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STRONGLY OSCILLATORY AND NONOSCILLATORY SUBSPACES OF LINEAR EQUATIONS

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Consider the nth order linear equation

$$(1) \quad y^{(n)} + \sum_{k=1}^{n} p_k y^{(n-k)} = 0 \quad \text{where } p_k \in C[a, \infty), n \geqq 2$$

and particularly the third order equation

$$(2) \quad y''' + \sum_{k=1}^{3} p_k y^{(3-k)} = 0 \quad \text{where } p_k \in C[a, \infty).$$

A nontrivial solution of $(1)_n$ is said to be oscillatory or nonoscillatory depending on whether it has infinitely many or finitely many zeros on $[a, \infty)$. Let $\mathscr{S}$, $\mathscr{O}$, $\mathscr{N}$ denote respectively the set of all solutions, oscillatory solutions, non-oscillatory solutions of $(1)_n$. $\mathscr{S}$ is an n-dimensional linear space. A subspace $\mathscr{T} \subseteq \mathscr{S}$ is said to be nonoscillatory or strongly oscillatory respectively if every nontrivial solution of $\mathscr{T}$ is nonoscillatory or oscillatory. If $\mathscr{T}$ contains both oscillatory and nonoscillatory solutions then $\mathscr{T}$ is said to be weakly oscillatory. In case $\mathscr{T} = \mathscr{S}$ satisfies any of the above mentioned properties of $\mathscr{T}$ we sometimes attribute the same title to the equation directly.

The oscillatory behavior of equation $(1)_n$ is the subject of a vast quantity of literature. Good bibliographies on this subject can be found in Barrett [**1**] and Swanson [**10**]. Qualitatively, the question of oscillation is simple for $n = 2$, because Sturm's Theorem implies $\mathscr{N} = \emptyset$ or $\mathscr{O} = \emptyset$. For $n \geqq 3$ however such a simple qualitative result is not true. The literature, for $n \geqq 3$, abounds with results which indicate conditions when one or both of $\mathscr{N}$ and $\mathscr{O}$ are not empty. Also many results indicate the number of linearly independent solutions contained in $\mathscr{N}$ or $\mathscr{O}$; see Jones [**5**; **6**], Kondratév [**7**], Hanan [**4**], Lazer [**8**] and Utz [**11**; **12**] for results of this type. Our first theorem shows that for all $n \geqq 2$ either $\mathscr{N}$ or $\mathscr{O}$ contains n linearly independent solutions.

It should be noted that linear combinations of oscillatory or nonoscillatory solutions need not be oscillatory or nonoscillatory respectively. Dolan [**2**], Kondratév [**7**], Hanan [**4**] and Lazer [**8**] have determined conditions for which there are two-dimensional non-oscillatory or strongly oscillatory subspaces of $\mathscr{S}$ for $n = 3$ and Dolan [**2**] considered the problem of decomposing $\mathscr{S}$ into the direct sum of such subspaces. Our second theorem shows that such a decomposition always exists for $n = 3$ and we leave open the question for

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higher dimensions. These results depend heavily on the cone structure of $\mathscr{N}$ and consequently the results include interesting facts about convex cones in three space with appropriate open questions about convex cones in n space for $n > 3$.

In the article by Dolan and Klaasen [3] numerous examples are given of third order equations for which $\mathscr{N}$ contains three linear independent solutions or $\mathscr{O}$ contains three linearly independent solutions. The class I and II of Hanan [4] are specific classes of these examples. The following theorem indicates that this property happens for all $n \geqq 2$.

THEOREM 1. *Either $\mathscr{O}$ contains n linearly independent solutions or $\mathscr{N}$ contains n linearly independent solutions.*

Proof. Let us suppose $\mathscr{O}$ contains exactly p linearly independent solutions where $1 \leqq p < n$. Then we can write a basis for $\mathscr{S}$ of the form $\{y_1, \ldots, y_p, z_{p+1}, \ldots, z_n\}$ where $y_i \in \mathscr{O}$, $1 \leqq i \leqq p$ and $z_i \in \mathscr{N}$, $p + 1 \leqq i \leqq n$. Consequently, $y_i + z_n \in \mathscr{N}$ for all $1 \leqq i \leqq p$ for if on the contrary there is an i such that $y_i + z_n \in \mathscr{O}$ then $\{y_1, \ldots, y_p, y_i + z_n\}$ is a set of $p + 1$ linearly independent solutions in $\mathscr{O}$ which violates the definition of p. Hence

$$\{y_1 + z_n, \ldots, y_p + z_n, z_{p+1}, z_{p+2}, \ldots, z_n\}$$

is a basis for $\mathscr{S}$ of elements of $\mathscr{N}$.

In order to prove our second theorem we introduce some notations and a lemma.

The set $\mathscr{N}$ of nonoscillatory solutions of (1) can be decomposed into two disjoint sets $\mathscr{N}^+$ and $\mathscr{N}^-$ which denote respectively the eventually positive and eventually negative nonoscillatory solutions of (1). If $\mathscr{T} \subseteq \mathscr{S}$, let $\mathscr{T}_0 = \mathscr{T} \cup \{0\}$ where 0 is the zero solution of (1).

The concepts of convex set theory which are used in this paper are developed in Valentine [13].

LEMMA 1. *If $\mathscr{N} \neq \emptyset$ for equation* (1), *then $\mathscr{N}_0^+$ and $\mathscr{N}_0^-$ are convex cones.*

Proof. Since $\mathscr{N}_0^+ = -\mathscr{N}_0^-$ it is sufficient to show that $\mathscr{N}_0^+$ is a convex cone. If $y, z \in \mathscr{N}_0^+$ and $0 \leqq \alpha \leqq 1$ then αy, $(1 - \alpha)z \in \mathscr{N}_0^+$ and hence $\alpha y + (1 - \alpha)z \in \mathscr{N}_0^+$. Also if $\alpha \geqq 0$, $\alpha y \in \mathscr{N}_0^+$.

Since $\mathscr{S}$ is an n-dimensional space it can be isomorphically identified with E_n. For example if $\{y_1, y_2, \ldots, y_n\}$ is a basis for $\mathscr{S}$ over the reals R then the mapping h of E_n onto $\mathscr{S}$ defined by $h(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^n \alpha_i y_i$ is an algebraic isomorphism of E_n onto $\mathscr{S}$. It follows that convex sets are mapped to convex sets and cones to cones. Hence any theorem about convex cones in n-space implies results about $\mathscr{N}_0^+$ and $\mathscr{N}_0^-$ in $\mathscr{S}$.

Suppose $\mathscr{T} \subseteq E_3$. $C(T)$ denotes the complement of T in E_3, ΔT denotes $T \cup -T$ where $-T = \{-t | t \in T\}$.

The following theorem about convex cones in 3-space does not seem to appear in the vast literature on convex set theory.

THEOREM 2. *If K is a convex cone in E_3, then there is a 2-dimensional subspace H of E_3 such that* N $\subset \Delta K$ *or* $H \subset C(\Delta K)_0$.

Proof. $\bar{K}$ is a closed convex cone with vertex at 0. If $\bar{K} = E_3$ then since K is convex it is easy to see that K contains a 2-dimensional subspace. Suppose $\bar{K} \neq E_3$. Then $\bar{K}$ is the intersection of the closed half spaces containing it and determined by the supporting planes of $\bar{K}$ [**9**, p. 71, Exercise 24]. We will consider three alternate cases. First suppose $\bar{K}$ is contained in the intersection of three half spaces determined by three planes which intersect only at 0. In this case there is a plane in $C(\Delta K)_0$. Secondly suppose the intersection of the family of all supporting planes of $\bar{K}$ is a line l. If the line l is not in ΔK then one of these supporting planes is in $C(\Delta K)_0$. If a point $x \neq 0$ of the line l is in K then the entire line l is in ΔK and the plane determined by l and a point y of K but not of l is a 2-dimensional subspace contained in ΔK. Finally suppose $\bar{K}$ is a half space. Then either the plane π supporting $\bar{K}$ is contained in $C(\Delta K)_0$ or it contains a line l of ΔK. If π contains a line l of ΔK then by the denseness of K in the half space and the convexity of K it follows that the plane determined by l and $y \in K$ such that $y \notin \pi$ is in ΔK.

THEOREM 3. *The solution space $\mathscr{S}$ of equation* (2) *possesses a 2-dimensional subspace which is either strongly oscillatory or nonoscillatory.*

The validity of Theorem 3 is a direct consequence of Lemma 1 and Theorem 2.

COROLLARY 1. *The solution space $\mathscr{S}$ of equation* (2) *possesses a decomposition $\mathscr{S} = H_1 \oplus H_2$ such that H_1 is strongly oscillatory and H_2 is nonoscillatory. One of H_1 may be degenerate.*

Proof. If $\mathscr{S}$ is nonoscillatory or strongly oscillatory then $H_1 = \{0\}$ and $H_2 = \mathscr{S}$ or $H_1 = \mathscr{S}$ and $H_2 = \{0\}$ respectively. If $\mathscr{S}$ is weakly oscillatory, let H be the 2-dimensional subspace determined by Theorem 2. Since $\mathscr{S}$ is weakly oscillatory there is a solution $y \in \mathscr{S} \cap C(H)$ which is oscillatory if H is nonoscillatory and which is nonoscillatory if H is strongly oscillatory. Hence with $[y] \equiv \{\alpha y | \alpha \in R\}$, it is easy to see that $\mathscr{S} = H \oplus [y]$ in accordance with the conclusions of this corollary.

The following example points the direction of generalizations of the previous two theorems to n-space. The solution set, $\mathscr{S}$, of the equation

$$y^{(\text{iv})} - 4y''' + 6y'' - 4y' = 0$$

has $\{1, e^{2x}, e^x \sin x, e^x \cos x\}$ as a basis and $\mathscr{S} = H \oplus K$ where $H = \{\alpha +$

$\beta e^{2x}|\alpha, \beta \in R\}$ and $K = \{\alpha e^x \sin x + \beta e^x \cos x|\alpha, \beta \in R\}$. Notice that $H \subseteq \mathscr{N}_0$ and $K \subseteq \mathscr{O}_0$. It is easy to argue from vector space theory that if $\mathscr{S} = H_1 \oplus K_1$ where $H_1 \subseteq \mathscr{N}_0$ and $K_1 \subseteq \mathscr{O}_0$ then H_1 and K_1 must have the same dimension as H and K respectively. Hence $\mathscr{S}$ contains no three dimensional strongly oscillatory or nonoscillatory subspace. In fact one can argue that if the p_k are constants in equation (1) then the corresponding solution set $\mathscr{S}$ always possesses a decomposition as a direct sum of a nonoscillatory and strongly oscillatory subspace. Of course one of these may be degenerate.

The following two conjectures accentuate the ideas obtained from this example. The first deals with nth order linear equations and the second is the analogue conjecture for convex cones in n-space.

Conjecture 1. The solution space, $\mathscr{S}$, of (1) possesses a decomposition $\mathscr{S} = H_1 \oplus H_2$ such that H_1 is strongly oscillatory and H_2 is nonoscillatory. One of H_1 and H_2 may be degenerate.

Conjecture 2. If K is a convex cone in E_n, then there are subspaces H_1 and H_2 of E_n such that $H_1 \subseteq \Delta K$, $H_2 \subset C(\Delta K)_0$ and $E_n = H_1 \oplus H_2$. One of H_1 and H_2 may be degenerate.

Another avenue of interest exposed by Corollary 1 is the possibility of decomposing the operator determined by equation (1) into two operators which in some sense relate to the subspace H_1 and H_2 of Corollary 1.

F. Neuman, in a paper submitted to the Journal of Differential Equations, states a theorem equivalent to Theorem 3 of this paper. The approach is quite different and appears not to use the theory of convex cones.

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