Delta^\Gamma + k(k+1)\big) = \big(\Delta^\Gamma + k(k+1)\big)\,S^\Gamma(0) = 0,$$
$$S^\Gamma(0) = \frac{1}{2\pi i} \int_\Gamma \big(\Delta^\Gamma - \lambda(1-\lambda)\big)^{-1} d\lambda,$$

and the fact that $-k(k+1)$ is an eigenvalue of order one of Δ^Γ, we obtain

$$S^\Gamma(0)V\psi(0) = 0.$$

As $\psi(0) \in \operatorname{Ran} S^\Gamma(0) = \operatorname{Ran} S(0)$, there exists $f_0 \in B_0$ such that $\psi(0) = S(0)f_0 = S^\Gamma(0)f_0$. Moreover $S^\Gamma(0)V\psi(0) \in B_\Gamma \subset B_1$, so we can evaluate

$$\langle S^\Gamma(0)V\psi(0), f_0\rangle = 0.$$

Next, as the Laplacian is a real operator, it is symmetric for $\langle\cdot,\cdot\rangle$. Thus the resolvent, $\widetilde{R}_0(\lambda)$, is symmetric too, first for $\operatorname{Re}(\lambda) > \frac{1}{2}$ and after in all D_N^+ by analytic continuation. In conclusion $S^\Gamma(0)$ is symmetric for $\langle\cdot,\cdot\rangle$. So

$$\langle V\psi(0), S^\Gamma(0)f_0\rangle = \langle V\psi(0), \psi(0)\rangle = 0,$$

and we have that equality for all $V \in \mathcal{F}$.

Thus, for all $m \in \mathbb{Z} \setminus \{0\}$ and all $V_m \in L^\infty_c(\mathbb{R}^+)$ we have

$$\int_0^{2\pi} e^{\mathrm{im}\,\theta} \int_{\mathbb{R}^+} V_m(r)\psi(0)^2(r,\theta)\,\mathrm{dvol}(g) = 0.$$

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This implies that for all the resonant states $\psi(0)$ of the free Laplacian, $\psi(0)^2$ does not depend on θ. But, considering the expression of the hyperbolic Laplacian and taking its decomposition corresponding to $\bigoplus_{\ell\in\mathbb{Z}}\big(L^2(\mathbb{R}^+, \operatorname{sh} r dr)\otimes e^{i\ell\theta}\big)$, we have resonant states of the form $\psi(0)(r,\theta) = \psi_\ell(r)e^{i\ell\theta}$ where $|\ell| \le k$ and ψ_ℓ are hypergeometric functions (see the annexe of [GZ95b]). Then for $\ell \neq 0$, $\psi(0)^2$ depend on θ; we have our contradiction.

Finally, there exists $V \in \mathcal{F}$ such that $-k$ is a resonance of $\Delta + V$ of order strictly greater than one. ∎

4 SO(n) Symmetries

This time we consider an isometric action of SO(n) on a complete Riemannian manifold (X, g) of dimension $n \ge 3$. Contrary to the case $\mathbb{S}^1$, SO(n) is not commutative, so we don't have a simple description of the isotypical components. To have a shift we add a hypothesis.

Condition C The isometric action of SO(n) on (X, g) has a fixed point O and the polar coordinates with pole O define a diffeomorphism from $X \setminus \{O\}$ to $\mathbb{R}^+ \setminus \{0\} \times \mathbb{S}^{n-1}$.

With this condition, in the polar coordinates the metric g becomes:

$$dr^2 + f(r)\mathrm{d}\omega^2, \quad (r,\omega) \in \mathbb{R}^+ \times \mathbb{S}^{n-1},$$

where $\mathrm{d}\omega^2$ is the metric on the $(n-1)$-sphere in $\mathbb{R}^n$. For example with $f(r) = r^2$ we have the Euclidean space and, with $f(r) = \operatorname{sh}(r)^2$, the hyperbolic space. If f is independent of r outside a compact then (X, g) is a manifold with a cylindrical end of section $\mathbb{S}^{n-1}$.

With the condition C we have

$$L^2(X) = \bigoplus_{k\in\mathbb{N}} L^2(\mathbb{R}^+) \otimes H^k,$$

where $H^k = \operatorname{Ker}\big(\Delta_{\mathbb{S}^{n-1}} - k(k+n-2)\big)$, $k \in \mathbb{N}$, be the eigenspaces of the Laplacian on $\mathbb{S}^{n-1}$. The action of SO(n) on X induces a representation of SO(n) on $L^2(X) \simeq L^2(\mathbb{R}^+)\otimes L^2(\mathbb{S}^{n-1})$ which only acts on the factor $L^2(\mathbb{S}^{n-1})$, so on the H^k. Moreover the restriction of this representation to each H^k is irreducible (*cf.* [BGM71]). The shift that we will use in order to construct isoresonant potentials, will appear on these H^k.

4.1 Representation

The group action of SO(n) on $L^2(X)$ induces an action of its Lie algebra, $\mathfrak{so}_n$. We can describe this action with the following operators,

$$D_\xi f(x) := \frac{\mathrm{d}}{\mathrm{d}t} f(e^{-t\xi}.x)_{|t=0}, \quad \xi \in \mathfrak{so}_n, f \in L^2(X), x \in X.$$

We consider the complexification of the Lie algebra $\mathfrak{so}_n$, $\mathfrak{g} := \mathfrak{so}_n^{\mathbb{C}} = \mathfrak{so}_n + i\mathfrak{so}_n$. Let $\mathfrak{h}$ be one of the Cartan subalgebras of $\mathfrak{g}$, *i.e.*, one of the maximal Abelian subalgebras

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of $\mathfrak{g}$. Let us describe $\mathfrak{h}$ as a subalgabra of $\mathfrak{gl}(\mathbb{C}^n)$. $\mathfrak{h}$ is the Lie algebra whose basis is $(\zeta_k)_{1\le k\le p}$, where p is the integer part of $\frac{n}{2}$ and ζ_k has all its entries null except the k^{th} 2×2-block, which is

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Let (ω_k) be the dual basis of (ζ_k) in $\mathfrak{h}^*$.

Remember that $\mathfrak{g}$ acts on itself by the adjoint representation:

$$\mathrm{ad}(Y)\colon Z \to [Y,Z], \quad (Y,Z)\in\mathfrak{g}^2.$$

We consider the following scalar product,

$$\langle Y,Z\rangle = \mathrm{Tr}\big(\mathrm{ad}(\overline{Y})\circ\mathrm{ad}(Z)\big), (Y,Z)\in\mathfrak{g}^2$$

where the conjugation is defined by $\overline{U+iV} = -U+iV$ with real U and V. So, for all $Y,Z\in\mathfrak{g}$, $\overline{[Y,Z]} = -[\overline{Y},\overline{Z}]$. With this we remark that, for all $\xi\in\mathfrak{h}$, we have $\overline{\xi}=\xi$ and thus $\mathrm{ad}(\xi)$ is selfadjoint (*cf.* [Sim96, p. 177]). So $\{\mathrm{ad}(\xi);\xi\in\mathfrak{h}\}$ is a family of selfadjoint operators on $\mathfrak{g}$ which commute together. We can simultaneously diagonalize them and decompose $\mathfrak{g}$ according to the eigenspaces.

We obtain $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus\mathfrak{g}_\alpha$ where the sum is over a finite set of $\alpha\in\mathfrak{h}^*$ which are the *roots* of $\mathfrak{g}$ and we denote $\mathfrak{g}_\alpha := \{X\in\mathfrak{g} : \mathrm{ad}(\xi)(X)=\alpha(\xi)X, \forall\xi\in\mathfrak{h}\}$, which are the *root spaces* (they are all one dimensional, *cf.* [Sim96, p. 180]). Let $\Lambda\subset\mathfrak{h}^*$ the integer lattice generated by the roots. In Λ we choose a lexicographical order "$\succeq$", with $\omega_1\succeq\cdots\succeq\omega_p$. Then we denote $\mathfrak{g}_+ := \bigoplus_{\alpha\succ 0}\mathfrak{g}_\alpha$ (respectively $\mathfrak{g}_- := \bigoplus_{\alpha\prec 0}\mathfrak{g}_\alpha$) the subalgebra of $\mathfrak{g}$ generated by root spaces with positive root (respectively negative). So we have $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{g}_+\oplus\mathfrak{g}_-$. For a general theory see [Sim96, chapter VIII].

We come back to the irreducible representation H^k. It has a decomposition according to the action of $\mathfrak{h}$:

$$H^k = \bigoplus_{\omega^k_{\mathrm{Min}}\preceq\omega\preceq\omega^k_{\max}} H^k_\omega,$$

where the sum is over a finite set of $\mathfrak{h}^*$, and these ω are the *weights* of H^k and the corresponding *weight spaces*, H^k_ω, are defined by

$$H^k_\omega = \{f\in H^k : D_\xi f = \omega(\xi)f, \forall\xi\in\mathfrak{h}\}.$$

We will need the following lemma.

Lemma 4.1 *If $f\in H^k_\omega$ and if $\xi\in\mathfrak{g}_\alpha$, then $D_\xi f\in H^k_{\omega+\alpha}$.*

Proof For all $\zeta\in\mathfrak{h}$, we have

$$D_\zeta(D_\xi f) = D_\xi(D_\zeta f) + D_{[\zeta,\xi]}f.$$

But $[\zeta,\xi] = \mathrm{ad}(\zeta)(\xi) = \alpha(\zeta)\xi$, because $\xi\in\mathfrak{g}_\alpha$, and $D_\zeta f = \omega(\zeta)f$ by definition of H^k_ω. So

$$D_\zeta(D_\xi f) = \omega(\zeta)D_\xi f + \alpha(\zeta)D_\xi f = (\omega+\alpha)(\zeta)(D_\xi f). \qquad \blacksquare$$

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We define particular vectors in the H^k, $k \in \mathbb{N}$, that will be used to create the necessary shift.

Definition 4.2 A nonzero vector $v \in H^k$ is a *highest weight vector* if it is an eigenvector for the action of all the D_ξ, $\xi \in \mathfrak{h}$, and it is in the kernel of all the D_ξ, $\xi \in \mathfrak{g}^+$.

As $\mathfrak{g}$ is semi-simple and H^k is an irreducible representation of it, there is a unique highest weight vector up to scalar: we note it by $v^k_{\max}$. In fact $H^k_{\alpha^k_{\max}}$ is one dimensional, generated by $v^k_{\max}$. With our choice, $v^k_{\max}$ can be calculated explicitly (see [Aut08]):

$$v^k_{\max} \circ \phi(x_1, \dots, x_n) = (x_1 + ix_2)^k,$$

where $\phi\colon \mathbb{R}^+ \setminus \{0\} \times \mathbb{S}^{n-1} \to X \setminus \{O\}$ is the diffeomorphism of Condition C and $(x_1, x_2, \dots, x_n)$ are the standard coordinates of $\mathbb{R}^n$ restricted to $\mathbb{S}^{n-1}$.

We will need the following lemma.

Lemma 4.3

$$H^k_{\omega^k_{\max}} = H^k \cap \Big(\bigcap_{\xi \in \mathfrak{g}_+} \operatorname{Ker} D_\xi \Big).$$

Proof The first inclusion $H^k_{\omega^k_{\max}} \subset H^k \cap \left(\bigcap_{\xi \in \mathfrak{g}_+} \operatorname{Ker} D_\xi \right)$ is the definition of a highest weight vector.

Let $u \in H^k \cap \left(\bigcap_{\xi \in \mathfrak{g}_+} \operatorname{Ker} D_\xi \right)$, so $u \in H^k = \bigoplus_{\omega^k_{\mathrm{Min}} \preceq \omega \preceq \omega^k_{\max}} H^k_\omega$ and we write $u = \sum_{\omega^k_{\mathrm{Min}} \preceq \omega \preceq \omega^k_{\max}} u_\omega$ with $u_\omega \in H^k_\omega$. For all $\xi \in \mathfrak{g}_\beta$ with $\beta \succ 0$, we have, thanks to Lemma 4.1, $D_\xi u_\omega \in H^k_{\alpha+\beta}$. By hypothesis, we have

$$D_\xi u = \sum_{\omega^k_{\mathrm{Min}} \preceq \omega \preceq \omega^k_{\max}} D_\xi u_\omega = 0,$$

and, as the previous sum is direct, we get for all ω,

$$\forall \xi \in \mathfrak{g}_+ \quad D_\xi u_\omega = 0.$$

Moreover, for all ω, by the definition of H^k_ω, u_ω is an eigenvector for the action of all the D_ξ, $\xi \in \mathfrak{h}$. Thus by the definition of a highest weight vector, we have $u_\omega = 0$ except for $u_{\omega^k_{\max}}$, and in conclusion $u \in H^k_{\omega^k_{\max}}$. ■

4.2 Isoresonant potentials

Let

$$L^2(X)^+ = \bigoplus_{k \in \mathbb{N}} L^2(\mathbb{R}^+) \otimes H^k_{\omega^k_{\max}}$$

and note P_k the corresponding projections into $L^2(\mathbb{R}^+) \otimes H^k_{\omega^k_{\max}}$.

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Theorem 4.4 *Let (X, g) a Riemannian manifold of dimension $n \geq 3$, with an isometric action of $\mathrm{SO}(n)$, verifying condition C. We assume that we have condition $A_{N,\rho}$ for some $N > 0$ with a function ρ invariant under the action of $\mathrm{SO}(n)$.*

Let V be the potential,

$$V = \sum_{k=1}^{\infty} V_k$$

where $V_k \in L^2(\mathbb{R}^+) \otimes H^k_{\omega^k_{\max}}$ and $\sup_k \|V_k\|_\infty < +\infty$. If V verifies condition $B_{N,\rho}$, and for all $\lambda \in D_N^+ \setminus \mathrm{Res}(\Delta)$, $\rho^{-(N+1)} V \widetilde{R}_0(\lambda) \rho^N$ is in a Schatten class $\mathcal{S}_q$, $q \in \mathbb{N} \setminus \{0\}$, then, in D_N^+, $\mathrm{Res}(\Delta + V) = \mathrm{Res}(\Delta)$ with the same multiplicities.

The Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$, asymptotically hyperbolic spaces and manifolds with asymptotically cylindrical ends with an action of $\mathrm{SO}(n)$, are examples where this theorem can be applied.

Remark 4.5 If X has an isometric action of $\mathrm{SO}(n)$ it has also an isometric action of $\mathbb{S}^1$. With Condition C, X is diffeomorphic to $\mathbb{R}^+ \setminus \{0\} \times \mathbb{S}^{n-1}$ and $\mathrm{SO}(n)$ acts on the factor $\mathbb{S}^{n-1}$. Taking, on $\mathbb{S}^{n-1}$, the hyperspherical coordinates $(\phi_1, \dots, \phi_{n-1}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^{n-2} \times [0, 2\pi)$, we can consider the action of $\mathbb{S}^1$ on X, corresponding to one of the inclusions $\mathbb{S}^1 \subset \mathrm{SO}(n)$, defined by

$$e^{i\theta}.(r, \phi_1, \dots, \phi_{n-1}) = (r, \phi_1, \dots, \phi_{n-1} - \theta).$$

If we consider the components $V_k \in L^2(\mathbb{R}^+) \otimes H^k_{\omega^k_{\max}}$ of V, they have the following form

$$V_k(r, \phi_1, \dots, \phi_{n-1}) = s_k(r) v^k_{\max}(\phi_1, \dots, \phi_{n-1}) = s_k(r) \Big(\prod_{i=1}^{n-2} \cos \phi_i \Big)^k e^{ik\phi_{n-1}}.$$

In fact we have $v^k_{\max} = (x_1 + ix_2)^k$ with $x_1 = (\prod_{i=1}^{n-2} \cos \phi_i) \cos \phi_{n-1}$ and $x_2 = (\prod_{i=1}^{n-2} \cos \phi_i) \sin \phi_{n-1}$. So V_k is $\mathbb{S}^1$ homogeneous of weight k for the previously described $\mathbb{S}^1$ action.

In conclusion, the family of potentials constructed thanks to the action of $\mathrm{SO}(n)$ is included into the potentials constructed with the action of $\mathbb{S}^1$.

So, why look at the $\mathrm{SO}(n)$ action? In fact, as we will see, using the $\mathrm{SO}(n)$ action simplifies the proof of isoresonance. In particular we don't need the lower bound of the spectrum of the Laplacian on functions of weight j, *i.e.*, Proposition 3.4. This allows us to add to the free Laplacian a real $\mathrm{SO}(n)$-invariant potential V_0 not compactly supported but just decreasing at infinity in order to continue R_{V_0} (compare with Remark 1.3 in the introduction).

In order to prove Theorem 4.4, first, note that V_k maps $L^2(\mathbb{R}^+) \otimes H^\ell_{\omega^\ell_{\max}}$ into $L^2(\mathbb{R}^+) \otimes H^{\ell+k}_{\omega^{\ell+k}_{\max}}$. As for the action of $\mathbb{S}^1$, this shift will be the key to the proof.

4.3 From $L^2(X)$ to $L^2(X)^+$

Let $\chi \in C_c^\infty(X)$ invariant under the action of SO(n). As for the circular action we begin studying $\mathrm{Res}(\Delta+\chi V)$, on D_N^+. We can write the Lipmann–Schwinger equation and get that if $\lambda_0 \in \mathrm{Res}(\Delta+\chi V) \cap \big(D_N^+ \setminus \mathrm{Res}(\Delta)\big)$ then there exists a nontrivial $u \in L^2(X)$ such that

$$\big(I + \rho^{-N}\chi V \widetilde{R}_0(\lambda_0)\rho^N\big)\, u = 0.$$

We want to prove that we can choose u in $L^2(X)^+$.

Lemma 4.6 *For $\lambda_0 \in \mathrm{Res}(\Delta+\chi V) \cap \big(D_N^+ \setminus \mathrm{Res}(\Delta)\big)$, there exists a nontrivial $w \in L^2(X)^+$ such that*

$$\big(I + \rho^{-N}\chi V \widetilde{R}_0(\lambda_0)\rho^N\big)\, w = 0.$$

Proof As $\rho^{-N}\chi V\widetilde{R}_0(\lambda_0)\rho^N \colon L^2(X) \to L^2(X)$ is a compact operator, we have $\mathcal{H}_{-1} := \mathrm{Ker}\big(I + \rho^{-N}\chi V\widetilde{R}_0(\lambda_0)\rho^N\big)$ is finite dimensional.

In addition, for all $\xi \in \mathfrak{g}_+$, D_ξ maps $\mathcal{H}_{-1}$ into itself. Indeed, by definition of highest weight vector, we have $D_\xi V = \sum_{k=1}^\infty D_\xi V_k = 0$. Moreover, as the action of SO(n) is isometric, D_ξ commutes with the Laplacian and so with $\widetilde{R}_0(\lambda_0)$. The operator D_ξ commutes also with ρ and χ because they are SO(n)-invariant. So, if $u \in \mathcal{H}_{-1}$ then $u = -\rho^{-N}\chi V\widetilde{R}_0(\lambda_0)\rho^N u$ and

$$\begin{aligned} D_\xi u &= -\rho^{-N}\chi D_\xi\big(V\widetilde{R}_0(\lambda_0)\rho^N u\big)\\ &= -\rho^{-N}\chi\big(D_\xi(V)\widetilde{R}_0(\lambda_0)\rho^N u + V D_\xi(\widetilde{R}_0(\lambda_0)\rho^N u)\big)\\ &= -\rho^{-N}\chi V\widetilde{R}_0(\lambda_0)\rho^N D_\xi u, \end{aligned}$$

and finally $D_\xi u \in \mathcal{H}_{-1}$.

So $\mathcal{H}_{-1}$ is a finite representation of $\mathfrak{g}_+$. Moreover $\mathfrak{g}_+$ is a nilpotent algebra. It comes from the fact that there is only a finite number of positive roots of $\mathfrak{g}$ and from the following calculation: if $\xi \in \mathfrak{g}_\alpha$ and $\zeta \in \mathfrak{g}_\beta$ then $\mathrm{ad}(\xi)(\zeta) \in \mathfrak{g}_{\alpha+\beta}$. To see this, for all $\sigma \in \mathfrak{h}$, we have

$$\begin{aligned} \mathrm{ad}(\sigma)([\xi,\zeta]) = \big[\sigma,[\xi,\zeta]\big] &= \big[\xi,[\sigma,\zeta]\big] + \big[[\sigma,\xi],\zeta\big]\\ &= [\xi,\beta(\sigma)\zeta] + [\alpha(\sigma)\xi,\zeta]\\ &= (\alpha+\beta)(\sigma)[\xi,\zeta]. \end{aligned}$$

Then by Engel's theorem (see [FH91, p. 125]) there exists a nonzero vector $w \in \mathcal{H}_{-1}$ such that $D_\xi w = 0$ for all $\xi \in \mathfrak{g}_+$.

We can decompose w:

$$w = \sum_{k\in\mathbb{N}} w_k, \quad w_k \in L^2(\mathbb{R}^+) \otimes H^k,$$

and for $\xi \in \mathfrak{g}_+$ we have

$$D_\xi w = \sum_{k\in\mathbb{N}} D_\xi w_k = 0,$$

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with $D_\xi w_k \in L^2(\mathbb{R}^+) \otimes H^k$. As the previous sum is direct, we have, for all k, $w_k \in (L^2(\mathbb{R}^+) \otimes H^k) \cap (\bigcap_{\xi \in \mathfrak{g}_+} \operatorname{Ker} D_\xi)$. But with Lemma 4.3 we have

$$\big(L^2(\mathbb{R}^+) \otimes H^k\big) \cap \Big(\bigcap_{\xi \in \mathfrak{g}_+} \operatorname{Ker} D_\xi\Big) = L^2(\mathbb{R}^+) \otimes H^k_{\omega^k_{\max}}.$$

In conclusion, for all k, $w_k \in H^k_{\omega^k_{\max}}$ and $w \in L^2(X)^+$. ■

4.4 The End of the Proof of Theorem 4.4

We assume that there exists $\lambda_0 \in \operatorname{Res}(\Delta + \chi V) \cap \big(D_N^+ \setminus \operatorname{Res}(\Delta)\big)$, and we take a nontrivial $w \in L^2(X)^+$, whose existence is given by Lemma 4.6 and which satisfies

$$\big(I + \rho^{-N}\chi V \widetilde{R}_0(\lambda_0)\rho^N\big)\, w = 0.$$

For all $j \in \mathbb{N}$, we denote $w_j := P_j w \in L^2(\mathbb{R}^+) \otimes H^j_{\omega^j_{\max}}$. The P_j commute with ρ, χ because they are both invariant under the action of $\mathrm{SO}(n)$ and $\widetilde{R}_0(\lambda_0)$ because the action is isometric. We have

$$w_j = P_j\big(-\rho^{-N}\chi V \widetilde{R}_0(\lambda_0)\rho^N w\big) = -\sum_{k=1}^{\infty} \rho^{-2N}\chi P_j\big(V_k \rho^N \widetilde{R}_0(\lambda_0)\rho^N w\big).$$

Moreover we have already seen that, for all $\xi \in \mathfrak{g}_+$, D_ξ commute with $\widetilde{R}_0(\lambda_0)$ and ρ. Thus, if $w \in L^2(X)^+$ then $\rho^N \widetilde{R}_0(\lambda_0)\rho^N w \in L^2(X)^+$ and we have $\rho^N \widetilde{R}_0(\lambda_0)\rho^N w = \sum_{\ell=0}^{\infty} P_\ell\big(\rho^N \widetilde{R}_0(\lambda_0)\rho^N w\big)$.

We use the shift created by highest weight vectors, *i.e.*:

$$V_k \colon L^2(\mathbb{R}^+) \otimes H^\ell_{\omega^\ell_{\max}} \to L^2(\mathbb{R}^+) \otimes H^{\ell+k}_{\omega^{\ell+k}_{\max}}.$$

So

$$w_j = -\sum_{k=1}^{\infty} \rho^{-2N}\chi V_k P_{j-k}\big(\rho^N \widetilde{R}_0(\lambda_0)\rho^N w\big) = -\sum_{k=1}^{\infty} \rho^{-2N}\chi V_k \rho^N \widetilde{R}_0(\lambda_0)\rho^N P_{j-k}(w).$$

Thanks to the hypothesis $\sup_k \|V_k\|_\infty < +\infty$, the operators $\rho^{-2N}\chi V_k \rho^N \widetilde{R}_0(\lambda_0)\rho^N$ are uniformly bounded in k, and consequently there exists a constant C such that for all $j \in \mathbb{N}$,

$$\|w_j\| \le C \sum_{k=1}^{\infty} \|w_{j-k}\|.$$

With this inequality and the fact that $w_j = 0$ for all $j \le 0$, we get $w_j = 0$ for all $j \in \mathbb{N}$, and thus $w = 0$ which is in contradiction with our hypothesis.

Finally we have proved that for all $\chi \in C_c^\infty(X)$ invariant under the action of $\mathrm{SO}(n)$, we have $D_N^+ \cap \operatorname{Res}(\Delta + \chi V) \subset \operatorname{Res}(\Delta)$.

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In a second part we pass from χV to V as in the case of the $\mathbb{S}^1$ action. We introduce a family of smooth and compactly supported functions (χ_r) invariant under the action of $\mathrm{SO}(n)$ such that $\|(\chi_r - 1)\rho\|_\infty$ tends to 0 when r tends to $+\infty$. We use the assumption $\rho^{-(N+1)} V \widetilde{R}_0(\lambda)\rho^N \in \mathcal{S}_q$ in order to characterize the resonances of $\Delta + V$ as the zeros of the holomorphic function in λ, $F(V,\lambda) := \det_q\big(I + \rho^{-N} V \widetilde{R}_0(\lambda)\rho^N\big)$. Finally with Rouché's theorem we prove that if $\widetilde{R}_V$ has a pole in $D_N^+ \setminus \mathrm{Res}(\Delta)$ then there exists r such that $\widetilde{R}_{\chi_r V}$ has also a pole which is in contradiction with the previous part. So $D_N^+ \cap \mathrm{Res}(\Delta + V) \subset \mathrm{Res}(\Delta)$.

To conclude, we prove that $D_N^+ \cap \mathrm{Res}(\Delta) \subset \mathrm{Res}(\Delta + V)$, and in fact the equality with multiplicity, using Agmon's perturbation theory of resonances exactly in the same way as in the case $\mathbb{S}^1$. This achieves the proof of Theorem 4.4.

5 Isoresonant Potentials on the Catenoid

We are going to construct isoresonant potentials on the catenoid. In this case we use complex scaling defined by Wunsch and Zworski in [WZ00] instead of Agmon's theory.

5.1 Statement of the Result

The catenoid is the surface X diffeomorphic to the cylinder $\mathbb{R} \times \mathbb{S}^1$ with the metric $g = \mathrm{d}r^2 + (r^2 + a^2)\mathrm{d}\alpha^2$, where $(r, e^{i\alpha}) \in \mathbb{R} \times \mathbb{S}^1$ and $a \in \mathbb{R} \setminus \{0\}$. We take $x = \frac{1}{|r|}$ outside $\{r = 0\}$ as the function defining the boundary at infinity of X, $\partial_\infty X$, which is two copies of $\mathbb{S}^1$. Near the boundary, we have

$$g = \frac{\mathrm{d}x^2}{x^4} + \frac{(1 + a^2x^2)\mathrm{d}\alpha^2}{x^2};$$

it's a *scattering metric* in Melrose's sense [Mel95].

The catenoid is an example in the Wunsch and Zworski article [WZ00], so we can use their results. They proved that there exists $\theta_0 > 0$ such that the resolvent of the free Laplacian, $(\Delta - z)^{-1}$, has a finite-meromorphic continuation from $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ to $\{z \in \mathbb{C} : \arg z < 2\theta_0\}$ with values in operators from $L^2_c(X)$ to $H^2_{\mathrm{loc}}(X)$. We denote this continuation by $\widehat{R}_0$. As in the previous part we call its poles resonances and we denote their set by $\mathrm{Res}(\Delta)$.

The group $\mathbb{S}^1$ acts isometrically on X by its trivial action on the factor $\mathbb{S}^1$: $e^{i\beta}.(r, e^{i\alpha}) = (r, e^{i(\alpha+\beta)})$. Using this action we are going to construct isoresonant potentials.

Theorem 5.1 *Let X be the catenoid $\big(\mathbb{R}\times\mathbb{S}^1, dr^2 + (r^2 + a^2)d\alpha^2\big)$ with $(r, e^{i\alpha}) \in \mathbb{R}\times\mathbb{S}^1$ and $a \in \mathbb{R} \setminus \{0\}$. Take $x = \frac{1}{|r|}$ outside $\{r = 0\}$ as the function defining the boundary at infinity of X. Let $V \in xL^\infty(X)$ defined by*

$$V(r, e^{i\alpha}) = \sum_{m=1}^{\infty} V_m(r) e^{\mathrm{i}m\alpha}, \quad (r, e^{i\alpha}) \in \mathbb{R} \times \mathbb{S}^1,$$

where, for all m, $V_m \in L^\infty(\mathbb{R})$*. Assume that, in a neighborhood of* $\partial_\infty X$*,* $V(x, e^{i\alpha})$ *with all its partial sums have an analytic continuation in* $U \times W$*, where* U *is an open set of* $\mathbb{C}$ *including* $\{\zeta \in \mathbb{C} : |\zeta| \le 1, 0 \le \arg\zeta \le \theta_0\}$ *with* $\theta_0 > 0$ *and* W *is a neighborhood of* $\mathbb{S}^1$ *in* $\mathbb{C}$*. Also assume that, in* X *and in all compacts of* $U \times W$*, the partial sums of* V *tend to* V *in infinite norm.*

Then the resolvent $(\Delta + V - z)^{-1}$ *has a finite-meromorphic continuation from* $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ *to* $\{z \in \mathbb{C} : \arg z < 2\theta_0\}$ *with values in operators from* $L^2_c(X)$ *to* $H^2_{\mathrm{loc}}(X)$*. Moreover, in this open set,* $\operatorname{Res}(\Delta + V) = \operatorname{Res}(\Delta)$ *with the same multiplicities.*

The following is an example of isoresonant potential on the catenoid:

$$V(x, e^{i\alpha}) = \frac{xe^{i\alpha}}{1 - \rho e^{i\alpha}} = x\sum_{m=1}^{\infty} \rho^{m-1} e^{\mathrm{im}\,\alpha},$$

with $0 \le \rho < 1$ and $U = \mathbb{C}$, $W = \{\omega \in \mathbb{C} : |\omega| < \rho^{-1}\}$.

5.2 Complex Scaling

We will use the complex scaling twice: to continue the resolvent of $\Delta + V$, and to study perturbations of resonances. So we begin with the description of this construction. We will follow [WZ00], where the construction is done for the free Laplacian. It is also valid when we add a potential $V \in xL^\infty(X)$.

We begin with the construction of a family $(X_\theta)_{0 \le \theta \le \theta_0}$ of submanifolds of $\mathbb{C} \times \mathbb{C}$ with the θ_0 of the statement of Theorem 5.1. They will be totally real (*i.e.*, for all $p \in X_\theta$, $T_pX_\theta \cap iT_pX_\theta = \{0\}$) and of maximal dimension. We define them as follows.

Let $\epsilon > 0$ and $(t_0, t_1) \in\,]0, 1[^2$ with $t_0 < t_1$. Then there exists a smooth deformation of $[0, 1)$ in U, denoted by $\gamma_\theta(t)$, $t \in [0, 1)$, satisfying the following properties:

$$\begin{gathered}
\gamma_\theta(t) = te^{i\theta} \quad \text{for } 0 \le t < t_0 \\
\gamma_\theta(t) \equiv t \quad \text{for } t > t_1 \\
\arg\gamma_\theta(t) \ge 0 \\
0 \le \arg\gamma_\theta(t) - \arg\gamma'_\theta(t) \le \epsilon \\
0 \le 2\arg\gamma_\theta(t) - \arg\gamma'_\theta(t) \le \theta + \epsilon.
\end{gathered} \tag{5.1}$$

Now we can define $X_\theta := (\gamma_\theta \times \partial_\infty X) \cup (X \cap \{x \ge 1\})$.

On a neighborhood of $\partial_\infty X$, the metric g has the form $\frac{\mathrm{d}x^2}{x^4} + \frac{h}{x^2}$ where $h = (1 + a^2x^2)\mathrm{d}\alpha^2$ continues holomorphically to $U \times W$. So consider $P^V := \Delta + V$, which is first an operator on X. If V verifies the hypothesis of Theorem 5.1, then its coefficients continue holomorphically in $U \times W$. We denote by $\widetilde{P}^V$ the differential operator coming from this continuation. Since X_θ is totally real and of maximal dimension, we can define without ambiguity (*cf.* [SZ91]) the differential operator P^V_θ by

$$\forall u \in C^\infty(X_\theta), \quad P^V_\theta u = (\widetilde{P}^V \widetilde{u})|_{X_\theta}$$

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where $\widetilde{u}$ is an almost analytic extension of u, that is

$$\widetilde{u} \in C^\infty(U \times W), \quad \widetilde{u}_{|X_\theta} = u, \quad \overline{\partial}\widetilde{u}_{|X_\theta} = \mathcal{O}\big(\mathrm{d}(\,\cdot\,, X_\theta)^N\big), \quad \text{for all } N.$$

We have

Proposition 5.2 *With $V \in xL^\infty(X)$, for all $0 \le \theta \le \theta_0$, P_θ^V has a discrete spectrum in $\mathbb{C} \setminus e^{2i\theta}\mathbb{R}^+$. Moreover, for θ such that $0 \le \theta_2 \le \theta \le \theta_0$, the spectrum of P_θ^V in $\{0 \le \arg z < 2\theta_2\}$ with its multiplicity do not depend on θ. This spectrum doesn't depend too on the choice of a γ_θ verifying* (5.2).

Proof Following exactly the Wunsch and Zworski proof in [WZ00] we can prove that, for all $z \in \mathbb{C} \setminus e^{2i\theta}\mathbb{R}^+$, $P_\theta^V - z\colon H^2(X_\theta) \to L^2(X_\theta)$ is a Fredholm operator with index zero. The unique difference is the presence of our potential V. But since it is null at the boundary of X_θ $(= \partial_\infty X)$, it doesn't change the principal symbol and the normal symbol of $\Delta_\theta - z$.

5.3 Continuation of the Resolvent

We want to get the meromorphic continuation of the resolvent $R_V(z) := (\Delta + V - z)^{-1}$ from $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ to $\{z \in \mathbb{C} : \arg z < 2\theta_0\}$ with values in operators from $L^2_c(X)$ to $H^2_{\mathrm{loc}}(X)$.

Let z with $\arg z < 2\theta_0$ which is not an eigenvalue of $P_{\theta_0}^V$. Take $f \in L^2_c(X)$. With Proposition 5.2, we can choose γ_{θ_0} and more precisely the t_1 in Definition 5.2 such that on the support of f, X_{θ_0} coincides with X. Then $f \in L^2(X_{\theta_0})$, and there exists an unique solution $u_{\theta_0} \in H^2(X_{\theta_0})$ of

$$(P_{\theta_0}^V - z)u_{\theta_0} = f.$$

We state a lemma whose proof is given in [SZ91].

Lemma 5.3 *Let $\Omega \subset \mathbb{C}^n$ be an open set, $K \subset \Omega$ compact, and X_t, $t \in [0,1]$, a continuous family of totally real submanifolds of Ω of maximal dimension such that $X_t \cap (\Omega \setminus K) = X_{t'} \cap (\Omega \setminus K)$ for all $t, t' \in [0,1]$. Let $\widetilde{P}$ a differential operator with holomorphic coefficients in Ω such that P_{X_t} (the restriction of $\widetilde{P}$ on X_t) is elliptic for all $t \in [0,1]$. If u is a distribution on X_0 and if $P_{X_0}u$ continues as a holomorphic function on a neighborhood of $\bigcup_{t\in[0,1]} X_t$, then the same is true for u.*

In our case, f has a holomorphic continuation to $\bigcup_{\theta\in[0,\theta_0]} X_\theta$, because deformations occur outside its support. Since $P_\theta^V - z$ is elliptic for all $\theta \in [0, \theta_0]$, we can apply Lemma 5.3 and get a holomorphic continuation of u_{θ_0} on $\bigcup_{\theta\in[0,\theta_0]} X_\theta$. We denote this continuation by G.

Then we define the continuation to the resolvent by

$$\widetilde{R}_V(z)f = G_{|X_0} \in H^2(X).$$

Now take z_0 to be an eigenvalue of $P^V_{\theta_0}$; then it is also an eigenvalue P^V_θ for all $\arg z_0 < 2\theta \le 2\theta_0$. For z near z_0 and θ such that $\arg z < 2\theta$ we have the Laurent expansion

$$(P^V_\theta - z)^{-1} = \sum_{j=1}^{M(z_0)} \frac{A^\theta_j(z_0)}{(z-z_0)^j} + H_\theta(z, z_0),$$

where $A^\theta_j(z_0)$ are finite rank operators and $H_\theta(z, z_0)$ is holomorphic in z near z_0. Still following [WZ00], we obtain that the continued resolvent has, near each of its poles, a Laurent expansion with exactly the same form.

In conclusion, a resonance $z_0 \in \{\arg z < 2\theta_0\}$ of $\Delta_X + V$, which is first defined as poles of the continuation of the resolvent, is also characterized as an element of the spectrum of a P^V_θ with $\arg z_0 < 2\theta \le 2\theta_0$. Multiplicities and orders are the same in the two visions, so, thanks to Proposition 5.2, they do not depend on the chosen θ.

5.4 Proof of the Isoresonance

5.4.1 Localization of Resonances for the Truncated Partial Sums of V

Let $S_M(r, e^{i\alpha}) = \sum_{m=1}^M V_m(r)e^{\mathrm{im}\,\alpha}$ and $\chi \in C^\infty_c(X)$, $\mathbb{S}^1$ invariant. In this part we will prove $\mathrm{Res}(\Delta + \chi S_M) \subset \mathrm{Res}(\Delta)$ in $D^+ := \{z \in \mathbb{C} : \arg z < 2\theta_0\}$.

We take another $\mathbb{S}^1$ invariant cutoff function $\chi_1 \in C^\infty_c(X)$ such that $\chi_1 = 1$ on the support of χ. Then we have, for $z \in D^+ \setminus \mathrm{Res}(\Delta)$,

$$(\Delta + \chi S_M - z)\widetilde{R}_0(z)\chi_1 = \chi_1\big(I + \chi S_M \widetilde{R}_0(z)\chi_1\big).$$

Also $\chi S_M \widetilde{R}_0(z)\chi_1$ is a holomorphic family of compact operators in $D^+ \setminus \mathrm{Res}(\Delta)$ such that

$$\|\chi S_M \widetilde{R}_0(z)\chi_1\| < 1$$

with $|z|$ sufficiently large in $\{z \in \mathbb{C} : \mathrm{Im}\, z < 0\}$. So we can apply the Fredholm analytic theory and get $(I + \chi S_M \widetilde{R}_0(z)\chi_1)^{-1}$ and thus $\widetilde{R}_{\chi S_M}(z) := (\Delta + \chi S_M - z)^{-1}$ meromorphic in $D^+ \setminus \mathrm{Res}(\Delta)$. Moreover, in $D^+ \setminus \mathrm{Res}(\Delta)$, we can characterize poles of $\widetilde{R}_{\chi S_M}$, that is resonances, by the existence of a nontrivial $u \in L^2(X)$ solution of

$$(I + \chi S_M \widetilde{R}_0(z)\chi_1)u = 0.$$

Now we prove that this nontrivial solution u can't exist. It is exactly the same proof than for Theorem 3.1. We use the shift created by the components $V_m(r)e^{\mathrm{im}\,\alpha}$ on the spaces $L^2_j(X)$. We just have to verify that $\widetilde{R}_0$ commutes with the action of $\mathbb{S}^1$ in order to have the commutation with the projectors P_j. For that, remember that the complex scaling doesn't touch the factor $\partial_\infty X$ of the catenoid and $\mathbb{S}^1$ only acts on this factor. So the action of $\mathbb{S}^1$ is isometric on all the X_θ and so it commutes with $\widetilde{R}_0$.

Finally we get $\mathrm{Res}(\Delta + \chi S_M) \subset \mathrm{Res}(\Delta)$ in D^+.

5.4.2 Localization of Resonances for V

We have to control perturbations of resonances when we pass from χS_M to V. Instead of using regularized determinants as before (they were adapted to the weight spaces $\rho^N L^2$ but not to cutoff functions), we use complex scaling in order to transform resonances into eigenvalues.

Assume, *ab absurdum*, $\Delta + V$ has a resonance z_0 in $D^+ \setminus \operatorname{Res}(\Delta)$. Using complex scaling, this means: z_0 is an eigenvalue of P_θ^V with $\arg z_0 < 2\theta \le 2\theta_0$. Let $\Omega \subset \{z \in \mathbb{C} : \arg z < 2\theta\} \setminus \operatorname{Res}(\Delta)$ an open set with a smooth boundary Γ, containing z_0 and such that $\overline{\Omega} \cap \operatorname{Res}(\Delta + V) = \{z_0\}$. Our aim is to show that there exists an $\mathbb{S}^1$ invariant cutoff function χ and M such that $P_\theta^{\chi S_M}$ has an eigenvalue in Ω. If we do that, $\Delta + \chi S_M$ will have a resonance in Ω which will be in contradiction with the previous part.

We have assumed in Theorem 5.1 that, for all M, S_M has an analytic continuation to $U \times W$, and V too. Hence we can restrict these two continuations to X_θ and now work on X_θ. $V \in xL^\infty(X)$, so V tends to 0 when we reach ∂X_θ. Consequently, there exists $\chi \in C_c^\infty(X_\theta)$, $\mathbb{S}^1$ invariant, which continues analytically in $U \times W$, such that $\|\chi V - V\|_{L^\infty(X_\theta)}$ is as small as we want. With the hypothesis of Theorem 5.1, we also have that the partial sums S_M tend to V on X_θ in infinite norm. Finally there exist χ as we have just described, and M such that

$$\|V - \chi S_M\|_{L^\infty(X_\theta)} < \frac{\delta^2}{\delta + \frac{\ell}{2\pi}},$$

where $\delta^{-1} = \max_{z\in\Gamma} \|(P_\theta^V - z)^{-1}\|$ and ℓ is the length of Γ. We have been inspired by Gohberg and Krejn in [GK71, Theorem 3.1].

We consider the projectors of $L^2(X_\theta)$ associated with the generalized eigenspaces of the two operators that we are comparing:

$$\Pi_V = \frac{1}{2\pi i}\int_\Gamma (P_\theta^V - z)^{-1}\,\mathrm{d}z,$$

$$\Pi_{\chi S_M} = \frac{1}{2\pi i}\int_\Gamma (P_\theta^{\chi S_M} - z)^{-1}\,\mathrm{d}z.$$

We have $(P_\theta^{\chi S_M} - z)^{-1} = (P_\theta^V - z)^{-1}\big(I + (\chi S_M - V)(P_\theta^V - z)^{-1}\big)^{-1}$. Since $\delta^2/(\delta + \frac{\ell}{2\pi}) < \delta$ for all $z \in \Gamma$, we can be certain of the convergence in

$$(P_\theta^{\chi S_M} - z)^{-1} = (P_\theta^V - z)^{-1}\Big(I + \sum_{j=1}^{\infty}[(V - \chi S_M)(P_\theta^V - z)^{-1}]^j\Big).$$

Look at the difference between the two projectors:

$$\Pi_{\chi S_M} - \Pi_V = \frac{1}{2\pi i}\int_\Gamma (P_\theta^V - z)^{-1}\sum_{j=1}^{\infty}[(V - \chi S_M)(P_\theta^V - z)^{-1}]^j\,\mathrm{d}z.$$

Hence

$$\|\Pi_{\chi S_M} - \Pi_V\| \leq \frac{\ell}{2\pi} \max_{z\in\Gamma} \frac{\|(P_\theta^V - z)^{-1}\|^2 \, \|V - \chi S_M\|_{L^\infty(X_\theta)}}{1 - \|(P_\theta^V - z)^{-1}\| \, \|V - \chi S_M\|_{L^\infty(X_\theta)}},$$

but $\|(P_\theta^V - z)^{-1}\| \leq \delta^{-1}$ by definition of δ and $\|V - \chi S_M\|_{L^\infty(X_\theta)} < \frac{\delta^2}{\delta + \frac{\ell}{2\pi}}$ by hypothesis, hence $1 - \|(P_\theta^V - z)^{-1}\| \, \|V - \chi S_M\|_{L^\infty(X_\theta)} > 1 - \frac{\delta}{\delta + \frac{\ell}{2\pi}}$ and so

$$\|\Pi_{\chi S_M} - \Pi_V\| < 1.$$

Consequently the ranges of Π_V and $\Pi_{\chi S_M}$ have the same dimension, which is not zero because z_0 is an eigenvalue of P_θ^V, so $P_\theta^{\chi S_M}$ has an eigenvalue in Ω, and we have our contradiction.

5.4.3 Persistence of Resonances

To finish the proof of Theorem 5.1 we have to show that $\mathrm{Res}(\Delta) \subset \mathrm{Res}(\Delta + V)$ in $D^+ = \{z \in \mathbb{C} : \arg z < 2\theta_0\}$. This time we use the complex scaling instead of Agmon's perturbation theory of resonances.

Let $z_0 \in D^+$ a resonance of Δ with multiplicity m. We take

$$\Omega \subset \{z \in \mathbb{C} : \arg z < 2\theta_0\}$$

a domain such that $\overline{\Omega} \cap \mathrm{Res}(\Delta) = \{z_0\}$. We consider the family of operators $\Delta + tV$ with $t \geq 0$. Observe that tV verifies the hypothesis of Theorem 5.1, so we can localize its resonances as in the previous part, $\mathrm{Res}(\Delta + tV) \subset \mathrm{Res}(\Delta)$, and thus:

$$\mathrm{Res}(\Delta + tV) \cap \Omega \subset \{z_0\}.$$

We are going to prove, by connectivity, that the set

$$E := \left\{ t_0 \geq 0 : \forall t \in [0, t_0], \mathrm{Res}(\Delta + tV) \cap \Omega = \{z_0\} \text{ with multiplicity } m \right\},$$

is equal to $[0, +\infty[$. It is not empty because $0 \in E$.

Take $t_0 \in E$, and θ such that $\arg z_0 < 2\theta \leq 2\theta_0$ and $\Omega \subset \{z \in \mathbb{C} : \arg z < 2\theta\}$. We know by complex scaling that the spectrum of $P_\theta^{t_0 V}$ in Ω exactly corresponds with the resonances of $\Delta + t_0 V$ with the same multiplicities. Hence

$$\mathrm{Spec}(P_\theta^{t_0 V}) \cap \Omega = \{z_0\}.$$

Moreover $t \to P_\theta^{tV}$ is a holomorphic family in the sens of Kato for t in a complex neighbourhood of t_0 because $P_\theta^{tV} = \Delta_\theta + tV_{|X_\theta}$ and V is bounded in X_θ. So its eigenvalues are continuous for t in a neighbourhood of t_0. But with the localization of the resonances of $\Delta + tV$ we obtain that for all t

$$\mathrm{Spec}(P_\theta^{tV}) \cap \Omega \subset \{z_0\},$$

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so there exists $\varepsilon > 0$, such that for all $t \in \,]t_0 - \varepsilon, t_0 + \varepsilon[$,

$$\mathrm{Spec}(P_\theta^{tV}) \cap \Omega = \{z_0\},$$

with multiplicity m. Then, thanks to the complex scaling parallel, we get that for all $t \in \,]t_0 - \varepsilon, t_0 + \varepsilon[$

$$\mathrm{Res}(\Delta + tV) \cap \Omega = \{z_0\}$$

with multiplicity m, and thus $]t_0 - \varepsilon, t_0 + \varepsilon[$ is included in E which is open.

We show that E is also closed doing the same work with the complementary set of E in $[0, +\infty[$.

In conclusion, $E = [0, +\infty[$, and taking $t = 1$ we have, in Ω, $\mathrm{Res}(\Delta + V) = \mathrm{Res}(\Delta)$ with the same multiplicities. Doing the same work in the neighbourhood of each resonance of the free Laplacian we obtain $\mathrm{Res}(\Delta + V) = \mathrm{Res}(\Delta)$ with the same multiplicities in all D^+, which finishes the proof of Theorem 5.1.

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