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A NOTE ON THE ENDPOINT REGULARITY OF THE HARDY–LITTLEWOOD MAXIMAL FUNCTIONS

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Abstract

In this note we give a simple proof of the endpoint regularity for the uncentred Hardy–Littlewood maximal function on $\mathbb{R}$. Our proof is based on identities for the local maximum points of the corresponding maximal functions, which are of interest in their own right.

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1. Introduction

Let d be a positive integer and $\mathbb{R}^d$ denote the d-dimensional Euclidean space. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, the centred Hardy–Littlewood maximal operator is defined by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

for any $x \in \mathbb{R}^d$, where $B(x,r)$ is the ball in $\mathbb{R}^d$ centred at x with radius r and $|B(x,r)|$ denotes the volume of $B(x,r)$. As is well known, the operator $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p \le \infty$ and maps $L^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$. In 1997, Kinnunen [6] first studied the regularity of $\mathcal{M}$ and showed that $\mathcal{M}$ is bounded on the Sobolev spaces $W^{1,p}(\mathbb{R}^d)$ for all $1 < p \le \infty$, where the Sobolev spaces $W^{1,p}(\mathbb{R}^d)$, $1 \le p \le \infty$, are defined by

$$W^{1,p}(\mathbb{R}^d) := \{f : \|f\|_{1,p} = \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla(f)\|_{L^p(\mathbb{R}^d)} < \infty\}$$

and $\nabla(f) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_d)$ is the weak gradient. See [4] for the basic properties of Sobolev functions. Subsequently, Kinnunen and Lindqvist [7] gave a local version of the original boundedness on $W^{1,p}(\Omega)$, where Ω is an open set of $\mathbb{R}^d$. This paradigm that an L^p-bound implies a $W^{1,p}$-bound was extended to a fractional version in [8] and

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to a bilinear version in [3] and to a multisublinear version in [10]. Later, the continuity of $\mathcal{M} : W^{1,p} \to W^{1,p}$ for $p > 1$ was established by Luiro in [11] and in [12] for its local version. (Continuity is not immediate from boundedness because of the lack of linearity.)

As usual, the endpoint case $p = 1$ is significantly different from the case $p > 1$, not only because $\mathcal{M}(f) \notin L^1(\mathbb{R}^d)$ whenever f is nontrivial, while the maximal operator acts boundedly on L^p for $p > 1$, but also because $L^1(\mathbb{R}^d)$ is not reflexive (so weak compactness arguments used when $1 < p < \infty$ are not available for $p = 1$). Since Kinnunen's result does not hold for $p = 1$, understanding the regularity in the endpoint case seems to be a deeper issue. In this regard, one of the main questions was posed by Hajlasz and Onninen in [5]:

Question 1.1. Is the operator $f \mapsto |\nabla \mathcal{M}(f)|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$?

In 2002, Tanaka [13] first gave a positive answer to Question 1.1 for the uncentred Hardy–Littlewood maximal function in the case $d = 1$. More precisely, Tanaka considered the uncentred Hardy–Littlewood maximal operator defined by

$$\widetilde{\mathcal{M}}(f)(x) = \sup_{s,t>0} \frac{1}{s+t} \int_{x-s}^{x+t} |f(y)|\, dy,$$

and showed that if $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{\mathcal{M}}(f)$ has a weak derivative in $L^1(\mathbb{R})$ and

$$\|(\widetilde{\mathcal{M}}(f))'\|_{L^1(\mathbb{R})} \le 2\|f'\|_{L^1(\mathbb{R})}.$$

This result was later refined by Aldaz and Pérez Lázaro [1] who showed, under the assumption that f is of bounded variation on $\mathbb{R}$, that $\widetilde{\mathcal{M}}(f)$ is absolutely continuous and

$$\|(\widetilde{\mathcal{M}}(f))'\|_{L^1(\mathbb{R})} \le \mathrm{Var}(f),$$

where $\mathrm{Var}(f)$ denotes the total variation of f. This yields

$$\|(\widetilde{\mathcal{M}}(f))'\|_{L^1(\mathbb{R})} \le \|f'\|_{L^1(\mathbb{R})}, \tag{1.1}$$

if $f \in W^{1,1}(\mathbb{R})$.

In this paper, we will continue to focus on Question 1.1. By studying the behaviour of the local maxima of $\widetilde{\mathcal{M}}(f)$ on $\mathbb{R}$, we will present a simple proof of the inequality (1.1) for $f \in W^{1,1}(\mathbb{R})$. More precisely, we shall prove the following result.

Theorem 1.2. *If $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{\mathcal{M}}(f)$ is absolutely continuous, and*

$$\|(\widetilde{\mathcal{M}}(f))'\|_{L^1(\mathbb{R})} \le \|f'\|_{L^1(\mathbb{R})}.$$

Remark 1.3. Obviously, Theorem 1.2 is an improvement of Tanaka's result in [13]. We remark that our proof is different from the proof of (1.1) given by Aldaz and Pérez Lázaro in [1], where the authors deduced that $\|(\widetilde{\mathcal{M}}(f))'\|_{L^1(\mathbb{R})} \le \|f'\|_{L^1(\mathbb{R})}$ by proving that $\mathrm{Var}(\widetilde{\mathcal{M}}(f)) \le \mathrm{Var}(f)$. Here, we will give a direct proof of the former inequality: $\widetilde{\mathcal{M}}(f)$ is weakly differentiable and the weak derivative is integrable on $\mathbb{R}$, which is equivalent to the absolute continuity of $\widetilde{\mathcal{M}}(f)$ on $\mathbb{R}$. This implies $\mathrm{Var}(\widetilde{\mathcal{M}}(f)) \le \mathrm{Var}(f)$ under the hypothesis $f \in W^{1,1}(\mathbb{R})$. Our method is quite elementary and simple. The main ingredients are identities at the local maximum points of $\widetilde{\mathcal{M}}(f)$, which are of interest in their own right (see Lemma 2.4). For the centred Hardy–Littlewood maximal operator $\mathcal{M}$, Kurka [9] recently showed that $\mathrm{Var}(\mathcal{M}(f)) \le C\,\mathrm{Var}(f)$ with a certain constant $C > 1$, if f is of bounded variation on $\mathbb{R}$. However, our method does not work for $\mathcal{M}$. An interesting question to ask is whether the corresponding result to Theorem 1.2 for $\mathcal{M}$ also holds, provided $f \in W^{1,1}(\mathbb{R})$ with constant $C = 1$.

The rest of this paper is organised as follows. After presenting some key lemmas in Section 2, we will prove Theorem 1.2 in Section 3. It should be pointed out that the main ideas in our proof are greatly motivated by [2, 13], but some new techniques are also necessary. Throughout this paper, the letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence, but independent of the essential variables.

2. Some notation and lemmas

Let us begin with the following definition.

DEFINITION 2.1. We say that a point x_0 is a local maximum of f if there exists $\alpha > 0$ such that

$$f(x_0) \ge f(x_0 - h), \quad f(x_0) \ge f(x_0 + h) \quad \text{for } 0 < h < \alpha.$$

LEMMA 2.2. *If f is continuous and integrable on $\mathbb{R}$, then $\widetilde{\mathcal{M}}(f)$ is continuous on $\mathbb{R}$ and $\widetilde{\mathcal{M}}(f)(x) \ge |f(x)|$ for all $x \in \mathbb{R}$. Moreover, if $f \in W^{1,1}(\mathbb{R})$, then both f and $\widetilde{\mathcal{M}}(f)$ vanish at infinity.*

PROOF. When $\|f\|_{L^1(\mathbb{R})} = 0$, it follows from the continuity of f that $f \equiv 0$ and the conclusions are obvious. Thus, we may assume that $\|f\|_{L^1(\mathbb{R})} > 0$ for the entire proof. It follows from the continuity of f and the Lebesgue differentiation theorem that

$$\widetilde{\mathcal{M}}(f)(x) \ge |f(x)| \quad \forall x \in \mathbb{R}.$$

We shall prove the continuity of $\widetilde{\mathcal{M}}(f)$. For any $x, h \in \mathbb{R}$, one can easily check that

$$|\widetilde{\mathcal{M}}(f)(x+h) - \widetilde{\mathcal{M}}(f)(x)| \le \sup_{s,t>0} \frac{1}{s+t} \int_{x-s}^{x+t} |f(y+h) - f(y)|\, dy.$$

For any $\epsilon > 0$, we set $\delta_1 = 2\|f\|_{L^1(\mathbb{R})}/\epsilon$. Since f is uniformly continuous on $[x - 2\delta_1, x + 2\delta_1]$, for any $\epsilon > 0$ there exists $0 < \delta < \delta_1$ such that $|f(y) - f(z)| < \epsilon$ for all $y, z \in [x - 2\delta_1, x + 2\delta_1]$ with $|y - z| < \delta$. We consider the following two cases:

(i) If $s + t \le \delta_1$, then for all $|h| < \delta$,

$$\frac{1}{s+t} \int_{x-s}^{x+t} |f(y+h) - f(y)|\, dy \le \frac{1}{s+t} \int_{x-s}^{x+t} \epsilon\, dy < \epsilon.$$

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(ii) If $s + t > \delta_1$, then for all h,

$$\frac{1}{s+t}\int_{x-s}^{x+t} |f(y+h) - f(y)|\,dy \le \frac{2}{s+t}\|f\|_{L^1(\mathbb{R})} < \epsilon.$$

Thus, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\widetilde{\mathcal{M}}(f)(x+h) - \widetilde{\mathcal{M}}(f)(x)| < \epsilon$$

for all $|h| < \delta$. The continuity of $\widetilde{\mathcal{M}}(f)$ follows from this.

Moreover, if $f \in W^{1,1}(\mathbb{R})$, then f is absolutely continuous on $\mathbb{R}$ and its classical derivative is equal to the weak derivative almost everywhere. From the fundamental theorem of calculus, for any $x \in \mathbb{R}$,

$$f(x) - f(0) = \int_0^x f'(t)\,dt.$$

Taking $x \to \infty$, by the dominated convergence theorem,

$$\lim_{x\to\infty} f(x) = \int_0^\infty f'(t)\,dt + f(0),$$

but then $\lim_{x\to\infty} f(x) = 0$ since we also have $f \in L^1(\mathbb{R})$. Similarly, $\lim_{x\to-\infty} f(x) = 0$. Thus f vanishes at infinity.

We claim that $\widetilde{\mathcal{M}}(f)$ vanishes at infinity. Since f vanishes at infinity, for any $\epsilon > 0$ there exists $B_1 > 0$ such that $|f(x)| < \epsilon$ for all $|x| > B_1$. Let $B_2 = B_1 + \|f\|_{L^1(\mathbb{R})}/\epsilon$. For any $|x| > B_2$,

$$\begin{aligned}\widetilde{\mathcal{M}}(f)(x) &\le \sup_{\substack{s,t>0\\ \max\{s,t\}>B_2-B_1}} \frac{1}{s+t}\int_{x-s}^{x+t}|f(y)|\,dy + \sup_{\substack{s,t>0\\ \max\{s,t\}\le B_2-B_1}} \frac{1}{s+t}\int_{x-s}^{x+t}|f(y)|\,dy \\ &\le \frac{\|f\|_{L^1(\mathbb{R})}}{B_2 - B_1} + \epsilon = 2\epsilon,\end{aligned}$$

which confirms our claim. □

REMARK 2.3. We remark that the first part of Lemma 2.2 can be obtained from [1, Lemma 3.4]. In the second part of Lemma 2.2, the condition $f \in W^{1,1}(\mathbb{R})$ can be weakened to f being an integrable function of bounded variation on $\mathbb{R}$. In fact, the function of bounded variation has one-sided limits everywhere, thus the limits at infinity must be zero because of the integrability of f. By the same argument as in the proof of Lemma 2.2, we can show that $\widetilde{\mathcal{M}}(f)$ vanishes at infinity.

The next lemma deals with the local maximum points of the corresponding maximal function. It will play a key role in the proof of Theorem 1.2.

LEMMA 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and integrable. If x_0 is a local maximum of $\widetilde{\mathcal{M}}(f)$, then $\widetilde{\mathcal{M}}(f)(x_0) = |f(x_0)|$.*

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PROOF. We shall prove the lemma by considering the following two cases.

Case 1. Suppose $\widetilde{\mathcal{M}}(f)(x_0)$ is attained for some $s_0 \geq 0, t_0 \geq 0$ such that

$$\widetilde{\mathcal{M}}(f)(x_0) = \frac{1}{s_0 + t_0} \int_{x_0 - s_0}^{x_0 + t_0} |f(x)| \, dx. \tag{2.1}$$

We may assume that $s_0, t_0 > 0$ (since the other cases can be obtained by a simple modification of our arguments). For any fixed $0 < \varepsilon < \min\{s_0, t_0\}$,

$$\begin{aligned}
\widetilde{\mathcal{M}}(f)(x_0) &= \frac{1}{s_0 + t_0} \int_{x_0 - s_0}^{x_0 + t_0} |f(x)| \, dx \\
&= \frac{s_0 - \varepsilon}{s_0 + t_0} \frac{1}{s_0 - \varepsilon} \int_{x_0 - s_0}^{x_0 - \varepsilon} |f(x)| \, dx + \frac{t_0 - \varepsilon}{s_0 + t_0} \frac{1}{t_0 - \varepsilon} \int_{x_0 + \varepsilon}^{x_0 + t_0} |f(x)| \, dx \\
&\quad + \frac{2\varepsilon}{s_0 + t_0} \frac{1}{2\varepsilon} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} |f(x)| \, dx \\
&\leq \frac{s_0 + t_0 - 2\varepsilon}{s_0 + t_0} \widetilde{\mathcal{M}}(f)(x_0) + \frac{2\varepsilon}{s_0 + t_0} \frac{1}{2\varepsilon} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} |f(x)| \, dx.
\end{aligned}$$

Therefore,

$$\widetilde{\mathcal{M}}(f)(x_0) \leq \frac{1}{2\varepsilon} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} |f(x)| \, dx.$$

Letting $\varepsilon \to 0$ gives $\widetilde{\mathcal{M}}(f)(x_0) \leq |f(x_0)|$. Combining this with Lemma 2.2 yields $\widetilde{\mathcal{M}}(f)(x_0) = |f(x_0)|$.

Case 2. Suppose there are no $s_0, t_0 \geq 0$ such that (2.1) holds. Without loss of generality, we may assume that $\widetilde{\mathcal{M}}(f)(x_0)$ is not attained for any $s \geq 0$. Then

$$\widetilde{\mathcal{M}}(f)(x_0) = \sup_{s > k, t > 0} \frac{1}{s + t} \int_{x_0 - s}^{x_0 + t} |f(x)| \, dx \quad \forall k = 1, 2, \ldots.$$

Otherwise, there exists some $M > 0$ such that

$$\widetilde{\mathcal{M}}(f)(x_0) = \sup_{0 < s \leq M, t > 0} \frac{1}{s + t} \int_{x_0 - s}^{x_0 + t} |f(x)| \, dx,$$

which gives a contradiction. Clearly,

$$\widetilde{\mathcal{M}}(f)(x_0) \leq \sup_{s > k} \frac{1}{s} \|f\|_{L^1(\mathbb{R})} \quad \forall k = 1, 2, \ldots,$$

which implies $\widetilde{\mathcal{M}}(f)(x_0) = 0$ and, from Lemma 2.2, $\widetilde{\mathcal{M}}(f)(x_0) = |f(x_0)| = 0$. This completes the proof. □

LEMMA 2.5. *Let f be a function on $\mathbb{R}$. Let (a, b) be an interval such that both f and $\widetilde{\mathcal{M}}(f)$ are continuous on (a, b). Suppose that $\widetilde{\mathcal{M}}(f)(x) > |f(x)|$ for any $x \in (a, b)$ and $\widetilde{\mathcal{M}}(f)$ is strictly monotonic on (a, b). Then $\widetilde{\mathcal{M}}(f)$ is absolutely continuous on (a, b) provided that one of the following conditions holds:*

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(i) $-\infty < a < b < \infty$;
(ii) $a = -\infty$, $b < \infty$ *and* $\lim_{x\to-\infty} \widetilde{\mathcal{M}}(f)(x)$ *exists;*
(iii) $a > -\infty$, $b = \infty$ *and* $\lim_{x\to\infty} \widetilde{\mathcal{M}}(f)(x)$ *exists;*
(iv) $a = -\infty$, $b = \infty$ *and both* $\lim_{x\to-\infty} \widetilde{\mathcal{M}}(f)(x)$ *and* $\lim_{x\to\infty} \widetilde{\mathcal{M}}(f)(x)$ *exist.*

Proof. One can easily check that there exists $C > 0$ such that $\widetilde{\mathcal{M}}(f)(x) \le C$ for any $x \in (a, b)$, provided one of the conditions (i)–(iv) holds. We may assume $\widetilde{\mathcal{M}}(f)$ is strictly increasing on (a, b). Let $G := \{x \in (a, b) : (\widetilde{\mathcal{M}}(f))'(x) = \infty\}$. It suffices to show that $|\widetilde{\mathcal{M}}(f)(G)| = 0$.

We shall prove $G = \emptyset$. Fix $x_0 \in (a, b)$. By the continuity of f at the point x_0, there exists $\delta > 0$ such that

$$|f(x)| < |f(x_0)| + \tfrac{1}{2}(\widetilde{\mathcal{M}}(f)(x_0) - |f(x_0)|)$$

for any $|x - x_0| \le \delta$. This together with the definition of $\widetilde{\mathcal{M}}(f)$ yields that, for any $\epsilon > 0$, there exist $s_0 > 0$ and $t_0 > 0$ such that $s_0 + t_0 > \delta$ and

$$\widetilde{\mathcal{M}}(f)(x_0) < \frac{1}{s_0 + t_0} \int_{x_0-s_0}^{x_0+t_0} |f(y)|\,dy + \epsilon.$$

For any $h \in (0, \delta)$,

$$\begin{aligned}
&\widetilde{\mathcal{M}}(f)(x_0) - \widetilde{\mathcal{M}}(f)(x_0 - h)\\
&\quad\le \frac{1}{s_0 + t_0} \int_{x_0-s_0}^{x_0+t_0} |f(y)|\,dy + \epsilon - \frac{1}{s_0 + t_0 + h} \int_{x_0-h-s_0}^{x_0+t_0} |f(y)|\,dy\\
&\quad\le \frac{1}{s_0 + t_0} \int_{x_0-s_0}^{x_0+t_0} |f(y)|\,dy + \epsilon - \frac{1}{s_0 + t_0 + h} \int_{x_0-s_0}^{x_0+t_0} |f(y)|\,dy\\
&\quad= \frac{h}{(s_0 + t_0)(s_0 + t_0 + h)} \int_{x_0-s_0}^{x_0+t_0} |f(y)|\,dy + \epsilon\\
&\quad\le \frac{h}{\delta(s_0 + t_0)} \int_{x_0-s_0}^{x_0+t_0} |f(y)|\,dy + \epsilon \le \frac{h}{\delta} C + \epsilon.
\end{aligned} \tag{2.2}$$

Since $\epsilon > 0$ is arbitrary, (2.2) implies

$$\frac{1}{h}[\widetilde{\mathcal{M}}(f)(x_0) - \widetilde{\mathcal{M}}(f)(x_0 - h)] \le \frac{1}{\delta} C \quad \text{for } h \in (0, \delta),$$

and hence $(\widetilde{\mathcal{M}}(f))'(x_0) \neq \infty$. Thus $G = \emptyset$, and then $|\widetilde{\mathcal{M}}(f)(G)| = 0$. This completes the proof. □

3. Proof of Theorem 1.2

For $f \in W^{1,1}(\mathbb{R})$, f is absolutely continuous on $\mathbb{R}$ and vanishes at infinity. By Lemma 2.2, $\widetilde{\mathcal{M}}(f)(x) \ge |f(x)|$ for all $x \in \mathbb{R}$ and $\widetilde{\mathcal{M}}(f)$ is continuous and vanishes at

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infinity. Therefore, the set $A := \{x \in \mathbb{R} : \widetilde{\mathcal{M}}(f)(x) > |f(x)|\}$ is open. We can write A as a countable union of disjoint open intervals:

$$A = \bigcup_j I_j := \bigcup_j (\alpha_j, \beta_j),$$

where $\widetilde{\mathcal{M}}(f)(\alpha_j) = |f(\alpha_j)|$ and $\widetilde{\mathcal{M}}(f)(\beta_j) = |f(\beta_j)|$. Moreover, if $\alpha_j = -\infty$ or $\beta_j = +\infty$, then $f(\alpha_j) = \widetilde{\mathcal{M}}(f)(\alpha_j) = 0$ and $f(\beta_j) = \widetilde{\mathcal{M}}(f)(\beta_j) = 0$.

Since constant segments are not allowed, we claim that each interval I_j satisfies only one of the following conditions:

(i) $\widetilde{\mathcal{M}}(f)$ is strictly increasing on I_j;

(ii) $\widetilde{\mathcal{M}}(f)$ is strictly decreasing on I_j;

(iii) there exists $b_j \in I_j$ such that $\widetilde{\mathcal{M}}(f)$ is strictly decreasing on (α_j, b_j) and is strictly increasing on (b_j, β_j).

Otherwise, there exist $\alpha_j < c_1 < c_2 < c_3 < \beta_j$ such that

$$\widetilde{\mathcal{M}}(f)(c_1) < \widetilde{\mathcal{M}}(f)(c_2), \quad \widetilde{\mathcal{M}}(f)(c_2) \geq \widetilde{\mathcal{M}}(f)(c_3) \quad \text{or}$$
$$\widetilde{\mathcal{M}}(f)(c_1) \leq \widetilde{\mathcal{M}}(f)(c_2), \quad \widetilde{\mathcal{M}}(f)(c_2) > \widetilde{\mathcal{M}}(f)(c_3).$$

From the Weierstrass extreme value theorem and the continuity of $\widetilde{\mathcal{M}}(f)$, there exists a local maximum of $\widetilde{\mathcal{M}}(f)$ in $[c_1, c_3]$ at least. But this contradicts Lemma 2.4.

Next, we shall conclude that $\widetilde{\mathcal{M}}(f)$ is absolutely continuous on I_j and

$$\mathrm{Var}(\widetilde{\mathcal{M}}(f); I_j) \leq \mathrm{Var}(f; I_j), \tag{3.1}$$

where $\mathrm{Var}(f; I_j)$ denotes the total variation of f on I_j. If I_j satisfies (i) or (ii), $\widetilde{\mathcal{M}}(f)$ is absolutely continuous on I_j by Lemma 2.5 and

$$\mathrm{Var}(\widetilde{\mathcal{M}}(f); I_j) = |\widetilde{\mathcal{M}}(f)(\beta_j) - \widetilde{\mathcal{M}}(f)(\alpha_j)| = \big|\,|f(\beta_j)| - |f(\alpha_j)|\,\big| \leq \mathrm{Var}(|f|; I_j) \leq \mathrm{Var}(f; I_j).$$

Thus (3.1) holds. If I_j satisfies (iii),

$$\begin{aligned}\mathrm{Var}(\widetilde{\mathcal{M}}(f); I_j) &= (\widetilde{\mathcal{M}}(f)(\beta_j) - \widetilde{\mathcal{M}}(f)(b_j)) + (\widetilde{\mathcal{M}}(f)(\alpha_j) - \widetilde{\mathcal{M}}(f)(b_j)) \\ &< (|f(\beta_j)| - |f(b_j)|) + (|f(\alpha_j)| - |f(b_j)|) \\ &\leq |f(\beta_j) - f(b_j)| + |f(\alpha_j) - f(b_j)| \leq \mathrm{Var}(f; I_j)\end{aligned}$$

and again (3.1) holds. On the other hand, by Lemma 2.5 again, $\widetilde{\mathcal{M}}(f)$ is absolutely continuous on (α_j, b_j) and on (b_j, β_j). Then $\widetilde{\mathcal{M}}(f)$ maps sets of measure zero in (α_j, b_j) and in (b_j, β_j) onto sets of measure zero. Thus, for any set $G \subset I_j$ of measure zero,

$$|\widetilde{\mathcal{M}}(f)(G)| \leq |\widetilde{\mathcal{M}}(f)(G \cap (\alpha_j, b_j))| + |\widetilde{\mathcal{M}}(f)(G \cap (b_j, \beta_j))| = 0,$$

which implies that $\widetilde{\mathcal{M}}(f)$ maps sets of measure zero in I_j onto sets of measure zero. Since $\widetilde{\mathcal{M}}(f)$ is continuous and of bounded variation on I_j, $\widetilde{\mathcal{M}}(f)$ is absolutely

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continuous on I_j and thus has a weak derivative v on each I_j. Moreover, the weak derivative coincides with the classical derivative almost everywhere.

Next, similarly to Tanaka's arguments in [13], we will prove that $\widetilde{\mathcal{M}}(f)$ is weakly differentiable on $\mathbb{R}$ with

$$(\widetilde{\mathcal{M}}(f))' = |f|'\chi_{A^c} + v\chi_A, \tag{3.2}$$

where χ_A and χ_{A^c} denote the indicator functions of the sets A and A^c. Indeed, for any $\varphi \in C_c^\infty(\mathbb{R})$, $\widetilde{\mathcal{M}}(f)\varphi$ is absolutely continuous on I_j. By integration by parts,

$$\int_{I_j} \widetilde{\mathcal{M}}(f)(x)\varphi'(x)\,dx = (|f(\beta_j)|\varphi(\beta_j) - |f(\alpha_j)|\varphi(\alpha_j)) - \int_{I_j} v(x)\varphi(x)\,dx.$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}} \widetilde{\mathcal{M}}(f)(x)\varphi'(x)\,dx &= \int_A \widetilde{\mathcal{M}}(f)(x)\varphi'(x)\,dx + \int_{A^c} \widetilde{\mathcal{M}}(f)(x)\varphi'(x)\,dx \\ &= \sum_j (|f(\beta_j)|\varphi(\beta_j) - |f(\alpha_j)|\varphi(\alpha_j)) - \int_A v(x)\varphi(x)\,dx + \int_{A^c} |f(x)|\varphi'(x)\,dx. \end{aligned} \tag{3.3}$$

Also, $f\varphi$ is absolutely continuous on I_j. By integration by parts again,

$$\begin{aligned}&\sum_j (|f(\beta_j)|\varphi(\beta_j) - |f(\alpha_j)|\varphi(\alpha_j)) - \int_A v(x)\varphi(x)\,dx + \int_{A^c} |f(x)|\varphi'(x)\,dx \\ &\quad= \int_A |f(x)|\varphi'(x)\,dx + \int_A |f|'(x)\varphi(x)\,dx - \int_A v(x)\varphi(x)\,dx + \int_{A^c} |f(x)|\varphi'(x)\,dx \\ &\quad= \int_{\mathbb{R}} |f(x)|\varphi'(x)\,dx + \int_A |f|'(x)\varphi(x)\,dx - \int_A v(x)\varphi(x)\,dx. \end{aligned} \tag{3.4}$$

Obviously, $|f|$ is weakly differentiable on $\mathbb{R}$ because $|f|$ is absolutely continuous on $\mathbb{R}$, so for any $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} |f(x)|\varphi'(x)\,dx = -\int_{\mathbb{R}} |f|'(x)\varphi(x)\,dx.$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}} |f(x)|\varphi'(x)\,dx + \int_A |f|'(x)\varphi(x)\,dx - \int_A v(x)\varphi(x)\,dx \\ &\quad= -\int_{\mathbb{R}} (|f|'(x)\chi_{A^c}(x) + v(x)\chi_A(x))\varphi(x)\,dx.\end{aligned}$$

Combining this with (3.3) and (3.4) yields (3.2).

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It follows from (3.1) and (3.2) that

$$\begin{aligned}
\|(\widetilde{\mathcal{M}}(f))'\|_{L^1(\mathbb{R})} &= \int_A |(\widetilde{\mathcal{M}}(f))'(x)|\,dx + \int_{A^c} |(\widetilde{\mathcal{M}}(f))'(x)|\,dx \\
&= \int_A |v(x)|\,dx + \int_{A^c} ||f|'(x)|\,dx = \sum_j \mathrm{Var}(\widetilde{\mathcal{M}}(f); I_j) + \int_{A^c} ||f|'(x)|\,dx \\
&\le \sum_j \mathrm{Var}(|f|; I_j) + \int_{A^c} ||f|'(x)|\,dx = \sum_j \int_{I_j} ||f|'(x)|\,dx + \int_{A^c} ||f|'(x)|\,dx \\
&\le \int_{\mathbb{R}} ||f|'(x)|\,dx = \mathrm{Var}(|f|) \le \mathrm{Var}(f) = \|f'\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Finally, since $\widetilde{\mathcal{M}}(f)$ is weakly differentiable on $\mathbb{R}$ and $(\widetilde{\mathcal{M}}(f))' \in L^1(\mathbb{R})$, $\widetilde{\mathcal{M}}(f)$ is absolutely continuous on $\mathbb{R}$.

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