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APPROXIMATE IDENTITIES IN BANACH ALGEBRAS OF COMPACT OPERATORS

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ABSTRACT. Let X be a Banach space and let $\mathcal{A}$ be a uniformly closed algebra of compact operators on X, containing the finite rank operators. We set up a general framework to discuss the equivalence between Banach space approximation properties and the existence of right approximate identities in $\mathcal{A}$. The appropriate properties require approximation in the dual X^* by operators which are adjoints of operators on X. We show that the existence of a bounded right approximate identity implies that of a bounded left approximate identity. We give examples to show that these properties are not equivalent, however. Finally, we discuss the well known result that, if X^* has a basis, then X has a shrinking basis. We make some attempts to generalize this to various bounded approximation properties.

0. **Introduction.** The subject of bounded approximate identities in algebras of compact operators was first mentioned in [J], where it arose in connection with amenability questions. In the monograph [D&W] general questions were raised, and some results relating to the approximation property and the existence of a basis for the underlying Banach space were given. In the paper [D] P. G. Dixon continued along this approach and gave a rather exhaustive description of left approximate identities in terms of the approximation property. A complete description was given for bounded left approximate identities for the Banach algebra of compact operators and the Banach algebra consisting of operators uniformly approximable by finite rank operators. However, there were no Banach space conditions known to be equivalent to the existence of right approximate identities (see [D&W, p. 261]). In Section 2 below we set up the framework to discuss this and find the corresponding descriptions for right approximate identities. As perhaps could be expected one has to look at approximation properties for the dual of X. The appropriate properties require approximation in X^* by operators which are adjoints of operators on X.

It is also shown that the existence of a bounded right approximate identity implies the existence of a bounded left approximate identity. Examples given in Section 4 show that these properties are not equivalent, however.

Just as approximation properties are generalizations of the notion of a basis, the approximation properties on the dual space we introduce may be regarded as generalizing the notion of a shrinking basis. This is in accordance with previous results on right approximate identities in the presence of a shrinking basis, see [B&P, p. 14] and

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[D&W, Theorem 30.4]. Now, if the dual space X^* has a basis, then X has a shrinking basis ([J,R&Z]). In Section 3 we make some attempts to generalize this result for various bounded approximation properties.

1. **Notation.** We begin by establishing notation. Throughout, X will denote a Banach space and X^* the space of bounded linear functionals on X with its usual norm. Small letters x will denote elements in X, whereas x^* will denote elements in X^*. We will consider the following classes of operators on X:

$$
\begin{aligned}
F(X) &= \{\text{finite rank operators}\}\\
\mathcal{N}(X) &= \{\text{nuclear operators}\}\\
\mathcal{F}(X) &= \text{ uniform closure of } F(X)\\
\mathcal{K}(X) &= \{\text{compact operators}\}\\
I(X) &= \{\text{integral operators}\}\\
\mathcal{W}(X) &= \{\text{weakly compact operators}\}\\
\mathcal{B}(X) &= \{\text{bounded operators}\}
\end{aligned}
$$

All these, except $F(X)$, are Banach algebras in their natural norms, and they are all two-sided ideals in $\mathcal{B}(X)$. We refer the reader to any of [D&U], [Pie], [Pis] for details.

If $S \in \mathcal{B}(X)$ we denote the adjoint map in $\mathcal{B}(X^*)$ by S^a, *i.e.*

$$\langle S(x), x^* \rangle = \langle x, S^a(x^*) \rangle \quad (x \in X, x^* \in X^*)$$

If $M \subseteq \mathcal{B}(X)$ we define $M^a \subseteq \mathcal{B}(X^*)$ by

$$M^a = \{T^a \mid T \in M\}$$

This should not be confused with the notation for dual space. For instance, if X has Grothendieck's approximation property, then $\mathcal{N}(X)^* = \mathcal{B}(X^*)$, whereas $\mathcal{N}(X)^a$ is the set of so-called X-nuclear operators on X^*.

We shall use the notation LAI, BLAI, RAI, BRAI, BAI for left approximate identity, bounded left approximate identity etc. We use these concepts in accordance with [B&D].

2. **Approximate identities.** Let $\mathcal{A}$ be an algebra of operators on a Banach space X. We are interested in the question whether $\mathcal{A}$ has a left approximate identity, bounded in the operator norm. This question has already been considered by P. G. Dixon in [D] for the cases $\mathcal{A} = \mathcal{K}(X)$ and $\mathcal{A} = \mathcal{F}(X)$. We shall use a more general approach, allowing for a discussion of right approximate identities as well, a question hitherto left virtually untouched in the literature [D&W]. As is apparent from P. G. Dixon's paper, the question is closely related to various versions of the approximation property, *i.e.* to the problem of obtaining the identity operator on X, 1_X, as a limit of certain nets converging in the topology of uniform convergence on compact sets. Let us denote this topology by τ and recall a convenient description of τ [L&T, p. 31].

Consider the projective tensor product $X\hat{\otimes}X^*$, and view it as a canonical Banach bimodule over $\mathcal{B}(X)$. Let tr: $X\hat{\otimes}X^* \longrightarrow \mathbb{C}$ be the canonical trace

$$\mathrm{tr}(\Sigma x_i \otimes x_i^*) = \Sigma\langle x_i, x_i^*\rangle.$$

PROPOSITION 2.1. *Let φ be a τ-continuous linear functional on $\mathcal{B}(X)$. Then there is a unique $u \in X\hat{\otimes}X^*$ such that*

$$\varphi(S) = \mathrm{tr}(S.u) \quad \big(S \in \mathcal{B}(X)\big).$$

Note that the Banach space X is implicitly given by τ.

In order to deal with various approximation properties simultaneously we make the following definition.

DEFINITION 2.2. *Let $\mathcal{A}_1$ denote the unit ball in $\mathcal{A}$. We say that*

(1) X has the $\mathcal{A}$-approximation property ($\mathcal{A}$-AP) if 1_X is in the τ-closure of $\mathcal{A}$.

(2) X has the λ-$\mathcal{A}$-approximation property (λ-$\mathcal{A}$-AP) if 1_X is in the τ-closure of $\lambda\mathcal{A}_1$ $\quad(\lambda \geq 1)$.

(3) X has the metric $\mathcal{A}$-approximation property (M-$\mathcal{A}$-AP), if X has the 1-$\mathcal{A}$-AP.

(4) X has the bounded $\mathcal{A}$-approximation property (B-$\mathcal{A}$-AP), if X has the λ-$\mathcal{A}$-approximation property for some λ.

Using Proposition 2.1 we then get the following description.

PROPOSITION 2.3.

(i) X has the $\mathcal{A}$-AP if and only if for each $u \in X\hat{\otimes}X^*$

$$\mathrm{tr}(A.u) = 0 \quad (A \in \mathcal{A}) \Rightarrow \mathrm{tr}(u) = 0$$

(ii) X has the λ-$\mathcal{A}$-AP if and only if for each $u \in X\hat{\otimes}X^*$

$$|\,\mathrm{tr}(A.u)| \leq \frac{1}{\lambda} \quad (A \in \mathcal{A}_1) \Rightarrow |\,\mathrm{tr}(u)| \leq 1$$

PROOF. This is a straight forward application of the Hahn-Banach theorem.

We are now ready to give the description of approximate identities for $\mathcal{A}$ in terms of appropriate approximation properties for the underlying Banach space.

THEOREM 2.4. *Suppose that $\mathcal{A}$ is contained in $\mathcal{K}(X)$.*

(i) If X has $\mathcal{A}$-AP, then $\mathcal{A}$ has a LAI.

(ii) If X has λ-$\mathcal{A}$-AP, then $\mathcal{A}$ has a BLAI *of norm not exceeding λ.*

PROOF. These statements follow from the obvious identity

$$\|EA - A\| = \sup\{\|Ex - x\| \mid x \in A(X_1)\},$$

where E and A are operators in $\mathcal{A}$. Using that $A(X_1)$ is compact since $\mathcal{A} \subseteq \mathcal{K}(X)$, it follows that a net converging to 1_X in the τ-topology will be a LAI or BLAI as the case may be.

Under appropriate conditions the converse is also true.

THEOREM 2.5. *Suppose that* span$\{\mathrm{rg}(A) \mid A \in \mathcal{A}\}$ *is dense in* X. *Then* X *has the* λ-$\mathcal{A}$-AP *if* $\mathcal{A}$ *has a* BLAI *of norm not exceeding* λ.

PROOF. Let $u \in X \hat{\otimes} X^*$ satisfy $|\,\mathrm{tr}(A.\,u)| \leq \frac{1}{\lambda}$ for all $A \in \mathcal{A}_1$. Let (E_α) be a BLAI for $\mathcal{A}$ of norm not exceeding λ. By the hypothesis on $\mathcal{A}$ the net converges to 1_X in the strong operator topology. It follows that

$$|\,\mathrm{tr}(u)| = \lim_\alpha |\,\mathrm{tr}(E_\alpha.\,u)| \leq 1.$$

Proposition 2.3.(ii) gives us the conclusion.

We now use the adjoint map to pass from left to right approximate identities.

COROLLARY 2.6. *Suppose that* span$\{\mathrm{rg}(A^a) \mid A \in \mathcal{A}\}$ *is dense in* X^*, *and that* $\mathcal{A} \subseteq \mathcal{K}(X)$. *Then* $\mathcal{A}$ *has a* BRAI *if and only if* X^* *has* λ-$\mathcal{A}^a$-AP.

PROOF. Since $\mathcal{A}$ is anti-isomorphic to $\mathcal{A}^a$, $\mathcal{A}$ has a BRAI if and only if $\mathcal{A}^a$ has a BLAI, and since taking adjoints preserves compactness we may use Theorems 2.4 and 2.5.

This suggests a certain symmetry between right and left. But, recalling that X has the Grothendieck approximation property if X^* has, it is not surprising that the existence of right approximate identities is the stronger of the two.

COROLLARY 2.7. *Suppose that* $\mathcal{F}(X) \subseteq \mathcal{A} \subseteq \mathcal{K}(X)$. *Then* $\mathcal{A}$ *has a* BRAI *if and only if* $\mathcal{A}$ *has a* BAI.

PROOF. Assume that $\mathcal{A}$ has a BRAI so that X^* has λ-$\mathcal{A}^a$-AP. Let $u \in X \hat{\otimes} X^*$ satisfy $|\,\mathrm{tr}(A.\,u)| \leq \frac{1}{\lambda}$ for all $A \in \mathcal{A}_1$. Let $\tilde{~}\colon X \hat{\otimes} X^* \longrightarrow X^* \hat{\otimes} X^{**}$ be the canonical map given by

$$(x \otimes x^*)\tilde{~} = x^* \otimes \iota(x),$$

where $\iota\colon X \longrightarrow X^{**}$ is the canonical embedding. Then $\mathrm{tr} \circ \tilde{~} = \mathrm{tr}$ and $\widetilde{A.\,u} = A^a.\,\tilde{u}$, so by hypothesis $|\,\mathrm{tr}(S.\,\tilde{u})| \leq \frac{1}{\lambda}$ for all $S \in \mathcal{A}_1^a$. It follows that $|\,\mathrm{tr}(\tilde{u})| = |\,\mathrm{tr}(u)| \leq 1$. Proposition 2.3 (ii) and Theorem 2.4 (ii) give the conclusion.

3. **More on approximation by operators on the dual space.** From Section 2 it is apparent that the important feature, when discussing BAI's, is approximation by adjoint maps in the dual space. This can be viewed as generalizing the notion of a shrinking basis. If X has a basis with associated sequence $(P_n)_{n\in\mathbb{N}}$ of projections, then the basis is shrinking exactly when the sequence $(P_n^a)_{n\in\mathbb{N}}$ determines a basis for X^*. From a more general stand these approximation properties may perhaps seem a little artificial. In the case $\mathcal{A} = \mathcal{F}(X)$, that is, when we are dealing with Grothendieck's approximation property, a much more satisfactory description is possible. This can be seen as parallelling the result that, if X^* has a basis, then X has a shrinking basis. However, as is known in other cases, it is much easier to establish an approximation property than the existence of a basis. First let us note

THEOREM 3.1. *If X^* has $F(X^*)$-AP, then X^* has $F(X)^a$-AP*

PROOF. Goldstine's theorem gives immediately that the τ-closure of $F(X)^a$ contains $F(X^*)$.

When dealing with the bounded approximation property it is necessary to work a little harder. Before stating the theorem we will establish a procedure to pass from bounded approximate identities in a Banach algebra to bounded approximate identities in some subalgebras.

LEMMA 3.2. *Let $\mathfrak{B}$ be a Banach algebra and let $\mathfrak{A}$ be a (closed) subalgebra. Let $\iota\colon \mathfrak{A} \longrightarrow \mathfrak{A}^{**}$ be the canonical embedding into the bi-dual. Consider $\mathfrak{A}$, $\mathfrak{A}^{**}$, and $\mathfrak{B}$ as Banach bi-modules over $\mathfrak{A}$. Suppose ι can be extended to a bounded right module homomorphism $\tilde{\iota}\colon \mathfrak{B} \longrightarrow \mathfrak{A}^{**}$ through the inclusion $\mathfrak{A} \hookrightarrow \mathfrak{B}$. If $\mathfrak{B}$ has a* BLAI *of norm not exceeding λ, then $\mathfrak{A}$ has a* BLAI *of norm not exceeding $\lambda\|\tilde{\iota}\|$.*

PROOF. By [B&D, Proposition 2.11.4] $\mathfrak{A}$ has a BLAI of norm not exceeding M, if and only if there is $E \in \mathfrak{A}^{**}$ with $\|E\| \leq M$ so that $E.a = a$ for all $a \in \mathfrak{A}$. Let (b_α) be a BLAI for $\mathfrak{B}$ and let E be a w^*-accumulation point of the net $\big(\tilde{\iota}(b_\alpha)\big)$. Then one checks easily that this E works.

This lemma will be applied to $\mathcal{F}(X)$, using a characterization of $\mathcal{F}(X)^*$. First notice that we may identify $\mathcal{F}(X)$ with the weak tensor product $X\check{\otimes}X^*$, as defined for instance in [B&D], by letting $F \in \mathcal{F}(X)$ act as a bilinear form on $X^* \times X^{**}$ in the way

$$(x^*, x^{**}) \longrightarrow \langle x^*, F^{**}(x^{**})\rangle.$$

We may thus identify $\mathcal{F}(X)^*$ with a space of integral operators on X^*. Since X^* is a dual space, a linear operator $\varphi \in \mathcal{B}(X^*)$ is integral according to Grothendieck [Gr], if and only if φ is integral according to Pietsch [Pie], see [D&U, Corollary VIII.2.10]. The duality is implemented by

$$\langle F, \varphi\rangle = \mathrm{tr}(F^a\varphi) \quad \big(F \in F(X), \varphi \in I(X^*)\big).$$

Hence we may extend the action of $\varphi \in I(X^*)$ to $\mathcal{F}(X^*)$ isometrically, simply by

$$\langle F, \varphi\rangle = \mathrm{tr}(F\varphi) \quad \big(F \in F(X^*), \varphi \in I(X^*)\big). \tag{1}$$

THEOREM 3.3. *Let X be a Banach space. The following statements are equivalent.*

(i) X^* has the λ-$\mathcal{F}(X^*)$-AP.

(ii) X^* has the λ-$\mathcal{F}(X)^a$-AP.

(iii) $\mathcal{F}(X^*)$ has a BLAI *of norm not exceeding λ.*

(iv) $\mathcal{F}(X)$ has a BRAI *of norm not exceeding λ.*

(v) $\mathcal{F}(X)$ has a BAI *of norm not exceeding λ.*

PROOF. The only implication that does not follow from Theorems 2.4 and 2.5 is (iii) $\Rightarrow$ (iv). We shall prove (iv) by showing that $\mathcal{F}(X)^a$ has a BLAI. We apply Lemma 3.2 with $\mathfrak{A} = \mathcal{F}(X)^a$ and $\mathfrak{B} = \mathcal{F}(X^*)$. Since $\mathfrak{A}$ then is isometrically isomorphic to $\mathcal{F}(X)$ we may identify $\mathfrak{A}^{**}$ with $I(X^*)^*$. The map $\tilde{\iota}\colon \mathfrak{B} \longrightarrow \mathfrak{A}^{**}$ is then given by the extension (1) above. One checks easily that the hypothesis of Lemma 3.2 is satisfied.

Of course, when X has the $F(X)$-AP then $\mathcal{F}(X) = \mathcal{K}(X)$. In a forthcoming paper by G. A. Willis ([W]) it will be shown that approximation by compact operators is genuinely weaker than by finite rank operators so we have to consider the situation concerning compact operators separately. It is seemingly much more complicated, partly because no convenient description of $\mathcal{K}(X)^*$ is available. Thus, the problem we address is the following.

Suppose that X^* has $\mathcal{K}(X^*)$-AP (B-$\mathcal{K}(X^*)$-AP). Can one prove that X^* has $\mathcal{K}(X)^a$-AP (B-$\mathcal{K}(X)^a$-AP)?

An example in Section 4 shows that this is not possible in general. However, when X is reflexive there is nothing to prove. This suggests that it may be fruitful to employ τ-approximation by weakly compact operators. But before we harvest, we need an observation which we state as

LEMMA 3.4. $\mathcal{K}(X)^a = \{K \in \mathcal{K}(X^*) \mid K^a(X^{**}) \subseteq \iota(X)\}$, *where* $\iota\colon X \longrightarrow X^{**}$ *is the natural embedding.*

PROOF. The inclusion '$\subseteq$' follows from [D&S, Theorem VI.4.2]. The converse follows from the obvious identity $K = L^a$, where $L \in \mathcal{K}(X)$ is the operator satisfying $K^a \circ \iota = \iota \circ L$.

We can now answer the problem addressed above affirmatively when τ-approximation by weakly compact operators is possible.

PROPOSITION 3.5. *Let X be a Banach space. Then*

(i) If X^* has $\mathcal{W}(X)^a$-AP *and* $\mathcal{K}(X^*)$-AP, *then X^* has* $\mathcal{K}(X)^a$-AP.

(ii) If X^* has λ_1-$\mathcal{W}(X)^a$-AP *and* λ_2-$\mathcal{K}(X^*)$-AP, *then X^* has* $\lambda_1\lambda_2$-$\mathcal{K}(X)^a$-AP.

PROOF. By [D&S, Theorem VI.4.2] and Lemma 3.4 above $\mathcal{K}(X^*)\mathcal{W}(X)^a$ is contained in $\subseteq \mathcal{K}(X)^a$. Hence the result follows by two consecutive approximations.

It is of some interest to notice that Proposition 3.5. (ii) also can be deduced from the extension Lemma 3.2. One would like to bring this lemma into use with $\mathfrak{A} = \mathcal{K}(X)^a$ and $\mathfrak{B} = \mathcal{K}(X^*)$ to parallel the proof for $\mathcal{F}(X)$. This is indeed possible:

PROPOSITION 3.6. *Suppose X^* has the* λ-$\mathcal{W}(X)^a$-AP. *Then there is a right* $\mathcal{K}(X)^a$-*module homomorphism*

$$\tilde{\iota}\colon \mathcal{K}(X^*) \longrightarrow \mathcal{K}(X)^{**}$$

extending the natural inclusion $\iota\colon \mathcal{K}(X)^a \longrightarrow \mathcal{K}(X)^{**}$.

PROOF. Let (U_α) be a bounded net in $\mathcal{W}(X)$ such that (U_α^a) converges to 1_{X^*} in the τ-topology on $\mathcal{B}(X^*)$. Then (U_α) converges to 1_X in the bw^*-topology on $\mathcal{B}(X)$. By [D&S, Corollary VI.1.5] there is a net in $\operatorname{conv}\{U_\alpha\}$ converging strongly to 1_X. Hence, by passing to a net of convex combinations of U_α's if necessary, we may assume that

$\lim_\alpha U_\alpha(x) = x$ and $\lim_\alpha U_\alpha^a(x^*) = x^*$ for all $x \in X, x^* \in X^*$. Now let $K \in \mathcal{K}(X^*)$. By [D&S, Theorem VI.4.2] and Lemma 3.4 above we have that $KU_\alpha^a \in \mathcal{K}(X)^a$ so we may define $\iota_\alpha \colon \mathcal{K}(X^*) \longrightarrow \mathcal{K}(X)^{**}$ simply by $\iota_\alpha(K) = \iota(KU_\alpha^a)$. Then $\|\iota_\alpha\| \le \|U_\alpha\|$. In particular, (ι_α) is bounded. Let $\tilde{\iota}$ be a w^*-limit point of the net (ι_α) in $\mathcal{B}\big(\mathcal{K}(X^*), \mathcal{K}(X)^{**}\big)$ (using the isometric identification with $\big(\mathcal{K}(X^*)\hat{\otimes}\mathcal{K}(X)^*\big)^*$). Since (U_α) converges to 1_X strongly and boundedly the map $\tilde{\iota} \colon \mathcal{K}(X^*) \longrightarrow \mathcal{K}(X)^{**}$ extends ι:

$$\begin{aligned}\langle \varphi, \tilde{\iota}(K^a)\rangle &= \lim_\alpha \langle K^a U_\alpha^a, \varphi\rangle \\ &= \lim_\alpha \langle (U_\alpha K)^a, \varphi\rangle \\ &= \langle K^a, \varphi\rangle \\ &= \langle \varphi, \iota(K^a)\rangle\end{aligned}$$

for all $K \in \mathcal{K}(X)$ and $\varphi \in \mathcal{K}(X)^*$.

Similarly, since U_α^a converges strongly and boundedly to 1_{X^*}, the map $\tilde{\iota}$ is a right $\mathcal{K}(X)^a$-module homomorphism. Let $\varphi \in \mathcal{K}(X)^*$, $K \in \mathcal{K}(X^*)$, and $L \in \mathcal{K}(X)^a$. Then

$$\begin{aligned}\langle \varphi, \tilde{\iota}(KL^a)\rangle &= \langle KL^a, \varphi\rangle \\ &= \lim_\alpha \langle K(LU_\alpha)^a, \varphi\rangle \\ &= \lim_\alpha \langle KU_\alpha^a, L^a.\varphi\rangle \\ &= \langle \varphi, \tilde{\iota}(K).L^a\rangle.\end{aligned}$$

The last two propositions leave us with the interesting

QUESTION. For which Banach spaces X is it true that X^* has the B-$\mathcal{W}(X)^a$-AP?

We conclude this section by restating Propositions 3.5 and 3.6 in terms of bounded approximate identities.

COROLLARY 3.7. *Let X be a Banach space such that X^* has the* λ_1-$\mathcal{W}(X)^a$-AP. *Then the following are equivalent*

(i) $\mathcal{K}(X^*)$ *has a* BLAI.

(ii) $\mathcal{K}(X)$ *has a* BRAI.

(iii) $\mathcal{K}(X)$ *has a* BAI.

If $\mathcal{K}(X^*)$ *has a* BLAI *of norm not exceeding* λ_2, *then the approximate identities of (ii) and (iii) may be chosen to have norms not exceeding* $\lambda_1\lambda_2$.

4. **Examples.** From Corollary 2.7 it follows that, if X is reflexive, then $\mathcal{K}(X)$ has a BRAI if and only if it has a BLAI. The first two examples show that this is not the case in general.

EXAMPLE 4.1. Let $X = \mathcal{N}(H)$, the trace class operators on a Hilbert space H. Then X is a dual space, with predual $X_* = \mathcal{K}(H)$. In [G,J&W] it is proved that $\mathcal{K}(X_*)$ is amenable. In particular, $\mathcal{K}(X_*)$ has a BAI, so that $\mathcal{K}(X)$ has a BLAI by Corollary 2.6

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and Theorem 2.4.(ii). But $X^* = \mathcal{B}(H)$, which does not even have the approximation property ([S]). In particular $\mathcal{K}(X)$ has a BLAI but not a BRAI. Thus in Corollary 2.7 'right' can not be replaced by 'left'.

EXAMPLE 4.2. As already pointed out by P. G. Dixon [D] Theorem 2.5 would be true also for possibly unbounded approximate identities if every compact subset in X were contained in the image of the unit ball by some $F \in \mathcal{A}$. He further observed that, if X has a quotient isomorphic to ℓ_1, this is true with $\mathcal{A} = \mathcal{F}(X)$ or $\mathcal{A} = \mathcal{K}(X)$. Using a space constructed by Szankowski he then produced an example, Z, of a space where neither $\mathcal{F}(Z)$ nor $\mathcal{K}(Z)$ has a LAI. It is possible to get such examples without going into the details of Szankowski's example. Simply let $Z = E \oplus \ell_1$, where E is a space without the $\mathcal{F}(E)$-AP. To get examples without RAI's, let E be a space with B-$\mathcal{F}(E)$-AP such that E^* does not have $\mathcal{F}(E^*)$-AP. Put $Y = E \oplus c_0$. Then any compact subset of Y^* is contained in the image of the unit ball by some $F \in \mathcal{F}(Y)^a$. We modify P. G. Dixon's argument to see this. First note that we may assume that the compact set has the form $\overline{\text{conv}}\{y_n^*\}$, where (y_n^*) is a sequence in Y^* tending to zero. Define an operator $F\colon Y \longrightarrow Y$ by

$$F(y) = \big(0, \phi(y)\big) \quad (y \in Y),$$

$\phi(y)_n = y_n^*(y)$. As in P. G. Dixon's paper one sees that F^a has the property called for. It follows that $\mathcal{F}(Y)$ $(= \mathcal{K}(Y))$ has a BLAI but does not have a RAI.

EXAMPLE 4.3. This example serves to illustrate that Theorem 3.3 is false when $\mathcal{F}(X)$ is replaced by $\mathcal{K}(X)$. Such an example must of course exploit the existence of a Banach space with the compact approximation property, but not the approximation property. By [W] an example like this exists. In fact, we may choose a separable reflexive space Y such that Y has the B-$\mathcal{K}(Y)$-AP but does not have $\mathcal{F}(Y)$-AP. Using the construction of [L&T, Theorem 1.d.3], there is a Banach space Z such that Z^{**} has a basis and Y^* is isomorphic to the canonical quotient space Z^{**}/Z. Let $X = Z^{**}$. By dualizing, we obtain a decomposition $X^* = Z^* \oplus Y$ such that X^* has the B-$\mathcal{K}(X^*)$-AP but not the $\mathcal{F}(X^*)$-AP. Since X has a basis, $\mathcal{F}(X) = \mathcal{K}(X)$. It follows that X^* has the B-$\mathcal{K}(X^*)$-AP, but not the $\mathcal{K}(X)^a$-AP. Hence $\mathcal{K}(X^*)$ has a BLAI, but $\mathcal{K}(X)$ does not have a BRAI. It further follows from Proposition 3.5 that X^* does not have the B-$\mathcal{W}(X)^a$-AP.

NOTE. After the present work was submitted it has come to our knowledge that some of our results have been obtained independently by C. Samuel [Sa]. Samuel proves Corollaries 2.6 and 2.7 in the case $\mathcal{A} = \mathcal{K}(X)$. He concludes his paper with three open questions. Question 1 is in our terminology: Does B-$\mathcal{K}(X)$-AP imply B-$\mathcal{F}(X)$-AP? It is answered in the negative by the second author in [W]. The main part of Question 2 is in our terminology: Suppose that X^* has the B-$\mathcal{K}(X^*)$-AP. Does X^* have the B-$\mathcal{K}(X)^a$-AP? Example 4.3 answers this.

REFERENCES

[B&P] E. Berkson and H. Porta, *Representations of* $\mathfrak{B}(X)$, J. Funct. Anal. **3**(1969) 1–34.

[B&D] F. F. Bonsall and J. Duncan, *Complete Normed Algebras,* Springer-Verlag, 1973, Berlin Heidelberg.

[Di] J. Diestel, *Sequences and Series in Banach Spaces,* Graduate Texts in Mathematics **92**, Springer-Verlag, New York, 1984.

[D&U] J. Diestel and J. J. Uhl, *Vector Measures,* Math. Surveys **15**, Amer. Math. Soc., 1977.

[D] P. G. Dixon, *Left approximate identities in algebras of compact operators on Banach spaces,* Proc. Royal Soc. Edinburgh **104A**(1989) 169–175.

[D&W] R. S. Doran and J. Wichmann, *Approximate identities and factorization in Banach modules,* Lecture Notes in Mathematics **768**, Springer-Verlag, 1979, Berlin Heidelberg.

[D&S] N. Dunford and J. T. Schwartz, *Linear Operators, Part 1,* Interscience, New York, 1958.

[Gr] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques,* Bol. Soc. Mat. São Paulo **8**(1953) 1–70.

[G,J&W] N. Grønbæk, B. E. Johnson and G. A. Willis, *Amenability of Banach algebras of compact operators,* forthcoming paper.

[J] B.E. Johnson, *Cohomology in Banach algebras,* Mem. Amer. Math. Soc. **127**(1972).

[J,R&Z] W. B. Johnson, H. P. Rosenthal and M. Zippin, *On bases, finite-dimensional decompositions and weaker structures in Banach spaces,* Israel J. Math. **9**(1971) 488–506.

[L&T] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I,* Springer-Verlag, 1977, Berlin Heidelberg.

[Pie] A. Pietsch, *Operator Ideals,* North-Holland, Berlin, 1980.

[Pis] G. Pisier, *Factorization of Linear Operators and Geometry of Banach Spaces,* Regional Conference Series in Mathematics **60**, Amer. Math. Soc., Providence, Rhode Island, 1986.

[S] A. Szankowski, *B(H) does not have the approximation property,* Acta. Math. **147**(1981) 89–108.

[Sa] C. Samuel, *Bounded approximate identities in the algebra of compact operators on a Banach space,* Proc. Amer. Math. Soc., to appear.

[W] G. A. Willis, *The compact approximation property does not imply the approximation property,* forthcoming paper.

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