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B_h-SETS OF REAL AND COMPLEX NUMBERS

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Abstract. Let $K = \mathbf{R}$ or $\mathbf{C}$. An n-element subset A of K is a B_h-set if every element of w has at most one representation as the sum of h not necessarily distinct elements of A. Associated to the B_h set $A = \{a_1, \ldots, a_n\}$ are the B_h-vectors $\mathbf{a} = (a_1, \ldots, a_n)$ in K^n. This paper proves that "almost all" n-element subsets of K are B_h-sets in the sense that the set of all B_h-vectors is a dense open subset of K^n.

1. Sumsets and B_h-sets

Let A be a nonempty subset of an additive abelian group or semigroup G. For every positive integer h, the h-fold *sumset* of A is the set of all sums of h not necessarily distinct elements of A:

$$hA = \underbrace{A + \cdots + A}_{h \text{ summands}} = \{a'_1 + \cdots + a'_h : a'_i \in A \text{ for all } i \in [1,h]\}.$$

For integers $h \geq 2$ and $n \geq 2$, let

$$\mathcal{X}_{h,n} = \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{N}_0^n : \sum_{i=1}^n x_i = h \right\}.$$

For all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}_{h,n}$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{X}_{h,n}$, we have

$$\|\mathbf{x} - \mathbf{y}\|_\infty = \max\{|x_i - y_i| : i = 1, \ldots, n\} \leq h. \tag{1}$$

Let $\mathbf{a} = (a_1, \ldots, a_n) \in G^n$ be a vector with distinct coordinates. Associated to the vector $\mathbf{a}$ is the subset $A = \{a_1, \ldots, a_n\}$ of G of cardinality n. For all $\mathbf{x} \in \mathcal{X}_{h,n}$ and $\mathbf{a} \in G^n$, we define the dot product

$$\mathbf{x} \cdot \mathbf{a} = \sum_{i=1}^n x_i a_i.$$

Then $\mathbf{x} \cdot \mathbf{a} \in G$. The h-fold sumset of A can be written in the form

$$hA = \{\mathbf{x} \cdot \mathbf{a} : \mathbf{x} \in \mathcal{X}_{h,n}\}.$$

The subset A of G is called a *B_h-set* if every element of G has at most one representation (up to permutation of the summands) as the sum of h not necessarily distinct elements of A. B_2-sets are also called *Sidon sets*. The study of B_h-sets of integers is a classical topic in combinatorial additive number theory (cf. [1]–[19]).

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This paper studies B_h-sets of real and complex numbers. Let $K = \mathbf{R}$ or $\mathbf{C}$. Associated to every vector $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$ with distinct coordinates is the subset $A = \{a_1, \ldots, a_n\}$ of K of cardinality n. The vector $\mathbf{a}$ is called a B_h-*vector* if its associated set A is a B_h-set. Let $\mathcal{B}_h$ be the set of B_h-vectors. If $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathcal{B}_h$, then $\sigma\mathbf{a} = \left(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}\right) \in \mathcal{B}_h$ for every permutation σ of $\{1, \ldots, n\}$.

The object of this paper is to prove that "almost all" n-element subsets of K are B_h-sets in the sense that the set $\mathcal{B}_h$ is an open and dense subset of K^n.

2. Open and dense subsets

Theorem 1. *Let $K = \mathbf{R}$ or $\mathbf{C}$. The set $\mathcal{B}_h$ is an open subset of K^n.*

Proof. Let $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$ be a B_h-vector. If $\mathbf{x}, \mathbf{y} \in \mathcal{X}_{h,n}$ with $\mathbf{x} \neq \mathbf{y}$, then $\mathbf{x} \cdot \mathbf{a} \neq \mathbf{y} \cdot \mathbf{a}$ and so

$$0 < \Delta = \min\left\{\|(\mathbf{x} - \mathbf{y}) \cdot \mathbf{a}\|_\infty : (\mathbf{x}, \mathbf{y}) \in \mathcal{X}_{h,n}^2 \text{ and } \mathbf{x} \neq \mathbf{y}\right\}.$$

We shall prove that, for all vectors $\mathbf{b} \in K^n$ with

$$0 < \|\mathbf{b}\|_\infty < \frac{\Delta}{h}$$

the vector

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$

is a B_h-vector and so the set of B_h-vectors contains the open ball with center at $\mathbf{a}$ and radius Δ/h.

Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}_{h,n}$ with $\mathbf{x} \neq \mathbf{y}$. If $\mathbf{x} \cdot \mathbf{c} = \mathbf{y} \cdot \mathbf{c}$, then

$$\mathbf{x} \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{y} \cdot (\mathbf{a} + \mathbf{b})$$

and

$$(\mathbf{x} - \mathbf{y}) \cdot \mathbf{a} = (\mathbf{y} - \mathbf{x}) \cdot \mathbf{b}$$

Applying inequality (1), we obtain

$$\begin{aligned} \Delta &\leq \|(\mathbf{x} - \mathbf{y}) \cdot \mathbf{a}\|_\infty = \|(\mathbf{y} - \mathbf{x}) \cdot \mathbf{b}\|_\infty \\ &\leq \|\mathbf{y} - \mathbf{x}\|_\infty \|\mathbf{b}\|_\infty \\ &< h\left(\frac{\Delta}{h}\right) = \Delta \end{aligned}$$

which is absurd. Therefore, $\mathbf{x} \neq \mathbf{y}$ implies $\mathbf{x} \cdot \mathbf{c} \neq \mathbf{y} \cdot \mathbf{c}$ and so $\mathbf{c}$ is a B_h-vector. This completes the proof. □

Lemma 1. *For all $\delta > 0$ there is a B_h-vector $\mathbf{b}$ such that $\|\mathbf{b}\|_\infty < \delta$.*

Proof. If $\mathbf{w} = (w_1, \ldots, w_n)$ is any B_h-vector in $\mathbf{R}^n$ or $\mathbf{C}^n$ with associated B_h-set $W = \{w_1, \ldots, w_n\}$, then, for every $\lambda \neq 0$, the "contraction" $\lambda * W = \{\lambda w_i : i = 1, \ldots, n\}$ is a B_h-set and the corresponding vector $\lambda\mathbf{w} = (\lambda w_1, \ldots, \lambda w_n)$ is also a B_h-vector. Choosing $0 < \lambda < \delta/\|\mathbf{w}\|_\infty$ and $\mathbf{b} = \lambda\mathbf{w}$ gives

$$\|\mathbf{b}\|_\infty = \|\lambda\mathbf{w}\|_\infty = |\lambda|\|\mathbf{w}\|_\infty < \left(\frac{\delta}{\|\mathbf{w}\|_\infty}\right)\|\mathbf{w}\|_\infty = \delta.$$

This completes the proof. □

Theorem 2. *Let $K = \mathbf{R}$ or $\mathbf{C}$. The set $\mathcal{B}_h$ is a dense subset of K^n.*

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Proof. Let $\mathbf{a} \in K^n$ be a vector that is not a B_h-vector. We shall prove that, for every $\varepsilon > 0$, there is a B_h-vector $\mathbf{c} \in K^n$ such that $\|\mathbf{c} - \mathbf{a}\|_\infty < \varepsilon$.

We partition the set of pairs of distinct vectors in $\mathcal{X}_{h,n}$ into two disjoint finite sets as follows:

$$\mathcal{U} = \{(\mathbf{x}, \mathbf{y}) \in \mathcal{X}_{h,n}^2 : \mathbf{x} \neq \mathbf{y} \text{ and } \mathbf{x} \cdot \mathbf{a} \neq \mathbf{y} \cdot \mathbf{a}\}$$

and

$$\mathcal{V} = \{(\mathbf{x}, \mathbf{y}) \in \mathcal{X}_{h,n}^2 : \mathbf{x} \neq \mathbf{y} \text{ and } \mathbf{x} \cdot \mathbf{a} = \mathbf{y} \cdot \mathbf{a}\}.$$

The set $\mathcal{V}$ is nonempty because $\mathbf{a}$ is not a B_h-vector. The set $\mathcal{U}$ is nonempty because $n \geq 2$ and $(\mathbf{x}, \mathbf{y}) \in \mathcal{U}$ with $\mathbf{x} = (h, 0, 0, \ldots, 0)$ and $\mathbf{y} = (0, h, 0, \ldots, 0)$. Then

$$0 < \Delta = \min\left\{\|(\mathbf{x} - \mathbf{y}) \cdot \mathbf{a}\|_\infty : (\mathbf{x}, \mathbf{y}) \in \mathcal{U}\right\}.$$

By Lemma 1, for all $\varepsilon > 0$ there is a B_h-vector $\mathbf{b}$ in K^n such that

$$\|\mathbf{b}\|_\infty < \min\left(\varepsilon, \frac{\Delta}{h}\right).$$

Applying inequality (1), for all pairs $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}_{h,n}^2$, we have

$$\|(\mathbf{x} - \mathbf{y}) \cdot \mathbf{b}\|_\infty \leq \|\mathbf{x} - \mathbf{y}\|_\infty \|\mathbf{b}\|_\infty < h\left(\frac{\Delta}{h}\right) = \Delta.$$

The vector

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$

satisfies

$$\|\mathbf{c} - \mathbf{a}\|_\infty = \|\mathbf{b}\|_\infty < \varepsilon.$$

We shall prove that $\mathbf{c}$ is a B_h-vector. Equivalently, we shall prove that $\mathbf{x} \cdot \mathbf{c} \neq \mathbf{y} \cdot \mathbf{c}$ for all pairs $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}_{h,n}^2$ with $\mathbf{x} \neq \mathbf{y}$.

If $(\mathbf{x}, \mathbf{y}) \in \mathcal{U}$, then $\mathbf{x} \cdot \mathbf{a} \neq \mathbf{y} \cdot \mathbf{a}$ and

$$\begin{aligned}
\|\mathbf{x} \cdot \mathbf{c} - \mathbf{y} \cdot \mathbf{c}\|_\infty &= \|(\mathbf{x} \cdot \mathbf{a} - \mathbf{y} \cdot \mathbf{a}) + (\mathbf{x} \cdot \mathbf{b} - \mathbf{y} \cdot \mathbf{b})\|_\infty \\
&\geq \|(\mathbf{x} - \mathbf{y}) \cdot \mathbf{a}\|_\infty - \|(\mathbf{x} - \mathbf{y}) \cdot \mathbf{b}\|_\infty \\
&> \Delta - \Delta = 0.
\end{aligned}$$

If $(\mathbf{x}, \mathbf{y}) \in \mathcal{V}$, then $\mathbf{x} \cdot \mathbf{a} = \mathbf{y} \cdot \mathbf{a}$. Because $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{b}$ is a B_h-vector, we have $\mathbf{x} \cdot \mathbf{b} \neq \mathbf{y} \cdot \mathbf{b}$. It follows that

$$\begin{aligned}
\|\mathbf{x} \cdot \mathbf{c} - \mathbf{y} \cdot \mathbf{c}\|_\infty &= \|(\mathbf{x} \cdot \mathbf{a} - \mathbf{y} \cdot \mathbf{a}) + (\mathbf{x} \cdot \mathbf{b} - \mathbf{y} \cdot \mathbf{b})\|_\infty \\
&= \|\mathbf{x} \cdot \mathbf{b} - \mathbf{y} \cdot \mathbf{b}\|_\infty \\
&> 0.
\end{aligned}$$

This completes the proof. □

Combining Theorems 1 and 2 gives the following result.

Theorem 3. *Let $K = \mathbf{R}$ or $\mathbf{C}$. The set $\mathcal{B}_h$ is a dense open subset of K^n.*

Theorem 4. *Let $\mathcal{B}_h$ be the set of B_h-vectors in K^n. The set*

$$\mathcal{B}_\infty = \bigcap_{h=1}^{\infty} \mathcal{B}_h$$

is a dense subset of K^n.

Proof. This is simply an application of Baire's theorem in functional analysis. □

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3. $B_h[g]$-SETS

Let g and h be positive integers. The subset A of an additive abelian semigroup G is called a $B_h[g]$*-set* if every element of G has at most g representations as the sum of h not necessarily distinct elements of A. The $B_2[1]$-sets are the B_2-sets. If A is a $B_h[g]$-set, then A is also a $B_h[g']$-set for all $g' \geq g$. In particular, every B_h-set is a $B_h[g]$-set.

Let $K = \mathbf{R}$ or $\mathbf{C}$. The vector $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$ is a $B_h[g]$*-vector* if the set $A = \{a_1, \ldots, a_n\}$ is an n-element $B_h[g]$-set in K. Every B_h-vector is a $B_h[g]$-vector. Because the set of B_h-vectors is dense in K^n, it follows that the set of $B_h[g]$-vectors is also dense in K^n.

Problem. *Is the set of $B_h[g]$-vectors also open in K^n?*

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