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CONVOLUTION WITH MEASURES ON CURVES IN $\mathbb{R}^3$

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ABSTRACT. We study convolution properties of measures on the curves $(t^{a_1}, t^{a_2}, t^{a_3})$ in $\mathbb{R}^3$.

For $0 < a_1 < a_2 < a_3$ let Γ be the curve in $\mathbb{R}^3$ defined by

$$\Gamma(t) = (t^{a_1}, t^{a_2}, t^{a_3}), \quad t > 0.$$

Let σ be the measure on Γ defined by $d\sigma = t^{|a|/6-1}\,dt$ where $|a| = a_1 + a_2 + a_3$. A natural conjecture is that

$$\sigma * L^{\frac{3}{2}}(\mathbb{R}^3) \subseteq L^2(\mathbb{R}^3) \tag{1}$$

whenever $0 < a_1 < a_2 < a_3$. The earliest result here is the theorem in [O1] which shows that (1) holds when $(a_1, a_2, a_3) = (1, 2, 3)$. This is the limiting case $\delta = 1$ of the Theorem below. The affine arclength measures $d\sigma = t^{|a|/6-1}\,dt$ were introduced into the study of this problem (and another) by Drury [D], where (1) is established for $(a_1, a_2, a_3) = (1, 2, k)$ with $k \geq 4$. The best result so far is due to Pan [P3] (see also [P1] and [P2]): a change of variable shows that it is enough to establish (1) when $|a| = 6$ and then Pan shows that (1) holds whenever $a_1 \leq 1$. It seems impossible to push the argument in [P3]—the method of "cut curves"—any farther. It is the purpose of this note to support the conjecture by showing that (1) holds in certain cases not covered by [P3].

THEOREM. *Suppose* $0 < \delta < 1$. *Then (1) holds if* $a_1 = 2 - \delta$, $a_2 = 2$, $a_3 = 2 + \delta$.

The method we will employ is an adaptation of the method in [O2]. We begin with a lemma that will lead to a favorable estimate for the Fourier transform of the measure σ.

LEMMA. *Fix* $\delta \in (0, 1)$ *and, for* $b_1, b_2, b_3 \in \mathbb{R}$, *let* $p(t) = p(b_1, b_2, b_3; t)$ *be defined by* $p(t) = b_1 t^{2-\delta} + b_2 t^2 + b_3 t^{2+\delta}$, $t > 0$. *There is a constant* $C = C(\delta)$ *such that*

$$\left| b_1 b_3 - \frac{b_2^2}{4 - \delta^2} \right|^{\frac{1}{4}} \leq C \cdot \inf_{t>0} \sum_{j=1}^{4} |p^{(j)}(t)|^{\frac{1}{j}}. \tag{2}$$

PROOF. First observe that it is enough to show that

$$\left| b_1 b_3 - \frac{b_2^2}{4 - \delta^2} \right|^{\frac{1}{4}} \leq C \sum_{j=1}^{4} |p^{(j)}(1)|^{\frac{1}{j}}$$

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for any b_1, b_2, b_3. For if $s > 0$ and $q(t) = p(st)$, then (3) for q is just

$$\left|b_1 s^{2-\varepsilon} b_3 s^{2+\varepsilon} - \frac{(b_2 s^2)^2}{4-\delta^2}\right|^{\frac{1}{4}} \le C \sum_{j=1}^{4} |s^j p^{(j)}(s)|^{\frac{1}{j}}. \tag{3}$$

The proof of (3) is facilitated by a change of variable: for $j = 1, \dots, 4$ let $B_j = p^{(j)}(1)$. Solving for b_1, b_2, b_3 in terms of B_1, B_2, B_3 and then substituting into $b_1 b_3 - b_2^2/(4-\delta^2)$ gives

$$\frac{B_2^2 - 2B_1B_3 - (1-\delta^2)B_1^2}{4\delta^2(-2+\delta)(2+\delta)}.$$

Thus (3) will follow from the inequalities

$$|B_2|^{\frac{1}{2}}, |B_1B_3|^{\frac{1}{4}}, |B_1|^{\frac{1}{2}} \le C \sum_{j=1}^{4} |B_j|^{\frac{1}{j}}.$$

The first of these is clear and the second follows from Jensen's inequality:

$$|B_1B_3|^{\frac{1}{4}} \le \frac{|B_1|}{4} + \frac{3|B_3|^{\frac{1}{3}}}{4}.$$

The third inequality could fail only if $|B_j| \le 1$ for $1 \le j \le 4$, so we examine this possibility. Allowing C to vary from line to line and remembering that $|B_j| \le 1$, we observe that

$$|b_1|, |b_3| \le C(|B_3| + |B_4|)$$

and so

$$|b_1|^{\frac{1}{2}}, |b_3|^{\frac{1}{2}} \le C(|B_3|^{\frac{1}{2}} + |B_4|^{\frac{1}{2}}) \le C(|B_3|^{\frac{1}{3}} + |B_4|^{\frac{1}{4}}).$$

Also $|b_2| \le C(|B_2| + |b_1| + |b_3|)$, which gives

$$|b_2|^{\frac{1}{2}} \le C(|B_2|^{\frac{1}{2}} + |b_1|^{\frac{1}{2}} + |b_3|^{\frac{1}{2}}) \le C(|B_2|^{\frac{1}{2}} + |B_3|^{\frac{1}{3}} + |B_4|^{\frac{1}{4}}).$$

Thus

$$|B_1|^{\frac{1}{2}} \le C(|b_1|^{\frac{1}{2}} + |b_2|^{\frac{1}{2}} + |b_3|^{\frac{1}{2}}) \le C(|B_2|^{\frac{1}{2}} + |B_3|^{\frac{1}{3}} + |B_4|^{\frac{1}{4}})$$

as desired.

The next step in the proof is the estimate

$$\left|\int_I e^{ip(t)}\, dt\right| \le \frac{C}{|b_1 b_3 - \frac{b_2^2}{4-\delta^2}|^{\frac{1}{4}}}, \tag{4}$$

to hold with $C = C(\delta)$ for any b_1, b_2, b_3. We first observe that

(5) given b_1, b_2, b_3, there exists a partition of $\mathbb{R}$ into at most 1000 disjoint intervals I_ℓ such that for each ℓ there is $j' = j'(\ell) \in \{1, 2, 3, 4\}$ satisfying $|p^{(j')}(t)|^{\frac{1}{j'}} = \sup_{1 \le j \le 4} |p^{(j)}(t)|^{\frac{1}{j}}$, $t \in I_\ell$.

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(To check (5) just count solutions to the equations

$$\big(p^{(j_1)}(t)\big)^{j_2} = \pm\big(p^{(j_2)}(t)\big)^{j_1}, \quad 1 \le j_1 < j_2 \le 4$$

using the fact that if $c_1 < c_2 < \cdots < c_k$, then any nontrivial equation

$$\sum_{j=1}^{k} a_j t^{c_j} = a_0, \quad a_j \in \mathbb{R}$$

has at most k nonnegative solutions in t.) Now

$$\Big|\int_I e^{ip(t)}\,dt\Big| \le \sum_\ell \Big|\int_{I\cap I_\ell} e^{ip(t)}\,dt\Big|$$

and, for each ℓ,

$$\Big|\int_{I\cap I_\ell} e^{ip(t)}\,dt\Big| \le \frac{C}{|b_1b_3 - \frac{b_2^2}{4-\delta^2}|^{\frac14}}$$

by (5), (2), and van der Corput's Lemma. Thus (4) is established. It follows that the measure σ satisfies

$$|\hat{\sigma}(b_1,b_2,b_3)| \le \frac{C}{|b_1b_3 - \frac{b_2^2}{4-\delta^2}|^{\frac14}}.$$

Since (1) is a convolution estimate with L^2 as range, (1) will follow from the fact that

$$\Big|b_1b_3 - \frac{b_2^2}{4-\delta^2}\Big|^{-\frac14}$$

is a Fourier multiplier from $L^{\frac32}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. But a linear change of variables transforms $b_1b_3 - b_2^2/(4-\delta^2)$ into $c_1^2 - c_2^2 - c_3^2$. And it is easy to show by standard arguments (see [O2]) that

$$|c_1^2 - c_2^2 - c_3^2|^{-\frac14}$$

is a Fourier multiplier from $L^{\frac32}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$.

ADDED IN PROOF. S. Secco has recently proved the conjecture (1) with no restriction on $a_1 < a_2 < a_3$. Her result will appear in Mathematica Scandinavia.

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