H. Ono
Nagoya Math. J.
Vol. 38 (1970), 63–69

SOME REMARKS ON EVALUATIONS OF THE PRIMITIVE LOGIC

HIROAKIRA ONO

Dedicated to Prof. K. Ono on his 60th birthday

In [4], K. Ono introduced the notion of *evaluations* of the primitive logic $\boldsymbol{LO}$ and proved that any *semi-evaluation* $\boldsymbol{E}$ is an evaluation of $\boldsymbol{LO}$ if $\boldsymbol{E}$ satisfies the following conditions:

$(E1)$ $p^* \longrightarrow 0 = 0,$

$(E2)$ $p^* \longrightarrow p^* = 0,$

$(E3)$ $0 \longrightarrow p^* = p^*,$

$(E4)$ $p^* \longrightarrow (p^* \longrightarrow q^*) = p^* \longrightarrow q^*,$

$(E5)$ $p^* \longrightarrow (q^* \longrightarrow r^*) = q^* \longrightarrow (p^* \longrightarrow r^*),$

$(E6)$ $p^* \longrightarrow q^* = 0$ *implies*

$$(r^* \longrightarrow p^*) \longrightarrow (r^* \longrightarrow q^*) = 0,$$

$(E7)$ $(x)p^*(x) \longrightarrow p^*(t) = 0$ *for any* t, *and*

$(E8)$ *if* $u^* \longrightarrow p^*(t) = 0$ *for any* t, *then*

$$u^* \longrightarrow (x)p^*(x) = 0.$$

That is, if $\boldsymbol{A}$ is provable in $\boldsymbol{LO}$, then for any semi-evaluation $\boldsymbol{E}$ satisfying the above conditions, $\boldsymbol{E}(\boldsymbol{A}) = 0$ holds identically. In this paper, we will show that in §1, *the condition* $(E8)$ *is so weak that we can not prove the above result and hence* $(E8)$ *must be replaced by the following condition* $(E8^*)$;

$(E8^*)$ *if* $u^* \longrightarrow (v^* \longrightarrow p^*(t)) = 0$ *for any* t, *then*

$$u^* \longrightarrow (v^* \longrightarrow (x)p^*(x)) = 0,$$

and that in §2, *converse of his result can be proved if* $(E8)$ *is replaced by* $(E8^*)$.

§1. We first define a *model* $\boldsymbol{D}$ of $\boldsymbol{LO}$, after the definition of evaluations. Let $\boldsymbol{D}$ be an algebraic structure $\langle D, 0, \longrightarrow, V \rangle$, where D and V are sets,

Received March 29, 1969

https://doi.org/10.1017/S0027763000013520 Published online by Cambridge University Press

$0 \in D$ and $\longrightarrow$ is a function from D^2 to D. If $\boldsymbol{D}$ satisfies the following conditions, we say that $\boldsymbol{D}$ is a *model* of $\boldsymbol{LO}$.

For each p, q and $r \in D$,

(D1) $p \longrightarrow 0 = 0$,

(D2) $p \longrightarrow p = 0$,

(D3) $0 \longrightarrow p = p$,

(D4) $p \longrightarrow (p \longrightarrow q) = p \longrightarrow q$,

(D5) $p \longrightarrow (q \longrightarrow r) = q \longrightarrow (p \longrightarrow r)$,

(D6) *if* $p \longrightarrow q = 0$ *then* $(r \longrightarrow p) \longrightarrow (r \longrightarrow q) = 0$.

Suppose that $a_t \in D$ for any $t \in V$. Then there exists an element $top\,[a_t \mid t \in V]$ in D which satisfies the following conditions[1].

(D7) $top\,[a_t \mid t \in V] \longrightarrow a_t = 0$ *for any* $t \in V$,

(D8) *if* $p \longrightarrow (q \longrightarrow a_t) = 0$ *for any* $t \in V$,

then $p \longrightarrow (q \longrightarrow top\,[a_t \mid t \in V] = 0$.

Let $\boldsymbol{D}$ be a model of $\boldsymbol{LO}$ and φ be a mapping from the class of primitive symbols of $\boldsymbol{LO}$ such that $\varphi(v) \in V$ for each variable or constant v and if p is an n-ary function symbol then $\varphi(p)$ is a mapping from V^n to D. Then we say that this φ is *an assignment over* $\boldsymbol{D}$. Although φ is defined only for primitive symbols, we can extend the domain of the mapping φ to the class of formulas in a natural way. That is,

$\varphi(p(t_1, \cdots, t_m)) = \varphi(p)[\varphi(t_1), \cdots, \varphi(t_m)]$ where right side of the equality means the value of $\varphi(p)$ for the m-tuple $[\varphi(t_1), \cdots, \varphi(t_m)]$,

$$\varphi(P \longrightarrow Q) = \varphi(P) \longrightarrow \varphi(Q) \text{ and}$$

$$\varphi((x)P(x)) = top\,[\varphi(P(x)) \mid x \in V].$$

$\boldsymbol{A}$ is *valid* in a model $\boldsymbol{D}$ if $\varphi(\boldsymbol{A}) = 0$ for any assignment φ over $\boldsymbol{D}$. Now we can prove the following lemma easily.

LEMMA 1. *For any semi-evaluation $\boldsymbol{E}$ satisfying from (E1) to (E7) and (E8*), there exists a model of $\boldsymbol{LO}$ and an assignment φ over $\boldsymbol{D}$ such that $\varphi(p) = \boldsymbol{E}(p)$ for any function symbol p. Conversely, for each model $\boldsymbol{D}$ of $\boldsymbol{LO}$ and each assignment φ over $\boldsymbol{D}$, there exists a semi-evaluation $\boldsymbol{E}$ such that $\varphi(p) = \boldsymbol{E}(p)$.*

[1] The word "top" is due to Henkin. See Henkin [2].

Notice that we can also prove Lemma 1 when we replace $(E8^*)$ by $(E8)$ and $(D8)$ by $(D8')$ defined as follows, in Lemma 1.

$(D8')$ *If* $p \longrightarrow a_t = 0$ *for any* $t \in V$, *then*
$$p \longrightarrow top\,[a_t \mid t \in V] = 0.$$

In the proof of Theorem 2 in [4], K. Ono asserted that it can be proved by using $(E8)$ that

if $p_1 \longrightarrow (p_2 \longrightarrow (\cdots (p_n \longrightarrow q(t)) \cdots)) = 0$ *for any* t,
then $p_1 \longrightarrow (p_2 \longrightarrow (\cdots (p_n \longrightarrow (x)q(x)) \cdots)) = 0$ (1).

However we can prove this is not the case, by constructing a structure which satisfies the conditions from $(D1)$ to $(D7)$ and $(D8')$, but in which (1) does not hold. We construct a structure $\boldsymbol{B} = \langle B, 0, \longrightarrow, V \rangle$ as follows. B is a Boolean lattice whose cardinality is 4 (see Fig. 1) and $V = \{v_1, v_2\}$. 0 is a minimal element of B. Define the value of $p \longrightarrow q$ and of $top[p_t \mid t \in V]$ as follows.

$$p \longrightarrow q = \begin{cases} 0 & \textit{if } p \geqq q \\ q & \textit{otherwise} \end{cases}$$

$top\,[p_t \mid t \in V] = p_{v_1} \cup p_{v_2}$ i.e., union of p_{v_1} and p_{v_2}

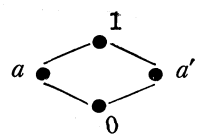

Figure 1

We can see easily that $\boldsymbol{B}$ satisfies the conditions from $(D1)$ to $(D7)$ and $(D8')$. Now suppose that an assignment φ over $\boldsymbol{B}$ is defined as follows.

$\varphi(x_1) = v_1$, $\varphi(x_i) = v_2$ if $x_i \neq x_1$, $\varphi(p_1) = a$, $\varphi(p_2) = a'$, and $\varphi(q) = f$ where $f(v_1) = a$ and $f(v_2) = a'$. Then we can prove the following lemma.

Lemma 2. $\varphi(p_1 \longrightarrow (p_2 \longrightarrow q(x))) = 0$ *for any* x, *but* $\varphi(p_1 \longrightarrow (p_2 \longrightarrow (x)q(x))) \neq 0$. *Therefore* (1) *can not be deduced from the conditions from* $(E1)$ *to* $(E8)$.

Now we will show that the condition $(E8^*)$ is sufficient to prove (1).

https://doi.org/10.1017/S0027763000013520 Published online by Cambridge University Press

LEMMA 3. (1) *is deducible from the conditions from* $(E1)$ *to* $(E7)$ *and* $(E8^*)$.

Proof. We use $[p_1, \cdots, p_n] \text{—}*\!\!\to q$ as an abbreviation for $p_1 \longrightarrow (p_2 \longrightarrow (\cdots (p_n \longrightarrow q) \cdots))$. More precisely, we define $[p_1, \cdots, p_n] \text{—}*\!\!\to q$ inductively as follows. For $n = 0$, $([p_1, \cdots, p_n] \text{—}*\!\!\to q) = q$ and for $n > 0$, $([p_1, \cdots, p_n] \text{—}*\!\!\to q) = (p_1 \longrightarrow ([p_2, \cdots, p_n] \text{—}*\!\!\to q))$.

We shall first show that

$$(x)([p_1, \cdots, p_m] \text{—}*\!\!\to q(x)) \longrightarrow ([p_1, \cdots, p_m] \text{—}*\!\!\to (x)q(x)) = 0 \qquad (2)$$

by using induction on m.
For $m = 0$, the left side of (2) is equal to $(x)q(x) \longrightarrow (x)q(x)$. Then (2) is deducible from $(E2)$.
Suppose that $m > 0$. By $(E7)$,

$$(x)(p_1 \longrightarrow ([p_2, \cdots, p_m] \text{—}*\!\!\to q(x))) \longrightarrow (p_1 \longrightarrow ([p_2, \cdots, p_m] \text{—}*\!\!\to q(t))) = 0$$

for any t.

Taking $(x)(p_1 \longrightarrow ([p_2, \cdots, p_m] \text{—}*\!\!\to q(x)))$ as u^*, p_1 as v^* and $[p_2, \cdots, p_m] \text{—}*\!\!\to q(t)$ as $p^*(t)$, and using $(E8^*)$, we have

$$(x)(p_1 \longrightarrow ([p_2, \cdots, p_m] \text{—}*\!\!\to q(x))) \longrightarrow (p_1 \longrightarrow (x)([p_2, \cdots, p_m] \text{—}*\!\!\to q(x))) = 0 \qquad (3).$$

By induction hypothesis,

$$(x)([p_2, \cdots, p_m] \text{—}*\!\!\to q(x)) \longrightarrow ([p_2, \cdots, p_m] \text{—}*\!\!\to (x)q(x)) = 0.$$

From $(E3)$, $(E6)$ and (3), we get

$$(x)([p_1, \cdots, p_m] \text{—}*\!\!\to q(x)) \longrightarrow ([p_1, \cdots, p_m] \text{—}*\!\!\to (x)q(x)) = 0.$$

Now, since $[p_1, \cdots, p_n] \text{—}*\!\!\to q(t) = 0$ for any t, $p_1 \longrightarrow (x)([p_2, \cdots, p_n] \text{—}*\!\!\to q(x)) = 0$ from $(E8^*)$. For, $(E8^*)$ implies $(E8)$. Using (2) and $(E6)$,

$$(p_1 \longrightarrow (x)([p_2, \cdots, p_n] \text{—}*\!\!\to q(x))) \longrightarrow ([p_1, \cdots, p_n] \text{—}*\!\!\to (x)q(x))$$
$$= 0 \longrightarrow ([p_1, \cdots, p_n] \text{—}*\!\!\to (x)q(x)) = 0.$$

Thus we have $[p_1, \cdots, p_n] \text{—}*\!\!\to (x)q(x) = 0$.

Now, we get the following theorem after K. Ono's proof of Theorem 2 in [4].

THEOREM 1. *If* $\boldsymbol{A}$ *is provable in* $\boldsymbol{LO}$, *then* $\boldsymbol{A}$ *is valid in any model of* $\boldsymbol{LO}$.

§2. We will prove the converse of Theorem 1. Suppose that A is valid in any model of LO. Since any *Brouwerian algebra* satisfies the conditions from ($D1$) to ($D8$), A is valid in any Brouwerian algebra[2]. Rasiowa proved in [6] that for any formula B in *the intuitionistic logic* LJ, if B is valid in any Brouwerian algebra then B is provable in LJ. Thus A is provable in LJ. Moreover A is a formula of LO, and hence A is provable in LO by using Gentzen's Hauptsatz[3].

THEOREM 2. *If A is valid in any model of LO, then A is provable in LO*[4].

K. Ono gave me a preprint of his new paper [5], in which he defines the *set-theoretical* (or *topological*) *interpretation* of LO. Also in this paper, the condition ($E8$) should be replaced by ($E8^*$). We will discuss the matter in §3.

§3. We must revise the conditions which make any pair of topologies ($\{T\}$, $[T]$) *logical* as follows.

DEFINITION. *Any pair of topologies* ($\{T\}$, $[T]$) *is logical if and only if* "$\longrightarrow$" *and* "(x)" *defined in* [5] *satisfy* ($E5$) *and* ($E8^*$) *for every closed set p, q, r and a_t for any t with respect to the topology* $\{T\}$.

LEMMA 4. *Suppose that p, q and r are closed sets with respect to* $\{T\}$. *Then* $[[r-q]^r \cap (r-p)]^r = 0$ *if and only if* $[r-q] \cap (r-p) = 0$.

Proof. For any set a, $a \subset [a]$. Hence if $[[r-q]^r \cap (r-p)]^r = 0$, then $[r-q] \cap r \cap (r-p) \cap r = [r-q] \cap (r-p) = 0$. Conversely, if $[r-q] \cap (r-p) = [r-q]^r \cap (r-p) = 0$ then $[[r-q]^r \cap (r-p)]^r = [0]^r = 0$.

COROLLARY[5]. *Any pair of topologies* ($\{T\}$, $[T]$) *is logical if*

($T5$) $[[r-p] \cap (r-q)] = [(r-p) \cap [r-q]]$

and

($T7$) *if* $[a_t - q] \cap (a_t - p) = 0$ *for any t, then*

$$[\{\cup a_x\} - q] \cap (\{\cup a_x\} - p) = 0$$

---

[2] For the definition of Brouwerian algebras, see, e.g., Rasiowa [6].

[3] See, Curry [1].

[4] We can prove this theorem directly, by using Henkin's method in his [2]. See Also [7].

[5] Cf. Theorem 8 of [5]. In this corollary, $\cup a_x$ denotes the union of a class of sets a_t where t runs over a set V.

https://doi.org/10.1017/S0027763000013520 Published online by Cambridge University Press

holds for p, q, r *and* a_t *with respect to* $\{T\}$.

We can prove each example of pair of topologies given in (2. 1) and (2. 2) of [5] to be logical by the above corollary. In particular, the following lemma holds.

LEMMA 5[6] *Any pair* ($[T]$, $[T]$) *of identical topology* $[T]$ *is logical* (*in our sense*) *if* $[T]$ *satisfies the condition* ($T6$).

From this, we shall prove the following theorem.

THEOREM 3. *If* A *is identically equal to* 0 *for any topological interpretation of* LO, *then* A *is provable in* LO.

Proof. McKinsey and Tarski showed that for any Brouwerian algebra B there exists a topological space X such that B is isomorphic to $\mathscr{C}(X)$ where $\mathscr{C}(X)$ is the class of closed subsets of X[7]. On the other hand, in Rasiowa's proof, she used the fact that if B is valid in the *Lindenbaum algebra* B_h then B is provable in LJ[8]. Of course, B_h is a Brouwerian algebra. Hence there exists a topological space X_0 such that B_h is isomorphic to $\mathscr{C}(X_0)$. Clearly, the pair ($[T_0]$, $[T_0]$) of topology $[T_0]$ whose class of all closed sets is $\mathscr{C}(X_0)$ determines a topological interpretation L of LO by Lemma 5[9]. It follows that A is identically equal to 0 for the interpretation L and hence A is valid in B_h. Thus A is provable in LO.

COROLLARY. *Following propositions are equivalent. For any formula* A *of* LO,

1) A *is provable in* LO,

2) A *is valid in any model of* LO,

3) A *is valid in the model* B_h,

4) A *is identically equal to* 0 *for any topological interpretation of* LO.

Although we know certain relations hold between the logic LO and the model which satisfies from ($E1$) to ($E7$) and ($E8^*$), by the above corollary, another point of view is possible. What can be said about the relations between LO and the formal system whose axioms are propositions

6) Cf. Theorem 12 of [5]. ($T6$) $[a]\cup[b]=[a\cup b]$.

7) See [3].

8) See [6].

9) See [5].

https://doi.org/10.1017/S0027763000013520 Published online by Cambridge University Press

from ($E1$) to ($E7$) and ($E8^*$)? It seems that answers of this problem imply some meaningful results about the relations between logics.

References

[1] H. Curry, A note on the reduction of Gentzen's calculus LJ, Bull. Amer. Math. Soc., **45** (1939) 288–293.
[2] L. Henkin, An algebraic characterization of quantifiers, Fund. Math., **37** (1950) 63–74.
[3] J.C.C. McKinsey and A. Tarski, On closed elements in closure algebras, Ann. of Math., **47** (1946) 122–162.
[4] K. Ono, On a class of truth-value evaluations of the primitive logic, Nagoya Math. J., **31** (1968) 71–80.
[5] ———, On a class of set-theoretical interpretations of the primitive logic, (to appear).
[6] H. Rasiowa, Algebraic treatment of the functional calculi of Heyting and Lewis, Fund. Math., **38** (1951) 99–126.
[7] H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, Warszawa, 1963, Monografie Matematyczne tom 41.

Research institute for mathematical sciences,
Kyoto University