COUNTING COLOURED GRAPHS II

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1. Introduction. As in my earlier paper **(2)**, a *graph on n labelled nodes* is a set of n objects called "nodes" distinguishable from each other and a set (possibly empty) of "edges," i.e. pairs of distinct nodes. Each edge is said to join its pair of nodes and no edge joins a node to itself. By a k-colouring of the nodes of such a graph we mean a mapping of the nodes onto a set of k distinct colours, such that no two nodes joined by an edge are mapped onto the same colour. By a colouring of the edges of such a graph we mean a mapping of the edges onto a set of colours. We shall suppose that there are just j different ways of "joining" each pair of nodes of different colours, i.e. we may not join them, we may join them by a red edge, we may join them by a blue edge, and so on.

If, in any particular k-colouring of the nodes, there are s_1 nodes of the first colour, s_2 of the second, and so on, we have

$$P(s_1, \ldots, s_k) = \frac{n!}{s_1!s_2! \ldots s_k!}$$

different colourings of this kind. The s_1 nodes of the first colour and the s_2 nodes of the second colour may be joined in $T(s_1 s_2)$ different ways, where $T(\alpha) = j^\alpha$. Hence we have

$$M_n = M_n(k, j) = \sum_{(n)} P(s_1, \ldots, s_k) T\left(\sum_{h \neq m} s_h s_m\right)$$

different coloured graphs, where $\sum_{(n)}$ denotes summation over all non-negative integers $s_1, \ldots, s_k$ such that $\sum s_h = n$. Here and subsequently $\sum$ alone denotes $\sum_{h=1}^{k}$. Since

$$2 \sum_{h \neq m} s_h s_m = \left(\sum s_h\right)^2 - \sum s_h^{\,2} = n^2 - \sum s_h^{\,2},$$

we have

$$M_n(k, j) = \sum_{(n)} P(s_1, \ldots, s_k) T(\tfrac{1}{2}n^2 - \tfrac{1}{2} \sum s_h^{\,2}). \tag{1.1}$$

In what follows we suppose j and k fixed and study the behaviour of M_n for large n.

2. Elementary methods. In this section we show how far we can get by elementary methods, in particular, without the use of any information

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about the asymptotic behaviour of $n!$ for large n. In what follows, A and B denote positive numbers, not always the same at each occurrence; of these, each A may depend on k and j but is independent of n, while each B may depend on k, j, and n but is always bounded above and below by an A, so that $A < B < A$.

We write $K = \frac{1}{2}\{1 - (1/k)\}$, $N = [n/k]$, $a = n - kN$ and P_0 for the value of P when $s_1 = \ldots = s_a = N + 1$ and $s_{a+1} = \ldots = s_k = N$. We can easily verify that

$$n^2 - \sum s_h^2 = 2Kn^2 - \sum\{s_h - (n/k)\}^2 \tag{2.1}$$

and so, by (1.1),

$$M_n T(-Kn^2) \leqslant \sum_{(n)} P(s_1, \ldots, s_k) = k^n. \tag{2.2}$$

Again, by (2.1),

$$n^2 - a(N+1)^2 - (k-a)N^2 > 2Kn^2 - A$$

and so

$$M_n \geqslant AP_0 T(Kn^2). \tag{2.3}$$

Now

$$\begin{aligned} P_0 &= n!\{(N+1)!\}^{-a}(N!)^{a-k} \\ &= (kN+a)\ldots(kN+1)(N+1)^{-a}(kN)!(N!)^{-k} \\ &= B(kN)!(N!)^{-k}. \end{aligned} \tag{2.4}$$

Again

$$\begin{aligned} (kN + k - 1)! &= (k-1)! \prod_{l=0}^{k-1} \prod_{t=1}^{N} (kt + l) \\ &\geqslant \left\{ \prod_{t=1}^{N} (tk) \right\}^k = k^{kN}(N!)^k \end{aligned}$$

and so, by (2.4),

$$P_0 n^{k-1} = B(kN + k - 1)!(N!)^{-k} > Ak^{kN} > Ak^n.$$

Hence, by (2.2) and (2.3), we have

$$An^{1-k}k^n T(Kn^2) \leqslant M_n \leqslant k^n T(Kn^2). \tag{2.5}$$

But we can, with very little extra complication, improve on (2.5) substantially and prove the following theorem.

THEOREM 1.

$$M_n = Bn^{-\frac{1}{2}(k-1)}k^n T(Kn^2), \tag{2.6}$$

so that

$$\log M_n = Kn^2 \log j + n \log k - \tfrac{1}{2}(k-1)\log n + O(1).$$

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If $s_1 - s_2 \geqslant 2$, we have

$$P(s_1, s_2, s_3, \ldots, s_k) = s_1^{-1}(s_2+1)P(s_1-1, s_2+1, s_3, \ldots, s_k) < P(s_1-1, s_2+1, s_3, \ldots, s_k).$$

Hence the largest values of P are those in which no two s differ by more than 1, and these are the P equal to P_0. Thus, by (2.1),

$$M_n T(-Kn^2) \leqslant P_0 \sum_{(n)} T\left(-\tfrac{1}{2}\sum \{s_h - (n/k)\}^2\right)$$
$$\leqslant P_0\left\{\sum_{s=-\infty}^{\infty} T(-\tfrac{1}{2}\{s-(n/k)\}^2)\right\}^k$$
$$\leqslant P_0\left\{2\sum_{s=0}^{\infty} T(-\tfrac{1}{2}s^2)\right\}^k = AP_0.$$

From this and (2.3), we have

$$M_n = BP_0T(Kn^2).$$

To complete the proof of (2.6), it remains to show that

$$P_0 = Bk^n n^{-\frac{1}{2}(k-1)}. \tag{2.7}$$

We write $p = [\frac{1}{2}(k-1)]$. By (2.4), we have

$$\frac{P_0 n^p}{k^n} = \frac{Bn^p(kN)!}{k^n(N!)^k} = \frac{B(kN+p)!}{k^{kN}p!(N!)^k} = BR_1R_2, \tag{2.8}$$

where

$$R_1 = \prod_{t=1}^{N}\prod_{q=-p}^{p}\left(\frac{kt+q}{kt}\right) = \prod_{t=1}^{N}\prod_{q=1}^{p}\left(1-\frac{q^2}{k^2t^2}\right)$$

for all k, $R_2 = 1$ if k is odd, and

$$R_2 = \prod_{t=1}^{N}\left(\frac{kt-\frac{1}{2}k}{kt}\right) = \prod_{t=1}^{N}\left(\frac{2t-1}{2t}\right)$$

if k is even. Clearly

$$1 > R_1 \geqslant \prod_{q=1}^{p}\left(1-\frac{q^2}{k^2}\sum_{t=1}^{N}\frac{1}{t^2}\right) \geqslant \left(1-\frac{\pi^2}{24}\right)^p > A,$$

so that $R_1 = B$. Again, if k is even, we have

$$4R_2^2N = 4N\prod_{t=1}^{N}\left(\frac{2t-1}{2t}\right)^2 = \prod_{t=2}^{N}\frac{(2t-1)^2}{2t(2t-2)}$$
$$= \prod_{t=2}^{N}\left(1-\frac{1}{(2t-1)^2}\right)^{-1} = B.$$

Hence, whether k is odd or even, $R_2 = Bn^{p-\frac{1}{2}(k-1)}$ and so (2.7) follows from (2.8).

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3. A more detailed result. We can, however, use the asymptotic expansion of the Γ function, just as we did in **(2)** for the case $j = 2$, to obtain a much more exact result. The arguments and calculations are the same for general j and we do not repeat them. We have

THEOREM 2. *As* $n \to \infty$,

$$M_n = k^n j^{Kn^2} \left(\frac{k}{n \log j}\right)^{\frac{1}{2}(k-1)} \left\{ \sum_{t=0}^{H-1} C_t n^{-t} + O(n^{-H}) \right\},$$

where $C_t = C_t(k, j, a)$ *depends on* k, j, t, *and the residue* a *of* n (mod k), *but not otherwise on* n. *In particular,*

$$C_0(k, j, a) = k^{\frac{1}{2}} \{(\log j)/2\pi\}^{\frac{1}{2}(k-1)} L(a), \tag{3.1}$$

where

$$L(a) = \sum_{((a))} T\left(-\frac{1}{2} \sum s_h^2 + \frac{a^2}{2k}\right)$$

and the sum $\sum_{((a))}$ *is taken over all integral values of the* s_h, *positive, negative, or zero, subject to the condition that*

$$\sum s_h = a. \tag{3.2}$$

4. The coefficient $C_0(k, j, a)$ **and the sum** $L(a)$. In **(2)**, we showed that $C_0(k, 2, a)$ was very near to 1. In fact, for $k < 1000$ and all a,

$$|C_0(k, 2, a) - 1| < 1.33 \times 10^{-6}. \tag{4.1}$$

Nonetheless, at least for $k = 2$, $C_0(k, 2, a)$ was not independent of a and, in fact,

$$C_0(2, 2, 0) - C_0(2, 2, 1) > 2.6194 \times 10^{-6},$$

so that $C_0(2, 2, 0)$ and $C_0(2, 2, 1)$ differ by almost as much as (4.1) allows. But we did not study $C_0(k, 2, a)$ any further.

Here we find a transformation for $C_0(k, j, a)$ which, at least for the smaller values of j, gives $C_0(k, j, a)$ in a form which shows the nature of its dependence on a very clearly and, for $j = 2$ and values of k greater than 2, greatly improves on (4.1).

We use $\sum'$ to denote summation over all values of h such that $1 \leqslant h \leqslant k-1$, $\sum''$ over all values of h, m such that $1 \leqslant h < m \leqslant k-1$ and $\sum_{k-1}$ over all values of $s_1, s_2, \ldots, s_{k-1}$ positive, negative, or zero. We write

$$\gamma = 2\pi^2/\log j, \qquad Z_1 = \sum' s_h, \qquad Z_2 = \sum' s_h^2,$$

$$\begin{aligned} k\Delta = k\Delta(s_1, \ldots, s_{k-1}) &= kZ_2 - Z_1^2 \\ &= (k-1)\sum' s_h^2 - 2\sum'' s_h s_m. \end{aligned}$$

We shall prove the following theorem.

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THEOREM 3. *$C_0(k, j, a) = \sum_{k-1} e^{\nu}$, where $\nu = -\gamma\Delta - 2\pi i a Z_1/k$.*

In the ring of all $k-1$ by $k-1$ matrices, we write I for the unit matrix and E for the matrix all of whose elements are 1. If we write Q for the matrix of the quadratic form Δ, we have $Q = I - E/k$. Since

$$(kI - E)(I + E) = kI + (k-1)E - E^2 = kI,$$

we see that $Q^{-1} = I + E$, so that, if we write Δ^{-1} for the quadratic form whose matrix is Q^{-1}, we have

$$\Delta^{-1}\left(s_1 - \frac{a}{k}, \ldots, s_{k-1} - \frac{a}{k}\right) = \sum{}'\left(s_h - \frac{a}{k}\right)^2 + \left\{\sum{}'\left(s_h - \frac{a}{k}\right)\right\}^2 = \sum\left(s_h - \frac{a}{k}\right)^2 = \sum s_h^2 - \frac{a^2}{k}, \tag{4.2}$$

where $\sum s_h = a$. Now, by **(1**, (69.2)**)**

$$\sum{}_{k-1} \exp\{-\gamma\Delta(s_1, \ldots, s_{k-1}) - 2\pi i a Z_1/k\} = (\pi/\gamma)^{\frac{1}{2}(k-1)}|Q|^{-\frac{1}{2}} \sum{}_{k-1} \exp\left\{-\frac{\pi^2}{\gamma}\Delta^{-1}\left(s_1 - \frac{a}{k}, \ldots, s_{k-1} - \frac{a}{k}\right)\right\} = C_0(k, j, a). \tag{4.3}$$

by (4.2), since $|Q| = 1/k$. This proves Theorem 3.

5. Calculation of the leading terms in $C_0(k, j, a)$. We can readily verify that $\nu(-s_1, -s_2, \ldots, -s_{k-1})$ is the complex conjugate of $\nu(s_1, \ldots, s_k)$. Hence we may replace each e^{ν} in Theorem 3 by its real part, viz.

$$e^{-\gamma\Delta(s_1, \ldots, s_{k-1})} \cos(2\pi a Z_1/k), \tag{5.1}$$

so that

$$C_0(k, j, a) = \sum{}_{k-1} e^{-\gamma\Delta} \cos(2\pi a Z_1/k). \tag{5.2}$$

Now

$$k\Delta(s_1, \ldots, s_{k-1}) = k\sum{}' s_h^2 - Z_1^2 = \sum{}' s_h^2 + \sum{}''(s_h - s_m)^2.$$

It follows from this that

$$\Delta(-s_1, s_2 - s_1, \ldots, s_{k-1} - s_1) = \Delta(s_1, s_2, \ldots, s_{k-1}).$$

Since also

$$Z_1(-s_1, s_2 - s_1, \ldots, s_{k-1} - s_1) = Z_1(s_1, s_2, \ldots, s_{k-1}) - ks_1,$$

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we see that (5.1) is unchanged in value if $s_1, s_2, \ldots, s_{k-1}$ are replaced by $s_1, s_1 - s_2, \ldots, s_1 - s_{k-1}$. Again ν is symmetrical in the s_h. Thus, in the sum in (5.2), substantial numbers of equal terms can be grouped together. This greatly facilitates calculation of the terms which contribute effectively to the value of C_0.

A result which helps us to classify the terms in (5.2) conveniently is the following theorem.

THEOREM 4. *If*

$$x = \max\{|s_h|(1 \leqslant h \leqslant k-1), |s_h - s_m| \ (1 \leqslant h < m \leqslant k-1)\},$$

we have

$$k\Delta \geqslant (k-2)[\tfrac{1}{2}(x^2+1)] + x^2$$

and there is equality for at least one set of s_h.

The theorem is trivial when $x = 0$. We may, therefore, suppose that $x \geqslant 1$. Since

$$\Delta(s_1, \ldots, s_{k-1}) \geqslant \Delta(|s_1|, \ldots, |s_{k-1}|),$$

it is enough to prove our theorem when every s_h is non-negative. In that case, x is one of the s_h and $0 \leqslant s_h \leqslant x$. Let us suppose that the integer u occurs just α_u times among the s_h. Then

$$\alpha_x \geqslant 1, \qquad k - 1 = \sum_{0 \leqslant u \leqslant x} \alpha_u,$$

and

$$k\Delta = \sum_{0 \leqslant u \leqslant x} \alpha_u u^2 + \sum_{0 \leqslant u < v \leqslant x} \alpha_u \alpha_v (u-v)^2$$

$$\geqslant \alpha_x x^2 + \sum_{u=0}^{x-1} \alpha_u\{u^2 + (x-u)^2\}.$$

But

$$u^2 + (x-u)^2 = \tfrac{1}{2}\{x^2 + (x-2u)^2\} \geqslant [\tfrac{1}{2}(x^2+1)].$$

Hence

$$k\Delta \geqslant \alpha_x x^2 + [\tfrac{1}{2}(x^2+1)] \sum_{0 \leqslant u \leqslant x-1} \alpha_u$$

$$= (k - 1 - \alpha_x)[\tfrac{1}{2}(x^2+1)] + \alpha_x x^2 \geqslant (k-2)[\tfrac{1}{2}(x^2+1)] + x^2.$$

To show that this is best possible, we write $w = [\frac{1}{2}(x+1)]$ and remark that

$$k\Delta(w, w, \ldots, w, x) = x^2 + (k-2)w^2 + (k-2)(x-w)^2$$
$$= (k-2)[\tfrac{1}{2}(x^2+1)] + x^2.$$

Hence this lower bound is attained.

By Theorem 4, we have $\Delta \geqslant \xi(x)$, where $\xi(x) = \frac{1}{2}x^2$ if x is even and

$$\xi(x) = \frac{1}{2}x^2 + \frac{1}{2} - \frac{1}{k}$$

if x is odd.

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We now write U_s for the sum of all those terms on the right-hand side of (5.2) which correspond to sets $(s_1, \ldots, s_{k-1})$ for which $x = s$. Let $\omega(s)$ be the number of these sets and

$$\Omega(s) = \sum_{t=0}^{s} \omega(t),$$

the number of the sets for which $x \leqslant s$, i.e. the number of sets $(s_1, \ldots, s_{k-1})$ satisfying

$$(5.3) \qquad |s_h| \leqslant s, \qquad |s_h - s_m| \leqslant s$$

for all h and m. Let $G(t_1, t_2)$ be the family of sets satisfying $t_1 \leqslant s_h \leqslant t_2$ for all h. Then any set satisfying (5.3) belongs to one or more of the families

$$(5.4) \qquad G(-s, 0), G(1 - s, 1), \ldots, G(0, s).$$

Again any set belonging to one or more of the families (5.4) satisfies (5.3). Hence $\Omega(s)$ is the number of different sets in the union of the families (5.4). But a little consideration shows that any set that belongs to just t of the families (5.4) belongs to just $t - 1$ of the families

$$(5.5) \qquad G(1 - s, 0), G(2 - s, 1), \ldots, G(0, s - 1)$$

and conversely. Hence $\Omega(s)$ is the sum of the number of members of each of the families (5.4) less the sum of the number of members of each of the families (5.5), i.e.

$$\Omega(s) = (s + 1)^k - s^k,$$

since $G(u - s, u)$ has $(s + 1)^{k-1}$ members and $G(u + 1 - s, u)$ has just s^{k-1} members. Hence, for any x,

$$\begin{aligned} |U_x| \leqslant \omega(x)e^{-\gamma\xi(x)} &= \{\Omega(x) - \Omega(x - 1)\}e^{-\gamma\xi(x)} \\ &= \{(x + 1)^k + (x - 1)^k - 2x^k\}e^{-\gamma\xi(x)}. \end{aligned}$$

U_0 contains just one term, namely $e^{\nu(0, \ldots, 0)} = 1$. Again $x = 1$ when just y of the s_h are each equal to 1 or each equal to -1 and the remaining $k - 1 - y$ of the s_h are each zero. For each $y (1 \leqslant y \leqslant k - 1)$, there are $2\binom{k-1}{y}$ such terms, in each of which $k\Delta = y(k - y)$ and $Z_1 = \pm y$. Hence

$$\begin{aligned} U_1 &= 2 \sum_{y=1}^{k-1} \binom{k-1}{y} e^{-\gamma y(k-y)/k} \cos \frac{2\pi a y}{k} \\ &= 2 \sum_{y=1}^{[\frac{1}{2}(k-1)]} \left\{ \binom{k-1}{y} + \binom{k-1}{k-y} \right\} e^{-\gamma y(k-y)/k} \cos \frac{2\pi a y}{k} + \beta_1 \\ &= 2 \sum_{y=1}^{[\frac{1}{2}(k-1)]} \binom{k}{y} e^{-\gamma y(k-y)/k} \cos \frac{2\pi a y}{k} + \beta_1, \end{aligned}$$

where $\beta_1 = 0$ if k is odd and

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$$\beta_1 = 2\binom{k-1}{\frac{1}{2}k} e^{-\gamma k/4} \cos \pi a$$

if k is even.

Similar but more complicated calculations show us that

$$U_2 = \sum_{u=1}^{[\frac{1}{2}k]} \frac{k!}{(u!)^2(k-2u)!} e^{-2u\gamma} + 2 \sum_{u=1}^{[\frac{1}{2}(k-1)]} \sum_{v=1}^{k-u} \frac{k!}{u!(u+v)!(k-2u-v)!} e^{-\gamma\{2u+v(k-v)/k\}} \cos \frac{2\pi av}{k},$$

where, as usual, $0!$ denotes 1. Of course, many of the terms in U_1 and U_2 can be neglected if we are neglecting U_x for $x \geqslant 3$, since they contain a factor $e^{-\gamma\xi(3)}$ or smaller.

For the smaller values of j, the value of $e^{-\gamma}$ is very small and, since U_2 contains a factor $e^{-2\gamma}$, it is only for fairly large k that U_2 matters at all. We have thus

$$C_0(2, j, a) = 1 + 2e^{-\frac{1}{2}\gamma} \cos \pi a,$$
$$C_0(3, j, a) = 1 + 6e^{-2\gamma/3} \cos(2\pi a/3),$$
$$C_0(4, j, a) = 1 + 8e^{-3\gamma/4} \cos \tfrac{1}{2}\pi a + 6e^{-\gamma} \cos \pi a,$$

with an error in each case of the order of $e^{-2\gamma}$. For $j = 2$, $e^{-\gamma} = 4.29 \times 10^{-13}$ and for $j = 10$, $e^{-\gamma} = 1.893 \times 10^{-4}$, so that the error is very small. For values of j in excess of 23, the original form of $C_0(k, j, a)$ may be as easy to evaluate as that of Theorem 3, since $e^{-\frac{1}{2}\log j}$ is then less than $e^{-\gamma}$.

References

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