BULL. AUSTRAL. MATH. SOC. 47H10
VOL. 34 (1986) 107-117.

CONVEX SETS, FIXED POINTS, VARIATIONAL AND MINIMAX INEQUALITIES

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Recently, Ky Fan extended his will known lemma (which is an extension of the classical theorem of Knaster, Kuratowski and Mazurkiewica) to the noncompact case. Using this result, another interesting lemma of Fan is generalized in this paper. As applications of our theorem, we obtain a generalization of Browder's variational inequality and derive Fan's other recent results directly from our theorem. Also, in this paper, we give a slight extension recent results of K. K. Tan, which themselves are generalizations of many well-known results on minimax and variational inequalities.

1. Introduction.

In 1961, Fan [4, Lemma 1] gave an extension of the classical Knaster-Kuratowski-Mazurkiewicz theorem [8] to an arbitrary Hausdorff topological vector space. Since then, this result has been widely used in nonlinear functional analysis, and is known as Fan's Lemma or K-K-M-Fans' Theorem (see [3]). Recently, Fan [7, Theorem 4] extended his well known lemma to the noncompact case. In this paper, we first use Fan's Theorem

Received 7 October 1985.

https://doi.org/10.1017/S000497270000455X Published online by Cambridge University Press

[7, Theorem 4] to obtain a generalization of another interesting lemma of Fan [4, Lemma 4]. As applications of our theorem, we obtain a generalization of Browder's variational inequality [2, Theorem 2] and directly from our theorem derive Theorems 7, 8 of Fan [7]. We also prove a slight generalization of Theorems 1, 2 of Tan [11] and other results contained therin, which itself is a generalization of many well-known results on minimax and variational inequalities.

We first state some definitions.

DEFINITION. Let X be a nonempty convex subset of a Hausdorff topological vector space E. A real-valued function f on X is said to be

(i) lower semicontinuous if for each t the set $\{x \in X | f(x) \leq t\}$ is closed;

(ii) convex if for x, y in X and $0 \leq r \leq 1$ we have $f((1-r)y + rx) \leq (1-r)f(y) + rf(x)$;

(iii) quasi-concave if for each t the set $\{x \in X | f(x) > t\}$ is convex or empty.

We will denote by $co\{x_1, \ldots, x_n\}$, the convex full of any finite subset $\{x_1, \ldots, x_n\}$ of E.

2. Main Results.

We use the following Lemma [7] to prove Theorem 1.

LEMMA 1. (Fan [7, Theorem 4]). *In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

Remark 1. Note that $\bigcap_{x \in X_0} F(x)$ cannot be empty since, if it were, taking $A(x)$ to be the complement of $F(x)$ in Y and applying Lemma 1 of [7] we would have a finite subset of X_0 whose convex hull contained a point outside the union of the corresponding $F(x)$'s.

THEOREM 1. *Let* X *be a nonempty convex subset of a Hausdorff topological vector space* E . *Let* $A \subset X \times X$ *be a subset such that*

(a) *for each* $x \in X$, *the set* $\{y \in X \mid (x,y) \in A\}$ *is closed in* X;

(b) *for each* $y \in X$, *the set* $\{x \in X \mid (x,y) \notin A\}$ *is convex or empty;*

(c) $(x,x) \in A$ *for each* $x \in X$;

(d) X *has a nonempty compact convex subset* X_0 *such that the set* $B = \{y \in X \mid (x,y) \in A \text{ for all } x \in X_0\}$ *is compact.*

Then there exists a point $y_0 \in B$ *such that* $X \times \{y_0\} \subset A$.

Proof. For each $x \in X$, let $F(x) = \{y \in X \mid (x,y) \in A\}$. By assumption (a), $F(x)$ is closed in X . By assumptions (b), (c), $\mathrm{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$ for any finite subset $\{x_1, \ldots, x_n\}$ of X . Indeed, let $z = \sum_{i=1}^{n} \alpha_i x_i$, $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geq 0$, $i = 1, \ldots, n$. If $z \notin \bigcup_{i=1}^{n} F(x_i)$, then $(x_i, z) \notin A$ for $i = 1, \ldots, n$. By assumption (b) applied to this z , the set $\{x \in X \mid (x,z) \notin A\}$ is convex. Therefore $(z,z) \notin A$, which contradicts (c). By assumption (d), the intersection $\bigcap_{x \in X_0} F(x)$ is contained in B and is compact. By Lemma 1, there exists a point $y_0 \in \bigcap_{x \in X} F(x)$, which means $X \times \{y_0\} \subset A$.

Remark 2. (i) Condition (d) of Theorem 1 can be replaced by the following condition:

(d1) let X_0 be a nonempty compact convex subset of X , and K a nonempty compact subset of X . If for every $y \in X \setminus K$, there is a point $x \in X_0$ such that $(x,y) \notin A$.

We remark that, under the assumption (a) of the theorem, (d1) is a special case of (d). Indeed, by (d1), the set $\{y \in X \mid (\forall x \in X_0)((x,y) \in A)\} \subset K$. By (a), the set $\{y \in X \mid (\forall x \in X_0)((x,y) \in A)\}$ is closed, and is compact.

From the above remark, we see that, under condition (d1), our conclusion will be: there exists a point $y_0 \in K$ such that $X \times \{y_0\} \subset A$.

(ii) As in Remark 1, B is necessarily non-empty.

(iii) If $K = X$, then condition (d1) is automatically satisfied.

COROLLARY 1. (Fan [4, Lemma 4]). *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E. Let $A \subset X \times X$ be a subset such that the conditions (a), (b), (c) of Theorem 1 are satisfied.*

Then there exists a point $y_0 \in X$ such that $X \times \{y_0\} \subset A$.

Proof. From Theorem 1 and Remark 2.

Now we will see some applications of Theorem 1. We first obtain a generalization of a variational inequality of Browder [2, Theorem 2].

THEOREM 2. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E, T a continuous mapping of X into E^*.*

(d) If X has a nonempty compact convex subset X_0 such that the set $B = \{y \in X \mid (\forall x \in X_0)((Ty, y-x) \geq 0)\}$ is compact, then there exists a point $y_0 \in B$ such that $(Ty_0, y_0 - x) \geq 0$ for all $x \in X$.

Proof. Let

$$A = \{(x,y) \in X \times X \mid (Ty, y-x) \geq 0\}.$$

Since T is continuous, the set $\{y \in X \mid (x,y) \in A\}$ is closed in X for each $x \in X$. It is clearly that $(x,x) \in A$ for each $x \in X$, and the set $\{x \in X \mid (x,y) \notin A\} = \{x \in X \mid (Ty, y-x) < 0\}$ is convex or empty for each $y \in X$. By Theorem 1, there exists a point $y_0 \in B$ such that $X \times \{y_0\} \subset A$, that is $(Ty_0, y_0 - x) \geq 0$ for all $x \in X$.

Remark 3. (i) Condition (d) of Theorem 2 can be replaced by the following condition:

(d2) let X_0 be a nonempty compact convex subset of X, and K a nonempty compact subset of X. If for every $y \in X \setminus K$, there is a point $x \in X_0$ such that $(Ty, y-x) < 0$.

(ii) We have the same remark as Remark 2 (ii), (iii). We will not repeat this statement for other remarks.

COROLLARY 2. (Browder [2, Theorem 2]). *Let* X *be a nonempty compact convex subset of a locally convex Hausdorff topological vector space* E, T *a continuous mapping of* X *into* E^*. *Then there exists a point* $y_0 \in X$ *such that* $(Ty_0, y_0 - x) \geq 0$ *for all* $x \in X$.

Proof. This follows from Theorem 2 and Remark 3.

Now we derive the following theorems of Fan ([7, Theorem 7,8]) directly from our Theorem 1.

THEOREM 3. [7, Theorem 7]. *Let* X *be a nonempty convex set in a normed vector space* E, *and let* $f: X \to E$ *be a continuous map.*

(d) If X *has a nonempty compact convex subset* X_0 *such that the*

$$\text{set } B = \{y \in X \mid (\forall x \in X_0) ||x - f(y)|| \geq ||y - f(y)||\} \text{ is compact},$$

then there exists a point $y_0 \in B$ *such that*

$$||y_0 - f(y_0)|| = \underset{x \in X}{Min} ||x - f(y_0)|| .$$

(In particular, if $f(y_0) \in X$, *then* y_0 *is a fixed point of* f*).*

Proof. Let

$$A = \{(x,y) \in X \times X \mid ||x - f(y)|| \geq ||y - f(y)||\}.$$

Since f is continuous, the set $\{y \in X \mid (x,y) \in A\}$ is closed in X for each $x \in X$. It is clear that $(x,x) \in A$ for each $x \in X$, and the set $\{x \in X \mid (x,y) \notin A\} = \{x \in X \mid ||x - f(y)|| < ||y - f(y)||\}$ is convex or empty. By Theorem 1, there exists a point $y_0 \in B$ such that $X \times \{y_0\} \subset A$,

$$||y_0 - f(y_0)|| = \underset{x \in X}{\text{Min}} ||x - f(y_0)|| .$$

Remark.4. (i) The condition (d) of Theorem 3 can be replaced by the following condition:

(d3) Let X_0 be a nonempty compact convex subset of X, and K a nonempty compact subset of X, such that for every $y \in X \backslash K$, there is a point $x \in X_0$ such that $||x - f(y)|| < ||y - f(y)||$.

Under the condition (d3), we can conclude that $y_0 \in K$ such that

$$||y_0-f(y_0)|| = \operatorname*{Min}_{x\epsilon X}||x-f(y_0)|| .$$

Actually, Fan [7] proved Theorem 3 under the condition (d3).
(ii) If $K = X$, this reduces to Fan [5, Theorem 2]. Other generalizations of this result for a closed convex subset $K = X$ in Hilbert space were obtained by the author [9,10] and also for a closed ball in a Banach space [9].

THEOREM 4. (Fan [7, Theorem 8]). *Let X be a nonempty paracompact convex set in a Hausdorff topological vector space. Let Ω ba a nonempty convex set (that is every convex combination of any two functions in Ω is in Ω) of lower semi-continuous convex functions on X. Let S be a subset of $X\times\Omega$ such that:*

(a) For each fixed $\phi\epsilon\Omega$, the section $S(\phi) = \{x\epsilon X | (x,\phi)\epsilon S\}$ is open in X.

(b) For any fixed $x\epsilon X$, the section $S(x) = \{\phi\epsilon\Omega | (x,\phi)\epsilon S\}$ is convex and nonempty.

Then either there exists $(y_1, \phi_1) \epsilon S$ satisfying

$$y_1\epsilon X \quad and \quad \phi_1(y_1) = \operatorname*{Min}_{x\epsilon X} \phi_1(x) ;$$

or for any nonempty compact convex subset X_0 of X and any compact subset K of X, there exists $(y_2, \phi_2) \epsilon S$ satisfying

$$y_2\epsilon X\backslash K \quad and \quad \phi_2(y_2) \le \phi_2(x) \quad for\ all \quad x\epsilon X_0 .$$

Proof. Our proof is a modification of Fan [7.Theorem 8]. By (b), for each $z\epsilon X$, there is a $\phi_z\epsilon\Omega$ such that $(z,\phi_z) \epsilon S$. By (a), $\{S(\phi_z)|z\epsilon X\}$ is an open cover of the paracompact space X. Let $\{\alpha_z|z\epsilon X\}$ be a continuous partition of unity subordinate to this open cover. Thus, for each $z\epsilon X$, α_z is a non-negative real continuous function on X, with its support $\operatorname{supp} \alpha_z \subset S(\phi_z)$. The family $\{\operatorname{supp} \alpha_z|z\epsilon X\}$ is a locally finite closed cover of X; and $\sum_{z\epsilon X} \alpha_z(x) = 1$ for all $x\epsilon X$, define $\psi(x)$ by

$$\psi(x) = \sum_{z\epsilon X} \alpha_z(x)\phi_z ,$$

https://doi.org/10.1017/S000497270000455X Published online by Cambridge University Press

which is a convex combination of a finite number of ϕ_z's . As Ω is convex, we have $\psi(x)\in\Omega$ for each $x\in X$. If x, z in X are such that $\alpha_z(x) > 0$, then $x\in\text{supp}\,\alpha_z \subset S(\phi_z)$ and therefore $\phi_z\in S(x)$. By (b), we have $\psi(x)\in S(x)$ for all $x\in X$.

Now, define

$$A = \{(x,y)\in X\times X \mid \psi(y)(y) \le \psi(y)(x)\} .$$

Since α_z is continuous and non-negative on X and ϕ_z is lower semi-continuous on X , then for each fixed $x\in X$, the function

$$h(x,y) = \psi(y)(y) - \psi(y)(x) = \sum_{z\in X} \alpha_z(y)\phi_z(y) - \sum_{z\in X} \alpha_z(y)\phi_z(x)$$

is also a lower semi-continuous function of y on X . Therefore the set

$$\{y\in X \mid (x,y)\in A\} = \{y\in X \mid \psi(y)(y) \le \psi(y)(x)\}$$

is closed in X for each $x\in X$. It is clear that $(x,x)\in A$ for all $x\in X$ and the set $\{x\in X \mid (x,y)\notin A\}$ is convex or empty for each $y\in X$ (since $\psi(y)\in\Omega$ and $\psi(y)$ is a convex function on X). By Theorem 1 and Remark 2, either there exists $y_1\in X$ such that $X\times\{y_1\} \subset A$; or for any non-empty compact convex subset X_0 of X and any compact subset K of X , there exists $y_2\in X\backslash K$ such that $(x,y_2) \in A$ for all $x\in X_0$. We take $\phi_1 = \psi(y_1)$ in the first case, and $\phi_2 = \psi(y_2)$ in the second case. In the first case, $X\times\{y_1\} \subset A$, this means

$$\phi_1(y_1) = \min_{x\in X} \phi_1(x) .$$

Since $\psi(y_1) \in S(y_1)$, $(y_1, \phi_1) \in S$. In the second case, $(x,y_2) \in A$ for all $x\in X_0$, it implies that

$$\phi_2(y_2) \le \phi_2(x) \quad \text{for all } x\in X_0 .$$

This completes the proof.

Now, we will use Lemma 1 to prove the following minimax inequalities.

THEOREM 5. *Let X be a nonempty convex set in a Hausdorff topological vector space E. Let f and g be two real-valued function on $X\times X$ having the following properties:*

(a) $g(x,y) \le f(x,y)$ for all $(x,y) \in X\times X$ and $f(x,x) \le 0$ for all $x\in X$;

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(b) *for each fixed* $X \in X$, $g(x,y)$ *is a lower semicontinuous function* y *on* X;

(c) *for each fixed* $y \in X$, *the set* $\{x \in X | f(x,y) > 0\}$ *is convex or empty;*

(d) X *has a nonempty compact convex subset* X_0 *such that the set* $B = \{y \in X | g(x,y) \le 0$ *for all* $x \in X_0\}$ *is compact.*

Then there exists a point $y_0 \in B$ *such that* $g(x,y_0) \le 0$ *for all* $x \in X$.

Proof. For each $x \in X$, let

$$G(x) = \{y \in X | g(x,y) \le 0\} ,$$
$$F(x) = \{y \in X | f(x,y) \le 0\} .$$

By (b), $G(x)$ is closed in X. From (a), (c), we have

$$\mathrm{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$$

for any finite subset $\{x_1, \ldots, x_n\}$ of X. Indeed, if $z = \sum_{i=1}^{n} \alpha_i x_i$, $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \ge 0$, and $z \notin \bigcup_{i=1}^{n} F(x_i)$, then $(f(x_i), z) > 0$, $i=1,\ldots,n$. By (c), $f(z,z) > 0$, which contradicts the assumption (a). By (a) $F(x) \subset G(x)$. Then $\mathrm{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$. Since $\bigcap_{x \in X_0} G(x)$ is a closed subset of the compact set B, $\bigcap_{x \in X_0} G(x)$ is compact. By Lemma 1, there exists a point $y_0 \in \bigcap_{x \in X} G(x)$, which means $g(x,y_0) \le 0$ for all $x \in X$.

Remark 5. Condition (d) of Theorem 5 can be replaced by the following condition:

(d5) let X_0 be a nonempty compact convex subset of X, and K a nonempty compact subset of X. If for every $y \in X \backslash K$, there is a point $x \in X_0$ such that $g(x,y) > 0$. Then our conclusion is that there exists a point $y_0 \in K$ such that

$$g(x,y_0) \le 0 \quad \text{for all} \quad x \in X .$$

https://doi.org/10.1017/S000497270000455X Published online by Cambridge University Press

COROLLARY 3. (Tan [11, Theorem 1]). *Let* X, E, f, *and* g *be the same as in Theorem 5 and satisfy conditions (a), (b), (c) and*

(d') *there exists a nonempty compact convex subset* K *of* X *such that for each* $y \in X \backslash K$, *there exists a point* $x \in K$ *with* $g(x,y) > 0$.

Then there exists a point $y_0 \in K$ *such that* $g(x,y_0) \leq 0$ *for all* $x \in X$.

Proof. Take $X_0 = K$ in (d5) .

Remark 6. If $f = g$ in Corollary 3, this result is due to Allen [1, Theorem 2].

THEOREM 6. *Let* X *be a nonempty convex set in a Hausdorff topological vector space. Let* f *and* g *be two real-valued function on* $X \times X$ *having the following properties:*

(a) $g(x,y) \leq f(x,y)$ *for all* $(x,y) \in X \times X$,

(b) *for each fixed* $x \in X$, $g(x,y)$ *is a lower semicontinuous function of* y *on* X,

(c) *for each fixed* $y \in X$, $f(x,y)$ *is a quasi-concave function of* x *on* X,

(d) *if* X *has a non-empty compact convex subset* X_0 *such that the set* $B = \{y \in X \mid (\forall x \in X_0)(g(x,y) \leq t)\}$ *is compact, if*

$$t = \sup_{x \in X} f(x,x) < +\infty .$$

Then the minimax inequality

$$\min_{y \in B} \sup_{x \in X} g(x,y) \leq \sup_{X \in X} f(x,x)$$

holds.

Proof. Without loss of generality, we can assume that $t = \sup_{x \in X} f(x,x) < +\infty$.

Applying Theorem 5 to

$$f_1(x,y) = f(x,y) = t, \quad g_1(x,y) = g(x,y) - t ,$$

the proof is complete.

Remark 7. Condition (d) of Theorem 6 can be replaced by the following condition:

(d6) let X_0 be a nonempty compact convex subset of X, and K a nonempty compact subset of X. If for every $y \in X \backslash K$, there is a point $x \in X_0$ such that

$$g(x,y) > t \quad \text{if} \quad t = \sup_{x \in X} f(x,x) < +\infty$$

then our conclusion is that the minimax inequality

$$\min_{y \in K} \sup_{x \in X} g(x,y) \leq \sup_{x \in X} f(x,x)$$

holds.

COROLLARY 4. (Tan [11, Theorem 2]). *Let X, f, g be the same as Theorem 6 and satisfy conditions (a), (b), (c) and*

(d') there exists a nonempty compact convex subset K of X such that for all $y \in X \backslash K$, there exists a point $x \in K$ with $g(x,y) > t$ if $t = \sup_{x \in X} f(x,x) < \infty$.

Then the minimax inequality

$$\min_{y \in K} \sup_{x \in X} g(x,y) \leq \sup_{x \in X} f(x,y)$$

holds.

Proof. Take $X_0 = K$ in (d6)

Remark 8. If $K = X$, the result is due to Yen [12, Theorem 1]. If $K = X$, $f = g$, the result is due to Fan [6].

Remark 9. Tan [11] applied corollaries 3, 4 to obtain some generalizations of variational inequalities ([11, Theorem 3, 5]) and fixed point theorems ([11, Theorem 6-8]). Applying our Theorems 5, 6 and using the same arguments as Tan [11], we can obtain a slight generalization of the corresponding Theorems 3, 5, 6, 7, 8 of Tan [11], just like we did for Theorems 5 and 6 and Remarks 5, 7.

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Added in Proof. I would like to thank Professor Sehie Park of Seoul National University (Korea) for calling my attention to the recent paper of Ky Fan [7], when we met at the International Conference on Approximation Theory held in Newfoundland, Canada, August 1984.