# SUBNORMAL SUBGROUPS OF DIVISION RINGS

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Let $K$ be a division ring. A subgroup $H$ of the multiplicative group $K'$ of $K$ is subnormal if there is a finite sequence $(H = A_0, A_1, \ldots, A_n = K')$ of subgroups of $K'$ such that each $A_i$ is a normal subgroup of $A_{i+1}$. It is known **(2, 3)** that if $H$ is a subdivision ring of $K$ such that $H'$ is subnormal in $K'$, then either $H = K$ or $H$ is in the centre $Z(K)$ of $K$. This leads to the following conjecture:

$P_{nD}$: *If $K$ is a division ring, $H$ a subdivision ring invariant under a subgroup $G_1, G_1 \lhd G_2 \lhd \ldots \lhd G_n = K'$, $G_1 \not\subset Z(K)$, then $H = K$ or $H \subset Z(K)$.*

This conjecture will be proved for $n = 2$ (the case $n = 1$ is the Cartan-Brauer-Hua theorem). Let $P_{nF}$ be the corresponding conjecture when $H$ is a subfield of $K$. It will be shown that $P_{nD}$ implies $P_{n+1,F}$, and that $P_{2D}$ is true. It follows that $P_{3F}$ is true. To prove the general conjecture, it remains only to show that $P_{nF}$ implies $P_{nD}$. In connection with the conjecture, one might even ask if any subnormal subgroup of $K'$ must be normal in $K'$.

The following notation will be used. If $K$ is a division ring, then $K'$ will denote its multiplicative subgroup. If $S$ is a subset of $K$, $C(S)$ will mean the centralizer of $S$ and $\bar{S}$ the subdivision ring generated by $S$. If $x$ and $y$ are non-zero elements of $K$, $[x, y] = xyx^{-1}y^{-1}$. If $F$ is a subfield of $K$ and $M$ a subdivision ring of $K$ containing $F$, then $[M : F]$ is the degree of $M$ over $F$. If $y \in K$ and $S$ is a subset of $K$, then $S^y = y^{-1}Sy$.

The following lemma follows immediately from Lemmas 1 and 2 of **(1)**.

LEMMA 1. *If $x \in K$, $y \in K$, $[y, x]$ commutes with both $x$ and $y$, $[y, x] \neq 1$, and $[y, [y, \ldots [y, 1 + x] \ldots]] = 1$, then $x$ is algebraic over $Z(K)$.*

A group is weakly nilpotent if any two of its elements generate a nilpotent subgroup. Huzurbazar **(1)** proved that $K'/Z(K)'$ has no weakly nilpotent normal subgroups, and every weakly nilpotent normal subgroup of $K'$ is in the centre. A minor remark permits the replacement of the word "normal" by "non-abelian subnormal" in this theorem. For convenience the remark will be formulated as a lemma.

LEMMA 2. *If $A_1 \lhd A_2 \lhd \ldots \lhd A_n = K'$, $x \in A_1$, $y \in K'$, $y_1 = [x, y]$, $y_{i+1} = [x, y_i]$, then $y_{n-1} \in A_1$.*

*Proof.* $y_1 \in A_{n-1}$ since $x \in A_{n-1} \lhd K'$. It follows by induction that $y_i \in A_{n-i}$.

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By using Lemma 2 at the appropriate places in Huzurbazar's proof (**1**, Theorem 1), the following theorem may be proved.

THEOREM 1. *If $K$ is a division ring with centre $Z$, then neither $K'$ nor $K'/Z'$ has any weakly nilpotent non-abelian subnormal subgroups.*

THEOREM 2. $P_{nD}$ *implies* $P_{n+1,F}$.

*Proof*. Deny the theorem. Let $K$ be a division ring, $H$ a subfield invariant under $G_1$, $G_1 \lhd G_2 \lhd \ldots \lhd G_{n+1} = K'$, $G_1 \not\subset Z(K)$, and $H \not\subset Z(K)$.

If $G_1 \subset C(H)$, then $\overline{G_1} \subset C(H)$. Thus $\overline{G_1}$ is a subdivision ring invariant under $G_2$. By $P_{nD}$, $\overline{G_1} = K$. Hence $C(H) = K$ and $H \subset Z(K)$. Therefore $G_1 \not\subset C(H)$.

*Case* 1. There is an $x \in H$ such that $x \notin Z(K)$ and $x$ is algebraic over $Z(K)$. Let $x_1, \ldots, x_n$ be the conjugates of $x$ in $H$. Then $Z(K)(x_1, \ldots, x_n$ is a field invariant under $G_1$ and not contained in $Z(K)$. Hence $K$, $H$, $G_1, \ldots, G_n$ may be assumed to be such that $[H : H \cap Z(K)]$ is finite and as small as possible.

Suppose that there are $y \in G_1$, $y \notin C(H)$, and $a \in H$, $a \notin Z(K)$, such that $[y, a] = 1$. Then the minimality of $[H : H \cap Z(K)]$ is contradicted, for $C(a)$ is a division ring, $H$ a subfield invariant under $G_1 \cap C(a)$, $G_i \cap C(a)$ is a normal subgroup of $G_{i+1} \cap C(a)$, $G_1 \cap C(a) \not\subset Z(C(a))$ since $y$ is in the former group but not the latter, $H \not\subset Z(C(a))$, and since $a \in Z(C(a)) \cap H$, $1 < [H : H \cap Z(C(a))] < [H : H \cap Z(K)]$.

Thus $G_1/G_1 \cap C(H)$ is isomorphic to a non-trivial group of automorphisms of $H$ over $H \cap Z(K)$ such that the fixed field of any automorphism ($\neq 1$) is $H \cap Z(K)$. It follows that $G_1/G_1 \cap C(H)$ and each of its non-trivial subgroups is the full Galois group of $H/H \cap Z(K)$. Therefore $G_1/G_1 \cap C(H)$ is of prime order. Hence the commutator subgroup $Q$ of $G_1$ is in $C(H)$. But $Q$ is normal in $G_2$, so $\bar{Q}$ is invariant under $G_2$. By $P_{nD}$, either $\bar{Q} = K$ or $Q \subset Z(K)$. If $\bar{Q} = K$, then $C(H) = K$, which is impossible. Hence $Q \subset Z(K)$. Therefore, $G_1$ is nilpotent. By Theorem 1, $G_1$ is abelian. Therefore $\overline{G_1}$ is a field invariant under $G_2$. Since $\overline{G_1} \neq K$, this contradicts $P_{nD}$.

*Case* 2. If $x \in H$ and $x \notin Z(K)$, then $x$ is transcendental over $Z(K)$.

First suppose that $H \cap G_1 \subset Z(K)$. Since $G_1 \not\subset C(H)$, there are $x \in G_1$ and $y \in H$ such that $[x, y] = a \neq 1$. Using the notation and result of Lemma 2, $y_n \in G_1$ and it is clear that each $y_i \in H$ since $H$ is invariant under $x$. Therefore $y_n \in G_1 \cap H \subset Z(K)$, $y_{n+1} = 1$. Therefore there is $u \in H$ ($y$ or an appropriate $y_i$) such that $[x, u] = b \neq 1$, $[x, b] = 1$, and $[u, b] = 1$ (this last because both $u$ and $b$ are in $H$). Clearly $(1 + u)_{n+1} = 1$ also. By Lemma 1, $u$ is algebraic over $Z(K)$, a contradiction.

Hence $H \cap G_1 \not\subset Z(K)$. If $(H \cap G_1)^u \subset C(H)$ for all $u \in G_2$, then the division ring $L$ generated by all $(H \cap G_1)^u$ with $u \in G_2$ is invariant under $G_2$, contradicting $P_{nD}$. Hence, for some $u \in G_2$, $(H \cap G_1)^u \not\subset C(H)$. Let $y \in (H \cap G_1)^u$, $y \notin C(H)$. For some $v \in H$, $[y, v] \neq 1$. Then $v_n \in H \cap G_1$ by

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Lemma 2, so $v_{n+1} \in H \cap G_1 \cap H^u$ since $H^u$ is invariant under $G_1{}^u = G_1$. Therefore $v_{n+2} = 1$ since $H^u$ is commutative. As in the preceding paragraph, this leads to a contradiction.

COROLLARY. *If $K$ is a division ring, $H$ a subfield invariant under a normal subgroup $G$ of $K'$, $G \not\subset Z(K)$, then $H \subset Z(K)$.*

*Proof.* $P_{1D}$ is the Cartan-Brauer-Hua theorem. By Theorem 2, $P_{2F}$ is true. But this is just the statement of the Corollary.

THEOREM 3. *If $K$ is a division ring, $H$ a subdivision ring invariant under a normal subgroup $G$ of $K'$, $G \not\subset Z(K)$, then either $H = K$ or $H \subset Z(K)$.*

*Proof.* Deny the assertion. We assert

(1) $$\text{If } h \in H,\ h \notin Z(K), \text{ then } C(h) \subset H.$$

*Subproof.* Deny the assertion. For some $y \notin H$, $yh = hy$. If $g \in G$, since $g^{1+y} \in G$, for some $h_1 \in H$,

$$(1 + y)g(1 + y)^{-1}h = h_1(1 + y)g(1 + y)^{-1},$$

and so

$$(1 + y)gh = h_1(1 + y)g.$$

Also, for some $h_2 \in H$,

$$ygy^{-1}h = h_2ygy^{-1},$$

and so

$$ygh = h_2yg.$$

Subtraction gives $gh - h_1g = (h_1 - h_2)yg$,

$$(ghg^{-1} - h_1) = (h_1 - h_2)y.$$

Since $y \notin H$, $h_1 = h_2 = ghg^{-1}$. Hence $ygh = h_2yg = ghg^{-1}yg$, or

$$y(ghg^{-1}) = (ghg^{-1})y.$$

Thus $y$ commutes with all elements of the form $ghg^{-1}$, $g \in G$. Since $yh \notin H$ and $yh \in C(h)$, $yh$ also commutes with all $ghg^{-1}$. Hence $h$ commutes with all $ghg^{-1}$. It follows by conjugation that any two conjugates of $h$ by elements of $G$ commute. Therefore these conjugates generate a field $F$ invariant under $G$. By the preceding corollary, $F \subset Z(K)$, a contradiction since $h \in F$. This proves (1).

Now $C(H) \subset H$ by (1), so $C(H)$ is a subfield invariant under $G$. By the corollary, $C(H) \subset Z(K)$. Thus $Z(H) = Z(K)$.

Suppose that $h \in H$, $h \notin Z(K)$, and that $h$ is algebraic over $Z(K)$. It is clear that $G \not\subset H$. Let $g \in G$, $g \notin H$. Then the fields $Z(K)(h)$ and $g^{-1}(Z(K)(h))g$ are isomorphic by an isomorphism leaving $Z(H)$ fixed. Hence **(4**, page 162**)** there is an $a \in H$ such that $a$ induces the same isomorphism. But then

$ag^{-1} \in C(h)$, so by (1), $ag^{-1} \in H$. Therefore, $g \in H$, a contradiction. Thus every element of $H$ outside $Z(K)$ is transcendental over $Z(K)$.

We assert that

$$(2) \qquad \text{if } y \in G, y \notin H, \qquad \text{then } H \cap H^{1+y} = Z(K).$$

For suppose this to be false. Then there are $h$, $h_1$, and $h_2$ in $H$ but not $Z(K)$ such that $(1 + y)h = h_1(1 + y)$ and $yh = h_2 y$. Therefore $h - h_1 = (h_1 - h_2)y$. Since $y \notin H$, $h = h_1 = h_2$, so $y \in C(h)$ in contradiction to (1).

Suppose $G \cap H \subset Z(K)$. If $y \in G$, $y \notin Z(K)$, $x \in H$, $x \notin Z(K)$, then $[y, x] = a \in Z(K)$, $a \neq 1$ by (1). Therefore $[y, [y, 1 + x]] = 1$, and $x$ is algebraic over $Z(K)$ by Lemma 1. Hence $G \cap H \not\subset Z(K)$.

For all $u \in K$, $G \cap H^u \not\subset Z(K)$. Let $u \in G$, $u \notin H$. There is an element $y \in G \cap H^{1+u}$, $y \notin Z(K)$. Since $C(H) = Z(K)$, there is $v \in H$ such that $[y, v] \neq 1$. Hence $[y, v] \in H \cap G$, $[y, [y, v]] \in H \cap H^{1+u} \subset Z(K)$ by (2). Therefore $[y, [y, [y, v]]] = 1$. We assert that there is $x \in H$ such that $[y, x] = a \neq 1$ and $a$ commutes with both $y$ and $x$. In fact, if $[y, [y, v]] \neq 1$, then $x = [y, v]$ will do. If $[y, [y, v]] = 1$, then $[y, v] \in C(y) \subset H^{1+u}$ by (1), so $[y, v] \in H \cap H^{1+u} \subset Z(K)$. Hence $x = v$ will do in this case, and such an $x$ always exists. Then, as before, $[y, [y, [y, 1 + x]]] = 1$. Hence $x$ is algebraic over $Z(K)$ by Lemma 1.

COROLLARY. *If $K$ is a division ring, $H$ a subfield invariant under $G$, $G \lhd L \lhd K'$, $G \not\subset Z(K)$, then $H \subset Z(K)$.*

*Proof.* This follows from Theorems 2 and 3.

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