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RESEARCH ARTICLE

Automorphy lifting with adequate image

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Abstract
Let F be a CM number field. We generalise existing automorphy lifting theorems for regular residually irreducible p-adic Galois representations over F by relaxing the big image assumption on the residual representation.

Contents

1. Introduction

This paper closely builds on [ACC+18], which proves modularity lifting theorems for regular n-dimensional Galois representations over a CM number field F without any self-duality condition. In this paper, we generalise the main results of [ACC+18] to relax the big image assumption on the residual representation from 'enormous image' to 'adequate image'. Following [Tho12], we define 'adequate image':

Definition 1.1. Let k be a finite field of characteristic p, such that $p \nmid n$, and let $G \subset \mathrm{GL}_n(k)$ be a subgroup which acts absolutely irreducibly on $V = k^n$. We suppose that k is large enough to contain all eigenvalues of all elements of G. If $g \in G$ and $\alpha \in k$ is an eigenvalue g, we write $e_{g,\alpha} : V \to V$ for the g-equivariant projection to the generalised α-eigenspace. We say that G is **adequate** if the following conditions are satisfied:

1. $H^0(G, \mathrm{ad}^0\, V) = 0$.
2. $H^1(G, k) = 0$.

3. $H^1(G, \mathrm{ad}^0 V) = 0$.
4. For every irreducible $k[G]$-submodule $W \subset \mathrm{ad}^0 V$, there exists an element $g \in G$ with an eigenvalue α, such that $\mathrm{tr}\, e_{g,\alpha} W \neq 0$.

Our main theorems are as follows:

Theorem 1.2. *Let F be an imaginary CM or totally real field, let $c \in \mathrm{Aut}(F)$ be complex conjugation and let p be a prime. Suppose given a continuous representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

1. *ρ is unramified almost everywhere.*
2. *For each place $v \mid p$ of F, the representation $\rho|_{G_{F_v}}$ is crystalline. The prime p is unramified in F.*
3. *$\overline{\rho}$ is absolutely irreducible and decomposed generic. The image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is adequate.*
4. *There exists $\sigma \in G_F - G_{F(\zeta_p)}$, such that $\overline{\rho}(\sigma)$ is a scalar. We have $p > n^2$.*
5. *There exists a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ satisfying the following conditions:*
   (a) *π is regular algebraic of weight λ, this weight satisfying*
   $$\lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau,n} - \lambda_{\tau c,n} < p - 2n$$
   for all τ.
   (b) *There exists an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, such that $\overline{\rho} \cong \overline{r_\iota(\pi)}$, and the Hodge-Tate weights of ρ satisfy the formula for each $\tau : F \hookrightarrow \overline{\mathbf{Q}}_p$:*
   $$HT_\tau(\rho) = \{\lambda_{\iota\tau,1} + n - 1, \lambda_{\iota\tau,2} + n - 2, \ldots, \lambda_{\iota\tau,n}\}.$$
   (c) *If $v \mid p$ is a place of F, then π_v is unramified.*

Then ρ is automorphic: there exists a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ, such that $\rho \cong r_\iota(\Pi)$. Moreover, if v is a finite place of F and either $v \mid p$ or both ρ and π are unramified at v, then Π_v is unramified.

Theorem 1.3. *Let F be an imaginary CM or totally real field, let $c \in \mathrm{Aut}(F)$ be complex conjugation and let p be a prime. Suppose given a continuous representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

1. *ρ is unramified almost everywhere.*
2. *Let $\mathbf{Z}^n_+ = \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n \mid \lambda_1 \geq \ldots \geq \lambda_n\}$. For each place $v \mid p$ of F, the representation $\rho|_{G_{F_v}}$ is potentially semistable, ordinary with regular Hodge-Tate weights. In other words, there exists a weight $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F, \overline{\mathbf{Q}}_p)}$, such that for each place $v \mid p$, there is an isomorphism*
   $$\rho|_{G_{F_v}} \sim \begin{pmatrix} \psi_{v,1} & * & * & * \\ 0 & \psi_{v,2} & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \psi_{v,n} \end{pmatrix},$$
   where for each $i = 1, \ldots, n$ the character $\psi_{v,i} : G_{F_v} \to \overline{\mathbf{Q}}_p^\times$ agrees with the character
   $$\sigma \in I_{F_v} \mapsto \prod_{\tau \in \mathrm{Hom}(F_v, \overline{\mathbf{Q}}_p)} \tau(\mathrm{Art}_{F_v}^{-1}(\sigma))^{-(\lambda_{\tau,n-i+1}+i-1)}$$
   on an open subgroup of the inertia group I_{F_v}.

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3. *$\overline{\rho}$ is absolutely irreducible and decomposed generic. The image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is adequate.*
4. *There exists $\sigma \in G_F - G_F(\zeta_p)$, such that $\overline{\rho}(\sigma)$ is a scalar. We have $p > n$.*
5. *There exists a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, such that π is ι-ordinary and $\overline{\rho} \cong \overline{r_\iota(\pi)}$.*

Then ρ is ordinarily automorphic of weight $\iota\lambda$: there exists a ι-ordinary cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\iota\lambda$, such that $\rho \cong r_\iota(\Pi)$. Moreover, if $v \nmid p$ is a finite place of F and both ρ and π are unramified at v, then Π_v is unramified.

The theorems above are very similar to [ACC+18, Theorem 6.1.1] and [ACC+18, Theorem 6.1.2], respectively. The only difference is replacing the *enormous* condition on image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ with *adequate*. This is a useful improvement, particularly in light of [GHTT12], which proves that when $p > 2(n+1)$, adequacy is equivalent to absolute irreducibility. This makes it a condition easy to work with in the context of automorphy of compatible systems, which we hope would help generalise [BLGGT14] to the context of [ACC+18] and this paper. We now give a brief overview of the argument. The main change in comparison to [ACC+18] is the usage of parahoric-level subgroups at Taylor-Wiles primes instead of Iwahori-level, the idea first introduced to relax the big image assumption in the setting of automorphy lifting theorems to 'adequate' in [Tho12]. To make the argument work in the parahoric setting, we need to analyse the representations of $\mathrm{GL}_n(F_v)$ with fixed vectors under various parahoric subgroups and their interactions with the local Langlands correspondence. A notable difficulty in comparison to [Tho12] is that we can no longer restrict to working with generic local representations, since they arise as components of cuspidal automorphic representations of unitary groups instead of GL_n. The local computations allow us to prove the necessary local-global compatibility results for Galois representations landing in Hecke algebras acting on cohomology of locally symmetric spaces with parahoric level. Another novel component is a proof of a 'growth of the space of cusp forms'-type result when adding Taylor-Wiles primes with parahoric level, which requires an investigation of representations of $\mathrm{GL}_n(F_v)$ over fields of finite characteristic.

1.1. Notation

We write GL_n for the usual general linear group (viewed as a reductive group scheme over $\mathbf{Z}$) and $T_n \subset B_n \subset GL_n$ for its subgroups of diagonal and of upper triangular matrices, respectively. We identify $X^*(T)$ with $\mathbf{Z}^n$ in the usual way and write $\mathbf{Z}^n_+ \subset \mathbf{Z}^n$ for the subset of B_n-dominant weights. If R is a local ring, we write $\mathfrak{m}_R$ for the maximal ideal of R. If Γ is a profinite group and $\rho : \Gamma \to GL_n(\overline{\mathbf{Q}}_p)$ is a continuous homomorphism, then we will let $\overline{\rho} : \Gamma \to GL_n(\overline{\mathbf{F}}_p)$ denote the *semisimplification* of its reduction, which is well defined up to conjugacy (by the Brauer-Nesbitt theorem). If M is a topological abelian group with a continuous action of Γ, then by $H^i(\Gamma, M)$, we shall mean the continuous cohomology. If G is a locally profinite group, $U \subset G$ is an open compact subgroup and R is a commutative ring, then we write $\mathcal{H}_R(G, U)$ for the algebra of compactly supported, U-biinvariant functions $f : G \to R$, with multiplication given by convolution with respect to the Haar measure on G which gives U volume 1. If $X \subset G$ is a compact U-biinvariant subset, then we write $[X]$ for the characteristic function of X, an element of $\mathcal{H}_R(G, U)$. When R is omitted from the notation, we take $R = \mathbf{Z}$. We write $\iota_{\mathcal{H}}$ for the anti-involution given by $\iota_{\mathcal{H}}(f)(g) = f(g^{-1})$.

If F is a perfect field, we let $\overline{F}$ denote an algebraic closure of F and G_F the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. We will use ζ_n to denote a primitive n-th root of unity when it exists. Let ϵ_l denote the l-adic cyclotomic character. We will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible complex admissible representation of $GL_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Frobenius semisimple Weil-Deligne representation of the Weil group W_K. If K is a finite extension of $\mathbf{Q}_p$ for some p, we write K^{nr} for its maximal unramified extension, I_K for the inertia subgroup of G_K, $\mathrm{Frob}_K \in G_K/I_K$ for the geometric Frobenius and W_K for the Weil group. We will write $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ for the Artin map normalised to send uniformisers to geometric Frobenius elements.

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We will write rec for rec_K when the choice of K is clear. We write rec_K^T for the normalisation of the local Langlands correspondence as defined in, for example [CT14, Section 2.1]; it is defined on irreducible admissible representations of $GL_n(K)$ defined over any field which is abstractly isomorphic to $\mathbf{C}$ (e.g. $\overline{\mathbf{Q}}_l$). If (r, N) is a Weil-Deligne representation of W_K, we will write $(r, N)^{F-ss}$ for its Frobenius semisimplification. If ρ is a continuous representation of G_K over $\overline{\mathbf{Q}}_l$ with $l \neq p$, then we will write $WD(\rho)$ for the corresponding Weil-Deligne representation of W_K. By a Steinberg representation of $GL_n(K)$, we will mean a representation $Sp_n(\psi)$ (in the notation of Section 1.3 of [HT01]), where ψ is an unramified character of $K^\times$.

If G is a reductive group over K and P is a parabolic subgroup with unipotent radical N and Levi component L, and if π is a smooth representation of $L(K)$, then we define $\mathrm{Ind}_{P(K)}^{G(K)}\pi$ to be the set of locally constant functions $f : G(K) \to \pi$, such that $f(hg) = \pi(hN(K))f(g)$ for all $h \in P(K)$ and $g \in G(K)$. It is a smooth representation of $G(K)$, where $(g_1 f)(g_2) = f(g_2 g_1)$. This is sometimes referred to as 'un-normalised' induction. We let δ_P denote the determinant of the action of L on Lie_N. Then we define the 'normalised' induction $\mathrm{ind}_{P(K)}^{G(K)}\pi$ to be $\mathrm{Ind}_{P(K)}^{G(K)}(\pi \otimes |\delta_P|_K^{1/2})$. We also define a parabolic restriction functor $r_{G(K)}^{P(K)}$ from $G(K)$-representations to $L(K)$-representations to be the composition of restriction to $P(K)$ and taking $N(K)$-coinvariants. If F is a CM number field and π is an automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, we say that π is regular algebraic if π_∞ has the same infinitesimal character as an irreducible algebraic representation W of $(\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n)_{\mathbf{C}}$. If $W^\vee$ has highest weight $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\mathbf{C})}$, then we say π has weight λ.

If $P(X) \in A[X]$ is a polynomial of degree n over any ring A, such that $P(0) \in A^\times$, we write $P^\vee(X)$ for $P(0)^{-1}X^n P(X^{-1})$. For two polynomials $P, Q \in A[X]$, we write $\mathrm{Res}(P, Q)$ to denote their resultant.

Given a Galois representation $\rho : G_{F,S} \to \mathrm{GL}_n(A)$, we will write $\rho^\perp := \rho^{c,\vee} \otimes \epsilon^{1-2n}$, and given a $G_{F,S}$-group determinant D, we will denote by $D^\perp$ the corresponding dual.

2. Representation theory of $\mathrm{GL}_n(F_v)$ in characteristic p

Let p be a rational prime and $k = \overline{\mathbf{F}}_p$. Let $F/\mathbf{Q}$ be a finite extension, and let x be a prime in F with residue field k_x of order q satisfying $q \equiv 1 \pmod p$ and the corresponding ring of integers $\mathcal{O}_x = \mathcal{O}_{F_x}$. Set $G_x = Gal(\overline{F}_x/F_x)$. Also set $G = GL_n$ with $p > n$, and let $T \subset B \subset G$ be the maximal torus and the corresponding Borel and $U \subset G$ be the unipotent subgroup. Let $K^1(x) \subset G(\mathcal{O}_x)$ be the full congruence subgroup. We also let $\mathrm{Iw}, \mathrm{Iw}_1 \subset G(\mathcal{O}_x)$ be the Iwahori and the Iwahori-1, respectively, and let $\mathrm{Iw}_1 \subset \mathrm{Iw}^p \subset \mathrm{Iw}$ be the subgroup, such that $[\mathrm{Iw}^p : \mathrm{Iw}_1]$ has order prime to p and $[\mathrm{Iw} : \mathrm{Iw}^p]$ has p-power order. Let $\mathfrak{p}(x)$ be a two-block parahoric subgroup of $G(\mathcal{O}_x)$ with blocks of sizes $n_1 + n_2 = n$ and P the corresponding parabolic. Let $W \cong S_n$ be the Weyl group for GL_n, and for a given parabolic subgroup $Q \subset G$, let $W_Q \subset W$ be the Weyl group of its Levi factor. Set $T_0 := T(\mathcal{O}_x)$ and $T_1 := \ker(T_0 \to T(\mathcal{O}_x/\varpi))$. Fix $\overline{\rho} : G_x \to \mathrm{GL}_n(k)$—a continuous unramified semisimple representation. We say that an irreducible admissible representation π of G over k is associated to $\overline{\rho}$ if π is a subquotient of $\mathrm{Ind}_B^G \chi_1 \otimes \ldots \otimes \chi_n$, where χ_i are unramified characters, such that $\{\chi_1(\varpi), \ldots, \chi_n(\varpi)\}$ is the set of eigenvalues of $\overline{\rho}(\mathrm{Frob}_x)$. We write $I(\chi)$ for $\mathrm{Ind}_B^G \chi_1 \otimes \ldots \otimes \chi_n$. The following lemma shows that if we do not fix the ordering of χ_i, then we can always consider π to be a subrepresentation of $I(\chi)$.

Proposition 2.1. *Let π be an irreducible admissible $k[G]$-module associated to $\overline{\rho}$. Then there exists an ordering of $\chi_1, \ldots, \chi_n$, such that π is a subrepresentation of $I(\chi)$.*

Proof. We use the adjunction between Ind_B^G and the parabolic restriction r_B^G to get an isomorphism

$$\mathrm{Hom}(\pi, I(\chi)) \cong \mathrm{Hom}(r_G^B(\pi), \chi).$$

Since π is associated to $\overline{\rho}$, we know that $r_G^B(\pi) \neq 0$. Since $r_G^B(\pi)$ is a representation of the torus, there exists a 1-dimensional quotient given by some character $\chi : T \to k^\times$. Then we get that $\mathrm{Hom}(\pi, I(\chi)) \neq 0$, and since π is irreducible, this implies that π is a subrepresentation of $I(\chi)$. Then χ forms the

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supercuspidal support of π and in fact has to be a permutation of the original $\chi_1, \ldots, \chi_n$. For the notion of supercuspidal support in positive characteristic, see [Vig96, II.2.6]. We would also like to remark, here, that in the case $q \equiv 1 \pmod p$, $p > n$, the notions of cuspidal and supercuspidal representations coincide (see [Vig96, II.3.9]). □

We now describe the Bernstein presentation of Iwahori-Hecke algebra $\mathcal{H}_k(G, \mathrm{Iw})$, following [Vig96, I.3.14]. Let

$$t_j = \mathrm{diag}(\underbrace{\varpi, \ldots \varpi}_{j}, 1 \ldots, 1),$$

and set $T_j = [\mathrm{Iw}\, t_j\, \mathrm{Iw}]$ and $X^j = T_j (T_{j-1})^{-1}$. We also let s_j be the permutation matrix corresponding to the transposition $(j, j+1)$ and set $S^j = [\mathrm{Iw}\, s_j\, \mathrm{Iw}]$. The elements X^j for $1 \leq j \leq n$ generate the group algebra $k[\mathbf{Z}^n]$ on which S_j acts by permuting the indices. The Bernstein presentation states that

$$\mathcal{H}_k(G, \mathrm{Iw}) \cong k[S_n \ltimes \mathbf{Z}^n]$$

under the action described above.

Now we introduce some useful Hecke operators. For any ring R, $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$ let $V^{j,2} \in \mathcal{H}_R(G, \mathfrak{p}(x))$ be the Hecke operator associated to the double coset

$$[\mathfrak{p}(x)\, \mathrm{diag}(\underbrace{1, \ldots, 1}_{n_1}, \underbrace{\varpi, \ldots, \varpi}_{j}, \underbrace{1, \ldots, 1}_{n_2 - j})\mathfrak{p}(x)]$$

and let $V^{i,1}$ be associated to

$$[\mathfrak{p}(x)\, \mathrm{diag}(\underbrace{\varpi, \ldots, \varpi}_{i}, 1 \ldots, 1)\mathfrak{p}(x)].$$

The following is part of [CHT08, Theorem B.1]:

Proposition 2.2. *Let V be an irreducible admissible $k[G]$-module, which is generated by its Iwahori-invariant vectors. Then $V^{\mathrm{Iw}} = V^{\mathrm{Iw}_1}$.*

Under the conditions of 2.2, we thus get an isomorphism

$$\begin{aligned} H^1(\mathrm{Iw}, V) &\cong H^1(B(k), V^{K^1(x)}) \cong H^1(T(k), V^{\mathrm{Iw}_1}) \\ &\cong H^1(T(k), V^{\mathrm{Iw}}) \cong \mathrm{Hom}(T(k), V^{\mathrm{Iw}}). \end{aligned} \tag{2.3}$$

Both sides of 2.3 can be endowed with the action of $\mathcal{H}_k(G, \mathrm{Iw})$. On $H^1(\mathrm{Iw}, V)$, we take the derived $\mathcal{H}_k(G, \mathrm{Iw})$-action, and on $\mathrm{Hom}(T(k), V^{\mathrm{Iw}})$, we consider the natural action on the target.

Proposition 2.4. *The isomorphism 2.3 is equivariant with respect to X^i for all $1 \leq i \leq n$.*

Proof. The action of X^i on $[f] \in H^1(\mathrm{Iw}, V)$ can be described as follows. Write

$$\mathrm{Iw}\, t_i\, \mathrm{Iw} = \bigsqcup_j g_{i,j}\, \mathrm{Iw}\,.$$

We now give an explicit description for $g_{i,j}$. Fix a set of representatives $S \subset \mathcal{O}_F$ for k. For each $m \in M_{i \times (n-i)}(S)$, let $g_{i,m}$ be the matrix, such that $g_{i,m}(k,k) = \varpi$ for $k \leq i$, $g_{i,m}(k,k) = 1$ for $k > i$ and $g_{i,m}(k,\ell) = m(k, \ell - i)$ for $k \leq i, \ell > i$. The rest of the entries are set to 0. Let us show that this is

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a full set of representatives. First we show that $g_{i,m}$ represent distinct cosets, that is that $g_{i,m}^{-1}g_{i,m'} \notin \mathrm{Iw}$ for $m \neq m'$. Suppose $m(k,\ell) \neq m'(k,\ell)$. Then

$$(g_{i,m}^{-1}g_{i,m'})(k,\ell+i) = \varpi^{-1}(m'(k,\ell) - m(k,\ell))$$

which is not in $\mathcal{O}_F$. Now we just need to verify that the number of cosets is $q^{i(n-i)}$. Indeed,

$$[\mathrm{Iw}\, t_i\, \mathrm{Iw} : \mathrm{Iw}] = [\mathrm{Iw} : \mathrm{Iw} \cap t_i\, \mathrm{Iw}\, t_i^{-1}] = q^{i(n-i)}$$

since $\mathrm{Iw} \cap t_i\, \mathrm{Iw}\, t_i^{-1}$ are just the elements of the Iwahori whose (k,ℓ)-coordinates for $k \leq i, \ell > i$ vanish mod ϖ.

Then

$$(X^i[f])(x) = \sum_j g_{i,\sigma(j)} f(g_{i,\sigma(j)}^{-1} x g_{i,j}),$$

where σ is the unique permutation, such that

$$g_{i,\sigma(j)}^{-1} x g_{i,j} \in \mathrm{Iw}$$

for all j. Denote by $\overline{} : \mathrm{Iw} \to T(k)$ the reduction map. Let s be the inverse of 2.3. For $[\tau] \in \mathrm{Hom}(T(k), V^{\mathrm{Iw}})$, we get

$$\begin{aligned}(X^i[s(\tau)])(x) &= \sum_j g_{i,\sigma(j)} s(\tau)(\overline{g_{i,\sigma(j)}^{-1} x g_{i,j}}) \\ &= \sum_j g_{i,\sigma(j)} s(\tau)(\overline{x}) = s(X^i[\tau])(x).\end{aligned}$$

The second equality is due to all the $g_{i,j}$ being in the Borel and having the same diagonal. □

Definition 2.5. A G-modules V over k is *locally admissible* if it is smooth, and for every $v \in V$ the subrepresentation generated by v is admissible. Let $\mathcal{C}$ denote the abelian category of locally admissible G-modules V over k, such that every irreducible subquotient of V is associated to $\overline{\rho}$.

The following is analogous to [CG18, Lemma 9.14]:

Proposition 2.6. *The category $\mathcal{C}$ has enough injectives, and the inclusion functor from $\mathcal{C}$ to locally admissible G-modules is exact.*

Proof. Inside the category of G-modules, the category $\mathcal{C}$ is fully contained inside the unipotent block (the block containing the trivial representation). By part 4) of [CHT08, Theorem B.1], the unipotent block is equivalent to the category of $\mathcal{H}_k(G, \mathrm{Iw}^p)$-modules. Via the Bernstein embedding[1], such modules can naturally be viewed as $\mathcal{H}_k(G, G(\mathcal{O}_x))$-modules, where $\mathcal{H}_k(G, G(\mathcal{O}_x))$ can be explicitly described via the Satake isomorphism as $k[X_1^{\pm1}, \ldots, X_n^{\pm1}]^W$. Here, we use the Satake isomorphism twisted by $|\det|^{(1-n)/2}$, which is defined over $\mathbf{Z}[q^{-1}]$. If V is any locally admissible element of the unipotent block, the associated Hecke module V^{Iw^p} is locally finite-dimensional over k, and thus we can write

$$V^{\mathrm{Iw}^p} = \bigoplus_{\mathfrak{m}} V_{\mathfrak{m}}^{\mathrm{Iw}^p},$$

where the sum is taken over all maximal ideals of $\mathcal{H}_k(G, G(\mathcal{O}_x))$. Let $\mathcal{D}$ denote the category of locally admissible representations in the unipotent block. Then we can write $\mathcal{D} = \bigoplus_{\mathfrak{m}} \mathcal{D}_{\mathfrak{m}}$, where $\mathcal{D}_{\mathfrak{m}}$ consists

[1]For the details on the Bernstein embedding $k[\mathbf{Z}^n] \to \mathcal{H}_k(G, I)$ in the case of an arbitrary open compact subgroup $I \subset \mathrm{Iw}$, such that $\mathrm{Iw}_1 \subset I$, see [ACC+18, Section 2.2.4]. We note that there the authors are working over some p-adic ring $\mathcal{O}$, but the results are valid over k as well since $q \equiv 1 \pmod{p}$.

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of representations whose associated $\mathcal{H}_k(G, G(\mathcal{O}_x))$-module is supported only at $\mathfrak{m}$. The maximal ideals of $\mathcal{H}_k(G, G(\mathcal{O}_x))$ have the form $(t_1 - \alpha_1, \dots, t_n - \alpha_n)$, where $\alpha_i \in k$ and $t_i = e_i(X_1, \dots, X_n)$ is the i-th elementary symmetric polynomial of $X_1, \dots, X_n$. If we now let $\mathfrak{n}$ be the ideal defined by $\alpha_i = e_i(\chi_1(\varpi), \dots, \chi_n(\varpi))$, then it is clear that $\mathcal{C} = \mathcal{D}_{\mathfrak{n}}$. The exactness is now clear, and to show that $\mathcal{C}$ has enough injectives, it is enough to check that the category $\mathrm{Mod}_G^{\mathrm{l.adm.}}(k)$ of locally admissible G-modules has enough injectives. The full category $\mathrm{Mod}_G(k)$ certainly has enough injectives, and the functor $\mathcal{L} : \mathrm{Mod}_G(k) \to \mathrm{Mod}_G^{\mathrm{l.adm.}}(k)$ taking a module to its smooth locally admissible vectors is right adjoint to the natural embedding $\mathrm{Mod}_G^{\mathrm{l.adm.}}(k) \to \mathrm{Mod}_G(k)$. This proves the claim. □

From now on, fix $\alpha = \chi_i(\varpi)$ for some $1 \le i \le n$. Let

$$P(X) = \prod_{i=1}^{n} (X - \chi_i(\varpi)).$$

For $1 \le j \le n_2$, let P_j be a polynomial whose roots with multiplicities are precisely

$$\sum_{\substack{J \subset S \\ \#J = j}} \prod_{a \in J} \chi_a(\varpi).$$

Factor $P_j = Q_j R_j$, where

$$R_j(X) = \left(X - \binom{n_2}{j} \alpha^j \right)^{k_j}$$

and Q_j, R_j are coprime. Set

$$e_\alpha := \lim_{m \to \infty} \left(\prod_{i=1}^{n_2} Q_j(V^{j,2}) \right)^{m!}.$$

Here, we consider e_α as an operator acting on $V^{\mathfrak{p}(x)}$ for $V \in \mathcal{C}$. Since objects in $\mathcal{C}$ are locally admissible, the limit makes sense.

We now define two functors $F, G : \mathcal{C} \to k-\mathbf{Vect}$. On objects, we set

$$F(V) := V^{G(\mathcal{O}_x)}, \qquad G(V) := e_\alpha V^{\mathfrak{p}(x)}.$$

Note that F, G are both left-exact and e_α is exact. Then we can form derived functors $R^k F, R^k G$ and identify

$$R^k F(V) = H^k(G(\mathcal{O}_x), V), \qquad R^k G(V) = e_\alpha H^k(\mathfrak{p}(x), V).$$

We have a natural transformation $\iota : F \to G$ given by composing the inclusion $V^{G(\mathcal{O}_x)} \hookrightarrow V^{\mathfrak{p}(x)}$ with e_α. We will make use of the following simple algebraic fact.

Lemma 2.7. *Let G be a profinite group and $H \lhd G$ be a normal subgroup. Let A be a p-torsion G-module for some positive integer p, and let H have pro-q order for a prime q satisfying $q \equiv 1 \pmod p$. Then the inflation map*

$$\inf : H^1(G/H, A^H) \to H^1(G, A)$$

is an isomorphism whose inverse sends a cocycle $[f] \in H^1(G, A)$ to

$$g \mapsto f(g) + (1 - g) a_f$$

for some $a_f \in A$.

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Proof. The condition $q \equiv 1 \pmod{p}$ ensures that $H^1(H, A)$ vanishes. Then it is enough to take $(g-1)a_f$ to be the coboundary trivialising the restriction of $[f]$ to H. □

Proposition 2.8. *Let π be an irreducible admissible $k[G]$-module associated to $\overline{\rho}$. Then the map*

$$f : H^1(G(k), \pi^{K^1(x)}) \to e_\alpha H^1(P(k), \pi^{K^1(x)})$$

is injective.

Proof. Both cohomology groups in question inject into $H^1(B(k), \pi^{K^1(x)})$ since

$$[G(k) : B(k)] \equiv n! \not\equiv 0 \pmod{p}$$

when $p > n$, so let us analyse that group. Since $q \equiv 1 \pmod{p}$, by inflation-restriction, we get

$$H^1(B(k), \pi^{K^1(x)}) \cong H^1(T(k), \pi^{\mathrm{Iw}_1}).$$

As a special case of 2.3, we have

$$H^1(\mathrm{Iw}, \pi) \cong H^1(B(k), \pi^{K^1(x)}) \cong \mathrm{Hom}(T(k), \pi^{\mathrm{Iw}}) \cong (\pi^{\mathrm{Iw}})^{\oplus n}. \tag{2.9}$$

The isomorphism above is equivariant with respect to the natural actions of $\{X^i\}$ on both sides arising from the actions of $\mathcal{H}_k(G, \mathrm{Iw})$ by Proposition 2.4. The space π^{Iw} injects into $I(\chi)^{\mathrm{Iw}}$, which has a basis $\{\varphi_w\}$ for $w \in W$, where φ_w is supported on $Bw\,\mathrm{Iw}$ and satisfies $\varphi_w(w) = 1$. It follows from the proof of [Tho12, Lemma 5.10], that on each component of $(I(\chi)^{\mathrm{Iw}})^{\oplus n}$, the operator e_α acts as a projection onto the space spanned by $\{\varphi_{w'} \mid w' \in W'\}$, where W' is the subset of W consisting of permutations which send $\{n_1 + 1, \ldots, n\}$ to the positions of α-s in the sequence $\chi_1(\omega), \ldots, \chi_n(\omega)$. On the level of cocycles, the isomorphism 2.9 sends $[s] \in H^1(B(k), \pi^{K^1(x)})$ to the map

$$g \mapsto s(g) + (1-g)\psi$$

for some $\psi \in I(\chi)$ (Lemma 2.7). Thus, a cocycle $[s] \in H^1(G(k), I(\chi)^{K^1(x)})$ being in the kernel of f means that for all $t \in T(k)$ and $w_0 \in W'$, we have

$$(s(t) + (1-t)\psi)(w_0) = 0. \tag{2.10}$$

For any $w \in W$, we have

$$(t\psi)(w) = \psi(w\tilde{t}) = \psi(w(\tilde{t})w) = \psi(w).$$

Here, $\tilde{t}$ is a lift of t to T_0 and w acts on the torus in a natural way. Note that here, we used that χ is unramified. Thus

$$((1-t)\psi)(w) = 0. \tag{2.11}$$

Combining 2.10 and 2.11 applied to w_0, we get

$$s(t)(w_0) = 0.$$

Now let us conjugate t by an arbitrary $w \in W$. Since the result is again in T, we use the cocycle condition and the transformation law of $I(\chi)$ with respect to the Borel to write

$$0 = s(wtw^{-1})(w_0) = (s(w) + w(s(t) + ts(w^{-1})))(w_0) \tag{2.12}$$

$$(wts(w^{-1}))(w_0) = ws(w^{-1})(w_0) = -s(w)(w_0). \tag{2.13}$$

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Combining 2.12 and 2.13, we get

$$0 = (ws(t))(w_0) = s(t)(w_0 w).$$

In other words, we now have $s(t)(w) = 0$ for all $t \in T(k)$ *and* for all $w \in W$. By 2.11, this implies that $[s] = 0$ since $\{\varphi_w\}$ make a basis for $I(\chi)^{\mathrm{Iw}}$. □

Theorem 2.14. *The natural transformation $\iota : F \to G$ given by $V^{G(\mathcal{O}_x)} \mapsto e_\alpha V^{\mathfrak{p}(x)}$ on objects is an isomorphism of functors. In particular, we get functorial isomorphisms*

$$\iota_* : H^k(G(\mathcal{O}_x), V) \xrightarrow{\sim} e_\alpha H^k(\mathfrak{p}(x), V)$$

for all $k \geq 0$.

Proof. In the proof of Proposition 2.6, we have identified $\mathcal{C}$ with a subcategory of $\mathcal{H}_k(G, \mathrm{Iw}^p)$-Mod. Thus, every element of $\mathcal{C}$ is a direct limit of finite length elements of $\mathcal{C}$, and it is, therefore, enough to establish the isomorphism for finite length V. The first step will be to show that $\iota(V)$ is an isomorphism for all $V \in \mathcal{C}$. For an irreducible subrepresentation $\pi \subset V$, consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(\pi) & \longrightarrow & F(V) & \longrightarrow & F(V/\pi) & \longrightarrow & R^1F(\pi) \\ & & \downarrow{\scriptstyle \iota(\pi)} & & \downarrow{\scriptstyle \iota(V)} & & \downarrow{\scriptstyle \iota(V/\pi)} & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & G(\pi) & \longrightarrow & G(V) & \longrightarrow & G(V/\pi) & \longrightarrow & R^1G(\pi). \end{array} \tag{2.15}$$

To show that $\iota(V)$ is injective, we can use the four lemmas and induct on the length of V. Thus, we only need to show that $\iota(\pi)$ is injective for irreducible π. This is done in [Tho12, Lemma 5.10].

Now we would like to show that $\iota(\pi)$ is an isomorphism. Consider the injection $\pi \subset I(\chi)$ and the associated diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(\pi) & \longrightarrow & F(I(\chi)) & \longrightarrow & F(I(\chi)/\pi) \\ & & \downarrow{\scriptstyle \iota(\pi)} & & \downarrow{\scriptstyle \iota(I(\chi))} & & \downarrow{\scriptstyle \iota(I(\chi)/\pi)} \\ 0 & \longrightarrow & G(\pi) & \longrightarrow & G(I(\chi)) & \longrightarrow & G(I(\chi)/\pi). \end{array} \tag{2.16}$$

We already know that $\iota(I(\chi)/\pi)$ is injective. Then to show that $\iota(\pi)$ is surjective by the four lemmas, we need to know that $\iota(I(\chi))$ is surjective. This follows once again from the proof of [Tho12, Lemma 5.10].

Finally, we are ready to see that $\iota(V)$ is an isomorphism for all $V \in \mathcal{C}$. We induct on the length of V using Eq. 2.15. Since f is injective by Proposition 2.8, the result follows. □

3. Representation theory of $\mathrm{GL}_n(F_v)$ in characteristic 0

Fix a finite extension $E/\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$ which contains the images of all embeddings $F \to \overline{\mathbf{Q}}_p$. We write $\mathcal{O}$ for the ring of integers of E and $\varpi \in \mathcal{O}$ for a choice of uniformiser. If v is a finite place of F prime to p, we write $\Xi_v := \mathbf{Z}^n$ and $\Xi_{v,1} := \langle \tau_v \rangle \times \mathbf{Z}^n$, where τ_v is the generator of $k_v^\times(p)$—the maximal p-power order quotient of $k_v^\times$. We have a natural homomorphism $\mathcal{O}_{F_v}^\times \to \mathbf{Z}[\Xi_{v,1}]$ induced by the homomorphism $\mathcal{O}_{F_v}^\times \to k_v^\times \to k_v^\times(p)$, which we denote by $\langle \cdot \rangle$. Consider a standard parabolic subgroup $P \subset \mathrm{GL}_n(F_v)$ corresponding to a partition $n = n_1 + \ldots + n_m$ which we will denote as μ. Given a partition of n, we will always let $s_{\mu,i} = n_1 + \ldots + n_i$, with $s_{\mu,0} = 0$. Let $P = MN$ and $\overline{P} = M\overline{N}$ be the Levi decompositions of P and its opposite parabolic. Let $\mathfrak{m}$ be the hyperspecial maximal compact subgroup of M. Define the subgroup of the symmetric group $S_\mu = S_{n_1} \times \ldots \times S_{n_m}$. For any positive integer k, let

$$\mathcal{S}_k : \mathcal{H}_{\mathbf{Z}[q_v^{1/2}]}(\mathrm{GL}_k(F_v), \mathrm{GL}_k(\mathcal{O}_{F_v})) \to \mathbf{Z}[q_v^{1/2}][X_1^{\pm 1}, \ldots, X_k^{\pm 1}]^{S_k}$$

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denote the (normalised) Satake isomorphism. We use those isomorphisms to identify

$$\mathcal{S}_\mu = \mathcal{S}_{n_1} \otimes \ldots \otimes \mathcal{S}_{n_k} : \mathcal{H}_{\mathbf{Z}[q_v^{1/2}]}(M, \mathfrak{m}) \xrightarrow{\sim} \mathbf{Z}[q_v^{1/2}][\Xi_v]^{S_\mu}.$$

Consider any open compact subgroup $\mathfrak{q}$ of $\mathrm{GL}_n(F_v)$, and set

$$\mathfrak{q}_M = \mathfrak{q} \cap M, \quad \mathfrak{q}^+ = \mathfrak{q} \cap N, \quad \mathfrak{q}^- = \mathfrak{q} \cap \overline{N}.$$

From now on, assume that $\mathfrak{q}$ has an Iwahori decomposition with respect to P, which means that $\mathfrak{q} = \mathfrak{q}^-\mathfrak{q}_M\mathfrak{q}^+$. We define a submonoid $M^+ \subset M$ of *positive* elements to consist of elements $m \in M$, such that

$$m\mathfrak{q}^+m^{-1} \subset \mathfrak{q}^+, \qquad m^{-1}\mathfrak{q}^-m \subset \mathfrak{q}^-.$$

Inside M^+, we have a further submonoid M^{++} of *strictly positive* elements consisting of $m \in M^+$ satisfying the following conditions:

- For any compact open subgroups $\mathfrak{n}_1, \mathfrak{n}_2$ of N, there exists a positive integer $x \geq 0$, such that

$$m^x\mathfrak{n}_1m^{-x} \subset \mathfrak{n}_2.$$

- For any compact open subgroups $\overline{\mathfrak{n}}_1$, $\overline{\mathfrak{n}}_2$ of $\overline{N}$, there exists a positive integer $x \geq 0$, such that

$$m^{-x}\overline{\mathfrak{n}}_1m^x \subset \overline{\mathfrak{n}}_2.$$

We denote by $\mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)^+$ the elements of $\mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)$ whose support is contained in M^+. From now on, we also assume that q_v has a square root in $\mathcal{O}$ and fix such square root.

Proposition 3.1.

1. *The map* $t_\mu^+ : \mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)^+ \to \mathcal{H}_\mathcal{O}(G, \mathfrak{q})$ *given by*

$$[\mathfrak{q}_M m \mathfrak{q}_M] \mapsto \delta_P^{1/2}(m)[\mathfrak{q}m\mathfrak{q}]$$

is an algebra homomorphism.
2. *The map* t_μ^+ *extends to a homomorphism* $t_\mu : \mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M) \to \mathcal{H}_\mathcal{O}(G, \mathfrak{q})$ *if and only if there exists a strictly positive element* $a \in Z(M)$*, such that* $[\mathfrak{q}a\mathfrak{q}]$ *is invertible in* $\mathcal{H}_\mathcal{O}(G, \mathfrak{q})$.
3. *Assuming the existence of the extension in (2), for any smooth* $\mathbf{C}[\mathrm{GL}_n(F_v)]$*-module* π*, the canonical map* $\pi^\mathfrak{q} \to \pi_N^{\mathfrak{q}_M}$ *is a homomorphism of* $\mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)$*-modules, where* $\mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)$ *acts on* $\pi^\mathfrak{q}$ *via the map* t_μ.

Proof. For the first two claims, see [Vig98, II.6]. For the third, see [Vig98, II.10.1]. □

Now we record some results about smooth admissible representations of $\mathrm{GL}_n(F_v)$ in characteristic 0. Let $\mathfrak{p}$ be a parahoric corresponding to the partition $n = n_1 + \ldots + n_k$ which we call μ, and let P be the underlying parabolic with the Levi decomposition $P = MN$. Let $\mathfrak{m} = M(\mathcal{O}_{F_v})$. We also let $\mathfrak{p}_1, \mathfrak{m}_1$ denote the kernels of the homomorphisms

$$\mathfrak{p} \to P(k_v) \to \mathrm{GL}_{n_k}(F_v) \xrightarrow{\det} k_v^\times \to k_v^\times(p)$$

$$\mathfrak{m} \to M(k_v) \to \mathrm{GL}_{n_k}(F_v) \xrightarrow{\det} k_v^\times \to k_v^\times(p).$$

Finally, let $\mathrm{Iw}' = \mathfrak{p}_1 \cap \mathrm{Iw}$.

Lemma 3.2. *The condition in part (2) of Proposition 3.1 is satisfied for* $\mathfrak{q} = \mathfrak{p}, \mathfrak{p}_1$.

Proof. This is a special case of [Whi22, Proposition 5.7]. □

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Fix a uniformiser ϖ_c of F_v. For any $1 \le j \le k$ and $1 \le i \le n_j$, consider the operators in $\mathcal{H}_{\mathcal{O}}(G, \mathfrak{p})$ given by

$$V^{i,j} = t_\mu(\mathcal{S}_\mu^{-1}(e_i(X_{s_{\mu,j-1}+1}, \dots, X_{s_{\mu,j}}))).$$

We will also consider operators in $\mathcal{H}_{\mathcal{O}}(G, \mathfrak{p}_1)$, such that their actions on $\pi^{\mathfrak{p}} \subset \pi^{\mathfrak{p}_1}$ agree with the action of $V^{i,j}$ for any smooth representation π. They can be constructed in the same way as $V^{i,j}$ above by replacing S_μ with the Satake isomorphism for $\mathfrak{m}_1$ from [Whi22, Theorem 5.1]. These operators will also be denoted $V^{i,j}$. We also define operators $T^{i,j}$ representing the images of the same elements under $\mathcal{S}_\mu^{-1}$ in $\mathcal{H}_{\mathcal{O}}(M, \mathfrak{m})$ and the corresponding operators on $\mathcal{H}_{\mathcal{O}}(M, \mathfrak{m}_1)$.

The following lemmas are straightforward generalisations of the lemmas in [Tho12, Section 5]. Given a parabolic subgroup Q of $\mathrm{GL}_n(F_v)$, we write $W_Q \subset W$ for the Weyl group of its Levi factor. Recall from [Cas] that the space $W_Q \backslash W / W_P$ has a canonical set of representatives $[W_Q \backslash W / W_P]$, consisting of minimal length elements from each double coset.

Lemma 3.3. *Let Q be a parabolic corresponding to the partition $n = m_1 + \ldots + m_r$. Then $[W_Q \backslash W / W_P]$ is isomorphic to the set of partitions*

$$m_i = n_1^i + \ldots + n_k^i, 1 \le i \le r,$$

such that

$$\sum_i n_j^i = n_j \text{ for all } 1 \le j \le k.$$

With Q as in the last lemma, let L_i denote the i-th component of the corresponding Levi subgroup. For $w \in [W_Q \backslash W / W_P]$ corresponding to the partition $n_1^i + \ldots + n_k^i$, let $\mathfrak{p}_i^w$ denote the parahoric subgroup of L_i corresponding to this partition, and let $\mathfrak{p}_{i,1}^w$ be the kernel of

$$\mathfrak{p}_i^w \to \mathrm{GL}_{n_k^i}(F_v) \xrightarrow{\det} k_v^\times \to k_v^\times(p).$$

Let $\mathfrak{q}$ be the parahoric corresponding to the partition $\{n_1^1, \dots, n_k^1, n_1^2, \dots, n_k^r\}$, and let $\mathfrak{n}$ be the hyperspecial maximal compact of the corresponding Levi subgroup. We define $\mathfrak{p}_{1,w}$ as a subgroup of $\mathfrak{q}$ defined by the conditions $\mathrm{im}(\det N_k^j \to k_v^\times(p)) = 1$ for all j, where N_k^j is the block corresponding to n_k^j.

Lemma 3.4. *For each $1 \le i \le r$, let π_i be a smooth representation of L_i. Then*

1. *For any $w \in [W_Q \backslash W / W_P]$, we have $L_i \cap w\mathfrak{p}w^{-1} = \mathfrak{p}_i^w$.*
2. *For any $w \in [W_Q \backslash W / W_P]$, we have $Q \cap w\mathfrak{p}_1 w^{-1} \supset \mathfrak{p}_{1,w}$.*
3. $$(\mathrm{ind}_Q^G \pi_1 \otimes \ldots \otimes \pi_r)^{\mathfrak{p}} \cong \bigoplus_{w \in [W_Q \backslash W / W_P]} \pi_1^{\mathfrak{p}_1^w} \otimes \ldots \otimes \pi_r^{\mathfrak{p}_r^w}.$$
4. $$(\mathrm{ind}_Q^G \pi_1 \otimes \ldots \otimes \pi_r)^{\mathfrak{p}_1} \subset \bigoplus_{w \in [W_Q \backslash W / W_P]} \pi_1^{\mathfrak{p}_{1,1}^w} \otimes \ldots \otimes \pi_r^{\mathfrak{p}_{r,1}^w}.$$

Let π be an irreducible admissible representation of G, such that $\pi^{\mathfrak{p}_1} \neq 0$. Since $\mathrm{Iw}' \subset \mathfrak{p}_1$, supercuspidal support of π consists of tamely ramified characters. We will now use the Bernstein-Zelevinsky classification [BZ77], following the conventions of [Rod82], as they are best suited for applications to local Langlands correspondence. We can write π as a quotient of

$$\mathrm{ind}_Q^G \mathrm{Sp}_{k_1}(\chi_1) \otimes \ldots \otimes \mathrm{Sp}_{k_r}(\chi_r),$$

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where $\mathrm{Sp}_n(\chi)$ for a tamely ramified character $\chi : F_v^\times \to \mathbf{C}^\times$ is the unique irreducible quotient of $\mathrm{ind}_B^{\mathrm{GL}_n} \chi \otimes \chi|\cdot| \otimes \ldots \otimes \chi|\cdot|^{n-1}$. The twisted Steinberg factors $\mathrm{Sp}_{k_i}(\chi_i)$ correspond to Zelevinsky segments $\Delta_i = (\chi, \chi(1), \ldots, \chi(k_i - 1))$.

Let $\mathcal{A}$ index the partitions of $sc(\pi)$ into k labeled subsets $S_1, \ldots, S_k$ satisfying the following conditions:

- $|S_i| = n_i$ for all i.
- characters from the same Zelevinsky segment always belong to different subsets.
- if $\chi \in S_i, \chi' \in S_j$ share a segment and $\chi' = \chi(a)$ for $a > 0$, then $i < j$.

For each partition $\alpha \in \mathcal{A}$, let $r(\alpha)$ be the representation of $T(F)$ given by tensoring the characters of $sc(\pi)$ in the following order: characters in S_i precede characters in S_j when $i < j$, and the ordering of characters within each S_i is induced by the ordering of Zelevinsky segments.

Lemma 3.5. *For each $1 \le i \le r$, let π_i be a smooth representation of L_i. Then*

$$(\mathrm{ind}_Q^G \pi_1 \otimes \ldots \otimes \pi_r)_N^{ss} = \bigoplus_{w \in [W_Q \backslash W / W_P]} \mathrm{ind}_{w^{-1}Qw \cap M}^M \, w^{-1}(\pi_1 \otimes \ldots \otimes \pi_r)_{L \cap wNw^{-1}}.$$

Lemma 3.6. *Let π be an irreducible admissible $GL_n(F_v)$-module, such that $\pi^{\mathfrak{p}_1} \neq 0$. Consider $\pi^{\mathfrak{p}_1}$ as a $\mathbf{Z}[\Xi_v]^{S_\mu}$-module via the map $t_\mu \circ \mathcal{S}_\mu^{-1}$. Then $(\pi^{\mathfrak{p}_1})^{ss}$ is a direct sum of 1-dimensional submodules indexed by a subset of $\mathcal{A}$. For a finite set S of characters and positive integer $k \le |S|$, let $e_k(S(\varpi))$ denote the k-th symmetric polynomial of elements of S evaluated at ϖ. Then on the component associated to $(S_1, \ldots, S_k) \in \mathcal{A}$, the action of $V^{i,j}$ is given by $e_i(S_j)$ for all $1 \le i \le n_j$.*

Proof. We have a surjection

$$\mathrm{ind}_Q^G \mathrm{Sp}_{k_1}(\chi_1) \otimes \ldots \otimes \mathrm{Sp}_{k_r}(\chi_r) \twoheadrightarrow \pi,$$

and the induced map

$$(\mathrm{ind}_Q^G \mathrm{Sp}_{k_1}(\chi_1) \otimes \ldots \otimes \mathrm{Sp}_{k_r}(\chi_r))^{\mathfrak{p}_1} \longrightarrow \pi^{\mathfrak{p}_1}$$

is also surjective. By Lemma 3.5, we can write

$$(\mathrm{ind}_Q^G \mathrm{Sp}_{k_1}(\chi_1) \otimes \ldots \otimes \mathrm{Sp}_{k_r}(\chi_r))_N^{ss} =$$

$$\sigma \oplus \bigoplus_{(S_1, \ldots, S_k) \in \mathcal{A}} \mathrm{ind}_{B \cap M}^M \left(\bigotimes_{\psi_1 \in S_1} \psi_1 \otimes \ldots \otimes \bigotimes_{\psi_k \in S_k} \psi_k \right).$$

Here, the summands indexed by $\mathcal{A}$ correspond to $w \in [W_Q \backslash W / W_P]$ represented by partitions $\{n_j^i\}$ satisfying $n_j^i \le 1$ for all i, j (cf. Lemma 3.3) and σ represents all other summands. We will now show that σ does not have $\mathfrak{m}_1$-invariants. Let $\mathfrak{m}_{i,1}^w \subset \mathfrak{p}_{i,1}^w$ be the subgroups of the Levi subgroup of L_i defined analogously to $\mathfrak{p}_{i,1}^w$.

Suppose $\sigma^{\mathfrak{m}_1}$ is nonzero. Let θ be a representation of $GL_{n_j^i}(F_v)$ which is a tensor factor of $(\mathrm{Sp}_{k_1}(\chi_1) \otimes \ldots \otimes \mathrm{Sp}_{k_r}(\chi_r))_{L \cap wNw^{-1}}$ for some $w \in [W_Q \backslash W / W_P]$ contributing to σ. Then θ has to be spherical if $j < k$ and has to have a fixed vector by $\ker(GL_{n_j^i}(\mathcal{O}_{F_v}) \to GL_{n_j^i}(k_v) \xrightarrow{\det} k_v^\times \to k_v^\times(p))$ if $j = k$. This would imply that $\mathrm{Sp}_{k_i}(\chi_i)^{\mathfrak{p}_{i,1}^w} \neq 0$ for all $1 \le i \le r$ and all w representing partitions $m_i = n_1^i + \ldots + n_k^i$, such that there exists at least one $1 \le i \le r$ for which $k_i > 1$ and $n_j^i > 1$ for some $1 \le j \le k$. To get a contradiction, it is therefore enough to show that $\mathrm{Sp}_{k_i}(\chi_i)^{\mathfrak{p}_{i,1}^w} = 0$.

Define the subgroup $\mathrm{Iw}_i' \subset \mathfrak{p}_{i,1}^w$ to be a subgroup of the L_i-Iwahori with 1's mod ϖ on the diagonal at indices $n_{k-1}^i + 1$ through n_k^i. There are two possibilities: either $\mathfrak{p}_{i,1}^w = \mathrm{GL}_{m_i}(\mathcal{O}_{F_v})$, or Iw_i' has at least one $*$ mod ϖ on the diagonal. In the former case, we are done since $\mathrm{Sp}_{k_i}(\chi_i)$ is never spherical. In the latter case, let $\mathfrak{t}'$ be the diagonal component of Iw_i'. Then

$$\mathrm{Sp}_{k_i}(\chi_i)^{\mathrm{Iw}_i'} = \mathrm{Sp}_{k_i}(\chi_i)_U^{\mathfrak{t}'} = (\chi_i \otimes \ldots \otimes \chi_i|\cdot|^{k_i-1})^{\mathfrak{t}'},$$

where U is the unipotent radical of the Borel. Since $\mathfrak{t}'$ has at least one $\mathcal{O}_{F_v}^\times$ factor, if this is nonzero, χ_i must be unramified. But in this case, any $\mathfrak{p}_{i,1}^w$-fixed vector would be automatically fixed by the parahoric $\mathfrak{p}_i^w$, which properly contains the Iwahori, and hence, does not fix any vector in $\mathrm{Sp}_{k_i}(\chi_i)$. □

For a partition $n = n_1 + \ldots + n_k$ which we call μ, define elements

$$P_{\mu,i} = \prod_{j=s_{\mu,i-1}+1}^{s_{\mu,i}} (T - X_j)$$

$$\mathrm{Res}_\mu = \prod_{i<j} \mathrm{Res}(P_{\mu,i}, P_{\mu,j}) \in \mathbf{Z}[\Xi_v]^{S_\mu}$$

$$\mathrm{Res}_{q_v,\mu} = \prod_{i<j} \mathrm{Res}(P_{\mu,i}(q_v T), P_{\mu,j}) \in \mathbf{Z}[\Xi_v]^{S_\mu}.$$

Then there exist unique polynomials $Q_{\mu,i} \in \mathbf{Z}[\Xi_v]^{S_\mu}[T]$, such that $\deg Q_{\mu,i} < n_i$ and

$$\sum_{i=1}^n Q_{\mu,i} \prod_{j\neq i} P_{\mu,j} = \mathrm{Res}_\mu .$$

Define

$$E_{\mu,i} = Q_{\mu,i} \prod_{j\neq i} P_{\mu,j}.$$

The following statement is elementary.

Lemma 3.7. *Take any $A \in M_n(\mathbf{C})$ with a factorisation*

$$\det(T - A) = \prod_{i=1}^k p_{\mu,i}(T),$$

where $p_{\mu,i} \in \mathbf{C}[T]$ are pairwise coprime and $\deg p_{\mu,i} = n_i$. Consider the homomorphism $\varphi : \mathbf{Z}[\Xi_v]^{S_\mu} \to \mathbf{C}$ defined by the polynomials $p_{\mu,i}$. By this, we mean the homomorphism sending $e_j(X_{s_{\mu,i-1}+1}, \ldots, X_{s_{\mu,i}})$ to $(-1)^j$ times the coefficient of T^j in $p_{\mu,i}$. This homomorphism can be extended to $\varphi : \mathbf{Z}[\Xi_v]^{S_\mu}[T, \mathrm{Res}_\mu^{-1}] \to \mathbf{C}[T]$. Then $\varphi(E_{\mu,i}/\mathrm{Res}_\mu)(A)$ projects $\mathbf{C}^n$ onto the sum of generalised eigenspaces of A corresponding to the roots of $p_{\mu,i}$.

Proposition 3.8. *Let π be an irreducible admissible $GL_n(F_v)$-module. Then either $\mathrm{Res}_{q_v,\mu}^{n!} \pi^{\mathfrak{p}_1} = 0$, or*

$$\mathrm{rec}_{F_v}(\pi) = (\chi_1 \oplus \ldots \oplus \chi_n, 0),$$

where $\chi_1, \ldots, \chi_{n_1+\ldots+n_{k-1}}$ are unramified and the rest are tamely ramified with equal restriction to inertia.

Proof. Using the notation from the discussion preceding Lemma 3.5, if there exists some $k_i > 1$, then $\mathrm{Res}_{q_v,\mu}^{n!} \pi^{\mathfrak{p}_1} = 0$ follows from Lemma 3.6. Otherwise, we can apply the proof of [CHT08, Lemma 3.1.6] for the second conclusion. □

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Proposition 3.9. *Let π be an irreducible admissible $GL_n(F_v)$-module. Let $(r, N) = \mathrm{rec}_{F_v}(\pi)$. Then either $(S_\mu \circ t_\mu^{-1} \circ \iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1})(\mathrm{Res}^{n!}_{q_v,\mu})\pi^{\mathfrak{p}_1} = 0$ or $N = 0$ and*

$$r^\vee = \chi_1 \oplus \ldots \oplus \chi_n,$$

where $\chi_1, \ldots, \chi_{n_1+\ldots+n_{k-1}}$ are unramified and the rest are tamely ramified with equal restriction to inertia.

Proof. Let $\pi^\vee$ be the contragradient of π. Then $\mathrm{rec}_{F_v}(\pi^\vee) = (r^\vee, -{}^tN)$. We have a perfect pairing $(\pi^\vee)^{\mathfrak{p}_1} \times \pi^{\mathfrak{p}_1} \to \mathbf{C}$ which is antisymmetric with respect to action of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ and $S_\mu \circ t_\mu^{-1} \circ \iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1}$. Therefore, $(S_\mu \circ t_\mu^{-1} \circ \iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1})(\mathrm{Res}^{n!}_{q_v,\mu})\pi^{\mathfrak{p}_1} = 0$ if and only if $\mathrm{Res}^{n!}_{q_v,\mu}(\pi^\vee)^{\mathfrak{p}_1} = 0$. Thus, we can assume both of these are nonzero, in which case, by Proposition 3.8, we get the desired result. □

Let $\varphi_v \in G_{F_v}$ be any lift of Frobenius.

Proposition 3.10. *Let π be an irreducible admissible $GL_n(F_v)$-module. Let $(r, N) = \mathrm{rec}_{F_v}(\pi)$. Let R be the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $\mathrm{End}_{\mathcal{O}}(\pi^{\mathfrak{p}_1})$ under the map $t_\mu \circ S_\mu^{-1}$. Then either $\mathrm{Res}^{n!}_{q_v,\mu}\pi^{\mathfrak{p}_1} = 0$ or the following relation holds over R : for all $\tau \in I_{F_v}$*

$$\mathrm{Res}_\mu^{n!}\left(\sum_{i=1}^{k-1} E_{\mu,i}(r(\varphi_v)) + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu,k}(r(\varphi_v)) - \mathrm{Res}_\mu r(\tau)\right) = 0.$$

Proof. Assume $\mathrm{Res}^{n!}_{q_v,\mu}\pi^{\mathfrak{p}_1} \neq 0$. It is enough to check our relation for each localisation of R at a maximal ideal $\mathfrak{m}$. If $\mathrm{Res}_\mu \in \mathfrak{m}$, then $\mathrm{Res}_\mu^{n!} = 0$ in $R_\mathfrak{m}$. Otherwise, $R_\mathfrak{m} = \mathbf{C}$ by [Sta18, Tag 00UA] and the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $R/\mathfrak{m}$ corresponds to the polynomials $\prod_{j=s_{\mu,i-1}+1}^{s_{\mu,i}} (T - \chi_j(\varphi_v))$ for $i = 1, \ldots, k$. Then the image of

$$\mathrm{Res}_\mu{}^{-1}\left(\sum_{i=1}^{k-1} E_{\mu,i}(r(\varphi_v)) + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu,k}(r(\varphi_v))\right)$$

in $M_n(R_\mathfrak{m})$ is a diagonal matrix with $n - n_k$ first entries equal to 1 and the rest equal to $\chi_n(\tau)$. This concludes the proof. □

Proposition 3.11. *Let π be an irreducible admissible $GL_n(F_v)$-module. Let $(r, N) = \mathrm{rec}_{F_v}(\pi)$. Let R' be the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $\mathrm{End}_{\mathcal{O}}(\pi^{\mathfrak{p}_1})$ via the map $\iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1}$. Then either $(\iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1})(\mathrm{Res}^{n!}_{q_v,\mu})\pi^{\mathfrak{p}_1} = 0$ or the following relation holds over R' : for all $\tau \in I_{F_v}$*

$$(\iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1})\left(\mathrm{Res}_\mu^{n!}\left(\sum_{i=1}^{k-1} E_{\mu,i}(r^\vee(\varphi_v)) + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu,k}(r^\vee(\varphi_v)) - \mathrm{Res}_\mu r^\vee(\tau)\right)\right) = 0.$$

Proof. This follows from Proposition 3.9 in the same way as Proposition 3.10 follows from Proposition 3.8. □

In what follows, we will use a twisted version of the propositions above. Define a map Σ^T : $\mathcal{O}[\Xi_{v,1}]^{S_\mu} \to \mathcal{H}_{\mathcal{O}}(GL_n(F_v), \mathfrak{p}_{v,1})$ given by

$$\Sigma^T(f)(g) = t_\mu(S_\mu^{-1}(f))(g)|\det(g)|^{(1-n)/2}.$$

Let us show that this map is in fact defined over $\mathbf{Z}[q_v^{-1}]$ and thus does not depend on the choice of square root of q_v^{-1}. Note that t_μ is defined over $\mathbf{Z}[q_v^{-1}]$ up to $\delta_{P_\mu}^{1/2}$ and S_μ is defined over $\mathbf{Z}[q_v^{-1}]$ up to

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$\prod_{i=1}^{k}\det(m_i)^{(1-n_i)/2}$, where $(m_i) \in M_\mu(F_v)$ with $m_i \in \mathrm{GL}_{n_i}(F_v)$. Thus, the desired rationality over $\mathbf{Z}[q_v^{-1}]$ follows from the fact that

$$\prod_{i=1}^{k}|\det(m_i)|^{(1-n)/2}\prod_{i=1}^{k}|\det(m_i)|^{(1-n_i)/2}\prod_{1\le i<j\le k}|\det(m_i)|^{n_j/2}|\det(m_j)|^{-n_i/2}$$

lies in $\mathbf{Z}[q_v^{-1}]$. Now let us restate Proposition 3.10 and Proposition 3.11.

Proposition 3.12. *Let π be an irreducible admissible $GL_n(F_v)$-module. Let $(r, N) = \mathrm{rec}^T_{F_v}(\pi)$. Let R be the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $\mathrm{End}_{\mathcal{O}}(\pi^{\mathfrak{p}_1})$ under the map Σ^T. Then either $\mathrm{Res}^{n!}_{q_v,\mu}\pi^{\mathfrak{p}_1} = 0$ or the following relation holds over R : for all $\tau \in I_{F_v}$*

$$\mathrm{Res}^{n!}_\mu\left(\sum_{i=1}^{k-1}E_{\mu,i}(r(\varphi_v)) + \langle\mathrm{Art}^{-1}_{F_v}(\tau)\rangle E_{\mu,k}(r(\varphi_v)) - \mathrm{Res}_\mu\, r(\tau)\right) = 0.$$

Proposition 3.13. *Let π be an irreducible admissible $GL_n(F_v)$-module. Let $(r, N) = \mathrm{rec}^T_{F_v}(\pi)$. Let R' be the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $\mathrm{End}_{\mathcal{O}}(\pi^{\mathfrak{p}_1})$ via the map $\iota_{\mathcal{H}}\circ\Sigma^T$. Then either $(\iota_{\mathcal{H}}\circ\Sigma^T)(\mathrm{Res}^{n!}_{q_v,\mu})\pi^{\mathfrak{p}_1} = 0$ or the following relation holds over R' : for all $\tau \in I_{F_v}$*

$$(\iota_{\mathcal{H}}\circ\Sigma^T)\left(\mathrm{Res}^{n!}_\mu\left(\sum_{i=1}^{k-1}E_{\mu,i}(r^\vee(\varphi_v)) + \langle\mathrm{Art}^{-1}_{F_v}(\tau)\rangle E_{\mu,k}(r^\vee(\varphi_v)) - \mathrm{Res}_\mu\, r^\vee(\tau)\right)\right) = 0.$$

4. Setup

Let F/F^+ be an imaginary CM-field with ring of integers $\mathcal{O}$. Let Ψ_n be the matrix with 1-s on the antidiagonal and 0-s elsewhere, and let

$$J_n = \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix}.$$

Define $\widetilde{G}$ to be the group scheme over $\mathcal{O}_{F^+}$ defined by the functor of points

$$\widetilde{G}(R) = \{g \in \mathrm{GL}_{2n}(R\otimes_{\mathcal{O}_{F^+}}\mathcal{O}_F) \mid {}^t g J_n g^c = J_n\}.$$

Then $\widetilde{G}$ is a quasisplit reductive group over F^+. It is a form of GL_{2n} which becomes split after the quadratic base change F/F^+. If v is a place of F lying above a place $\overline{v}$ of F^+ which splits in F, then we have a canonical isomorphism $\iota_v : \widetilde{G}(F^+_{\overline{v}}) \cong \mathrm{GL}_{2n}(F_v)$. There is an isomorphism $F^+_{\overline{v}}\otimes_{F^+}F \cong F_v\times F_{v^c}$ and ι_v is given by composition

$$\widetilde{G}(F_{\overline{v}^+}) \hookrightarrow \mathrm{GL}_{2n}(F_v)\times\mathrm{GL}_{2n}(F_{v^c}) \to \mathrm{GL}_{2n}(F_v),$$

where the second map is the projection on the first factor. We write $T \subset B \subset G$ for the subgroups consisting, respectively, of the diagonal and upper-triangular matrices in $\widetilde{G}$. Similarly, we write $G \subset P \subset \widetilde{G}$ for the Levi and parabolic subgroups consisting, respectively, of the block upper diagonal and block upper-triangular matrices with blocks of size $n\times n$. Then $P = U \rtimes G$, where U is the unipotent radical of P, and we can identify G with $\mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}}\mathrm{GL}_n$ via the map

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mapsto D \in \mathrm{GL}_n(R\otimes_{\mathcal{O}_{F^+}}\mathcal{O}_F).$$

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An element $(g_v)_v \in G(\mathbf{A}_{F^+}^\infty) = GL_n(\mathbf{A}_F^\infty)$ is called *neat* if the intersection $\cap_v \Gamma_v$ is trivial, where $\Gamma_v \subset \overline{\mathbf{Q}}^\times$ is the torsion subgroup of the subgroup of $\overline{F_v}^\times$ generated by the eigenvalues of g_v acting via some faithful representation of G. We call a neat open compact subgroup $K \subset G(\mathbf{A}_{F^+}^\infty)$ *good* if it has the form $K = \prod_v K_v$, where the product is running over the finite places of F. We make similar definitions with $\widetilde{G}$ in place of G.

After extending scalars to F^+, T and B form a maximal torus and a Borel subgroup, respectively, of $\widetilde{G}$ and G is the unique Levi subgroup of the parabolic subgroup P of $\widetilde{G}$ which contains T. We call an open compact subgroup $\widetilde{K}$ of $\widetilde{G}(\mathbf{A}_{F^+}^\infty)$ *decomposed* with respect to the Levi decomposition $P = GU$ if $\widetilde{K} = \widetilde{K}_G \ltimes \widetilde{K}_U$, where $\widetilde{K}_G$ is the image of $\widetilde{K}$ in G and $\widetilde{K}_U = \widetilde{K} \cap U(\mathbf{A}_{F^+}^\infty)$.

If K is a good subgroup of G, we let X_K be the corresponding locally symmetric space. Similarly, if $\widetilde{K}$ is a good open compact subgroup of $\widetilde{G}$, then $X_{\widetilde{K}}$ denotes the locally symmetric space. More generally, if H is a connected algebraic group over a number field L and $K_H \subset H(\mathbf{A}_M^\infty)$ is a good subgroup, then we write $X_{K_H}^H$ for the locally symmetric space of H of level K_H.

Fix a rational prime p and a finite extension $E/\mathbf{Q}_p$ which contains the images of all embeddings $F \hookrightarrow \overline{\mathbf{Q}}_p$. We write $\mathcal{O}$ for the ring of integers of E and $\varpi \in \mathcal{O}$ for a choice of uniformiser. For $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}}(F, E)$, we define an $\mathcal{O}[\prod_{v|p} \mathrm{GL}_n(\mathcal{O}_{F_v})]$-module $\mathcal{V}_\lambda$ as in [ACC+18, Section 2.2.1]. Similarly for $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}}(F^+, E)$, we have an $\mathcal{O}[\prod_{\overline{v}|p} \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})]$-module $\mathcal{V}_{\widetilde{\lambda}}$. Both $\mathcal{V}_\lambda$ and $\mathcal{V}_{\widetilde{\lambda}}$ are finite free $\mathcal{O}$-modules.

Let S be a set of places of F, such that $S = S^c$ and, such that S contains all places above p and all places of F which are ramified over F^+. Let $\overline{S}$ be the set of places of F^+ lying below a place in S. Let $K \subset G(\mathbf{A}_{F^+}^\infty)$ be a good subgroup, such that $K_{\overline{v}} = G(\mathcal{O}_{F^+_{\overline{v}}})$ for $\overline{v} \notin \overline{S}$, and similarly, let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup, such that $\widetilde{K}_v = \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})$ for $\overline{v} \notin \overline{S}$. Additionally, we define $\widetilde{\Xi}_{\overline{v}} \coloneqq \Xi_v \times \Xi_{v^c}$ and $\widetilde{\Xi}_{\overline{v},1} \coloneqq \Xi_{v,1} \times \Xi_{v^c}$.

Define the Hecke algebras

$$\mathcal{H}^S = \mathcal{H}_{\mathcal{O}}(G(\mathbf{A}_{F^+}^{\infty,\overline{S}}), K^{\overline{S}})$$

$$\widetilde{\mathcal{H}}^S = \mathcal{H}_{\mathcal{O}}(\widetilde{G}(\mathbf{A}_{F^+}^{\infty,\overline{S}}), \widetilde{K}^{\overline{S}})$$

$$\mathbf{T}^S \cong \bigotimes_{v\notin S}' \mathcal{O}[\Xi_v]^{S_n}$$

$$\widetilde{\mathbf{T}}^S \cong \bigotimes_{v\notin S}' \mathcal{O}[\widetilde{\Xi}_{\overline{v}}]^{S_{2n}}.$$

Using the isomorphism

$$G(\mathcal{O}_{F^+_{\overline{v}}}) \cong \mathrm{GL}_n(\mathcal{O}_{F_v})$$

together with the Satake isomorphisms, as well as the homomorphism

$$\mathcal{O}[\widetilde{\Xi}_{\overline{v}}]^{S_{2n}} \to \mathcal{H}_{\mathcal{O}}(\widetilde{G}(F^+_{\overline{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}}))$$

given by the polynomial $\widetilde{P}_v(X)$ defined in [ACC+18, Equation 2.2.6], we get homomorphisms $\mathbf{T}^S \to \mathcal{H}^S$ and $\widetilde{\mathbf{T}}^S \to \widetilde{\mathcal{H}}^S$. We also have homomorphisms

$$\mathbf{T}^S \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}_\lambda))$$

$$\widetilde{\mathbf{T}}^S \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}))$$

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defined in [ACC⁺18, Section 2.1.2], and we can denote by $\mathbf{T}^S(K,\lambda)$, $\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})$, respectively, the images of those homomorphisms. The functor H^* induces $\mathcal{O}$-algebra homomorphisms

$$\mathbf{T}^S(K,\lambda) \to \mathrm{End}_{\mathcal{O}}(H^*(X_K,\mathcal{V}_\lambda))$$

$$\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda}) \to \mathrm{End}_{\mathcal{O}}(H^*(\widetilde{X}_{\widetilde{K}},\mathcal{V}_{\widetilde{\lambda}})).$$

5. Boundary cohomology

Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^\infty_{F^+})$ be a neat compact open subgroup decomposed with respect to the Levi decomposition $P = GU$. We also assume that $\widetilde{K}_v = \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})$ for $\overline{v} \notin \overline{S}$. Define K as the image of $\widetilde{K}$ in $G(\mathbf{A}^\infty_{F^+})$, $\widetilde{K}_P = \widetilde{K} \cap P(\mathbf{A}^\infty_{F^+})$ and $K_U = \widetilde{K} \cap U(\mathbf{A}^\infty_{F^+})$. Both K and $\widetilde{K}_P$ are neat. We recall from [NT16, Section 3.1.2] that the boundary $\partial\widetilde{X}_{\widetilde{K}} = \overline{\widetilde{X}}_{\widetilde{K}}$ of the Borel-Serre compactification has a $\widetilde{G}(\mathbf{A}^\infty_{F^+})$-equivariant stratification indexed by the standard parabolic subgroups of $\widetilde{G}$. For each standard parabolic subgroup Q, label the corresponding stratum $\widetilde{X}^Q_{\widetilde{K}}$. We can write

$$\widetilde{X}^Q_{\widetilde{K}} = Q(F^+)\backslash(X^Q \times \widetilde{G}(\mathbf{A}^\infty_{F^+})/\widetilde{K}).$$

From now on, we will focus on the stratum $\widetilde{X}^P_{\widetilde{K}}$ corresponding to the Siegel parabolic. Let us establish some useful maps between the manifolds introduced above. The stratum $\widetilde{X}^P_{\widetilde{K}}$ can be described as a union of connected components indexed by the set $P(F^+)\backslash\widetilde{G}(\mathbf{A}^\infty_{F^+})/\widetilde{K}$. The locally symmetric space $X^P_{\widetilde{K}}$ is a union of the same components indexed by the set $P(F^+)\backslash P(\mathbf{A}^\infty_{F^+})/\widetilde{K}_P$. Thus, we have a natural open immersion $i : X^P_{\widetilde{K}} \to \widetilde{X}^P_{\widetilde{K}}$, such that $i^* : H^*(\widetilde{X}^P_{\widetilde{K}},\mathcal{O}) \to H^*(X^P_{\widetilde{K}},\mathcal{O})$ is a split epimorphism. We also have a proper map $j : X^P_{\widetilde{K}_P} \to X_K$ which has a section by [NT16, Section 3.1.1]. Thus, we get a split monomorphism $j^* : H^*(X_K,\mathcal{O}) \to H^*(X^P_{\widetilde{K}},\mathcal{O})$. We also recall the 'restriction to P' and 'integration along N' homomorphisms:

$$r_P : \mathcal{H}_{\mathcal{O}}(\widetilde{G}(\mathbf{A}^{\infty,\overline{S}}_{F^+}),\widetilde{K}^{\overline{S}}) \to \mathcal{H}_{\mathcal{O}}(\widetilde{P}(\mathbf{A}^{\infty,\overline{S}}_{F^+}),\widetilde{K}^{\overline{S}}_P)$$

$$r_G : \mathcal{H}_{\mathcal{O}}(\widetilde{P}(\mathbf{A}^{\infty,\overline{S}}_{F^+}),\widetilde{K}^{\overline{S}}_P) \to \mathcal{H}_{\mathcal{O}}(G(\mathbf{A}^{\infty,\overline{S}}_{F^+}),K^{\overline{S}})$$

defined in [NT16, Section 2.2]. We record the following proposition, which follows from the discussion above:

Proposition 5.1.

1. *For all* $t \in \widetilde{\mathbf{T}}^S$ *and* $h \in H^*(\widetilde{X}^P_{\widetilde{K}},\mathcal{O})$, *we have* $i^*(th) = r_P(t)i^*(h)$.
2. *For all* $t \in \mathcal{H}_{\mathcal{O}}(\widetilde{P}(\mathbf{A}^{\infty,\overline{S}}_{F^+}),\widetilde{K}^{\overline{S}}_P)$ *and* $h \in H^*(X_K,\mathcal{O})$, *we have* $j^*(r_G(t)h) = tj^*(h)$.

Consider the composite

$$\mathcal{S} = r_G \circ r_P : \mathcal{H}_{\mathcal{O}}(\widetilde{G}(\mathbf{A}^{\infty,\overline{S}}_{F^+}),\widetilde{K}^{\overline{S}}) \to \mathcal{H}_{\mathcal{O}}(G(\mathbf{A}^{\infty,\overline{S}}_{F^+}),K^{\overline{S}}).$$

By [NT16, Proposition-Definition 5.3], this map coincides with the tensor product of maps $\mathcal{O}[\widetilde{\Xi}_{\overline{v}}]^{S_{2n}} \to \mathcal{O}[\Xi_v]^{S_n}$ determined by the polynomial $\mathcal{S}_n(P_v(X)q_v^{n(2n-1)}P^\vee_{v^c}(q_v^{1-2n}X))$.

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Let $\mathfrak{m} \subset \mathbf{T}^S$ be a non-Eisenstein maximal ideal of Galois type with residue field k. We have an associated continuous semisimple representation $\overline{\rho}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(k)$, such that $\det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)) \equiv P_v(X) \mod \mathfrak{m}$. Fix a tuple $(Q, (\alpha_v)_{v \in Q})$, where

- $Q \subset S$ and $Q \cap Q^c = \varnothing$.
- Each place $v \in Q$ is split over F^+. Moreover, for each place $v \in Q$, there exists an imaginary quadratic subfield $F_0 \subset F$, such that q_v splits in F_0.
- For each place $v \in Q$, $\overline{\rho}_{\mathfrak{m}}$ is unramified at v and v^c and α_v is a root of $\det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)))$.

For each $v \in Q$, let d_v be multiplicity of α_v as a root of $\det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v))$. Fix the partitions

$$\mu_v : 2n = d_v + (n - d_v) + n$$

$$\nu_v : n = d_v + (n - d_v).$$

Let

$$\Delta_v = \bigsqcup_{m \in M^+_{\mu_v}} [\mathfrak{p}_{\mu_v,1} m \mathfrak{p}_{\mu_v,1}] \subset \mathrm{GL}_n(F_v).$$

nonumber

Now we recall the theory of Hecke algebras of a monoid from [ACC+18, Section 2.1.9]. Specifically, we consider the restriction from $\widetilde{G}$ to P

$$r_P : \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})) \to \mathcal{H}(P(F^+_{\overline{v}}), P(F^+_{\overline{v}}) \cap \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))$$

and integration along fibres

$$r_G : \mathcal{H}(P(F^+_{\overline{v}}), P(F^+_{\overline{v}}) \cap \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}) \to \mathcal{H}(G(F^+_{\overline{v}}), G(F^+_{\overline{v}}) \cap \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))$$

combined with the isomorphism

$$\mathcal{H}(G(F^+_{\overline{v}}), G(F^+_{\overline{v}}) \cap \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})) \cong \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})),$$

we get a map

$$\mathcal{S}^+_v : \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})) \to \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})).$$

Write $P_{n,n} = M_{n,n} L_{n,n}$ for the parabolic subgroup of $GL_{2n}(F_v)$ corresponding to the partition $2n = n+n$, together with its Levi decomposition. For a given $m \in M^{++}$, from [ACC+18, Section 2.1.9], we know that

$$\mathcal{S}^+_v(\iota_v^{-1}([\mathfrak{p}_{\mu_v,1} m \mathfrak{p}_{\mu_v,1}])) = |\delta_P(m)^{-1}| \iota_v^{-1}([(\mathfrak{p}_{\mu_v,1} \cap M_{n,n}) m (\mathfrak{p}_{\mu_v,1} \cap M_{n,n})]).$$

By the same argument as in the proof of Lemma 3.2, we see that there exists $m \in M^{++}$, such that the right-hand side is invertible in $\mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}}))$. Thus, we can extend the homomorphism to

$$\begin{aligned}\mathcal{S}_v : \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))[(\iota_v^{-1}([\mathfrak{p}_{\mu_v,1} m \mathfrak{p}_{\mu_v,1}]))^{-1}] \\ \to \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})).\end{aligned}$$

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This homomorphism fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{O}[\widetilde{\Xi}_{\overline{v},1}]^{S_{\mu_v}} & \longrightarrow & \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))[(\iota_v^{-1}([\mathfrak{p}_{\mu_v,1} m \mathfrak{p}_{\mu_v,1}]))^{-1}] \\ \downarrow \mathcal{S}_v^f & & \downarrow \mathcal{S}_v \\ \mathcal{O}[\Xi_{v,1}]^{S_{\nu_v}} \otimes_{\mathcal{O}} \mathcal{O}[\Xi_{v^c}]^{S_n} & \longrightarrow & \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})), \end{array}$$

where $\mathcal{S}_v^f$ is the unique homomorphism which corresponds the polynomial $\prod_{i=1}^{2n}(T - X_i)$ to the tuple of polynomials $\prod_{i=1}^{d_v}(T - X_i), \prod_{i=d_v+1}^{n}(T - X_i), \mathcal{S}_n(q_v^{n(2n-1)} P_{v^c}^\vee(q_v^{1-2n} X))$ and maps $\tau_{\overline{v}}$ to τ_v.

We can define global Hecke algebras associated to our Taylor-Wiles data:

$$\widetilde{\mathcal{H}}_Q^S = \widetilde{\mathcal{H}}^S \otimes_{\mathbf{Z}} \bigotimes_{v \in Q} \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))[(\iota_v^{-1}([\mathfrak{p}_{\mu_v,1} m \mathfrak{p}_{\mu_v,1}]))^{-1}]$$

$$\widetilde{\mathbf{T}}_Q^S = \widetilde{\mathbf{T}}^S \otimes_{\mathbf{Z}} \bigotimes_{v \in Q} \mathcal{O}[\widetilde{\Xi}_{\overline{v},1}]^{S_{\mu_v}}$$

$$\mathcal{H}_Q^S = \mathcal{H}^S \otimes_{\mathbf{Z}} \bigotimes_{v \in Q} \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}}))$$

$$\mathbf{T}_Q^S = \mathbf{T}^S \otimes_{\mathbf{Z}} \bigotimes_{v \in Q} \mathcal{O}[\Xi_{v,1}]^{S_{\nu_v}} \otimes_{\mathcal{O}} \mathcal{O}[\Xi_{v^c}]^{S_n}.$$

The following proposition follows from the discussion above:

Proposition 5.2. *There exist homomorphisms $\mathcal{S}_Q^f : \widetilde{\mathbf{T}}_Q^S \to \mathbf{T}_Q^S$ and $\mathcal{S}_Q : \widetilde{\mathcal{H}}_Q^S \to \mathcal{H}_Q^S$ fitting into a commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathbf{T}}_Q^S & \longrightarrow & \widetilde{\mathcal{H}}_Q^S \\ \downarrow \mathcal{S}_Q^f & & \downarrow \mathcal{S}_Q \\ \mathbf{T}_Q^S & \longrightarrow & \mathcal{H}_Q^S, \end{array}$$

where $\mathcal{S}_Q^f$ coincides with $\mathcal{S}_v^f$ at places $v \in Q$ and with the Satake isomorphism from [NT16, Proposition-Definition 5.3] at places $v \notin S$.

Let $\widetilde{K}$ be a good subgroup of $\widetilde{G}(\mathbf{A}_{F^+}^\infty)$, such that $\widetilde{K}^S = \widetilde{G}(\widehat{\mathcal{O}}_{F^+}^{\overline{S}})$ and $\widetilde{K}$ is decomposed with respect to P. We can define subgroups $\widetilde{K}_1(Q) \subset \widetilde{K}_0(Q) \subset \widetilde{K}$ as follows:

- If $\overline{v} \notin \overline{Q}$, then $\widetilde{K}_1(Q)_{\overline{v}} = \widetilde{K}_0(Q)_{\overline{v}} = \widetilde{K}_{\overline{v}}$.
- If $\overline{v} \in \overline{Q}$, then $\widetilde{K}_1(Q)_{\overline{v}} = \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})$ and $\widetilde{K}_0(Q)_{\overline{v}} = \iota_v^{-1}(\mathfrak{p}_{\mu_v})$.

Let $K_1(Q), K_0(Q), K$ be the images in $G(\mathbf{A}_{F^+}^\infty)$ of the intersections of $\widetilde{K}_1(Q), \widetilde{K}_0(Q), \widetilde{K}$ with $P(\mathbf{A}_{F^+}^\infty)$. From the definition, we can see that all the subgroups from the previous sentence are decomposed with respect to P.

Proposition 5.3. *For $i = 0, 1$, we have*

1. *The open immersion $i : X_{\widetilde{K}_i(Q)}^P \to \widetilde{X}_{\widetilde{K}_i(Q)}^P$ yields a split epimorphism* $i^* : H^*(\widetilde{X}_{\widetilde{K}_i(Q)}^P, \mathcal{O}) \to H^*(X_{\widetilde{K}_i(Q)}^P, \mathcal{O})$.
2. *The proper map $j : X_{\widetilde{K}_i(Q)_P}^P \to X_{K_i(Q)}$ yields a split monomorphism* $j^* : H^*(X_{K_i(Q)}, \mathcal{O}) \to H^*(X_{\widetilde{K}_i(Q)}^P, \mathcal{O})$.

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3. *For all* $t \in \mathcal{H}_{\mathcal{O}}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))$ *and* $h \in H^*(\widetilde{X}^P_{\widetilde{K}_i(Q)}, \mathcal{O})$*, we have* $i^*(th) = r_P(t) i^*(h)$.
4. *For all* $t \in \mathcal{H}_{\mathcal{O}}(\widetilde{P}(\mathbf{A}_{F^+}^{\infty,\overline{S}}), \widetilde{K}_i(Q)_P^{\overline{S}})$ *and* $h \in H^*(X_{K_i(Q)}, \mathcal{O})$*, we have* $j^*(r_G(t)h) = tj^*(h)$.

Proof. This follows from the discussion above Proposition 5.1 and [ACC+18, Lemma 2.1.14]. □

Now let $\mathfrak{m}_Q \subset \mathbf{T}^S_Q$ be the maximal ideal generated by $\mathfrak{m}$ and the kernels of the maps $\mathcal{O}[\widetilde{\Xi}_{\overline{v},1}]^{S_{\mu_v}} \to k$ associated to the polynomials $(x-\alpha_v)^{d_v}, \det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v))/(x-\alpha_v)^{d_v}, \det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{v^c}))$ for $v \in Q$. Also, let $\widetilde{\mathfrak{m}}_Q = S_Q^{f\,-1}(\mathfrak{m}_Q)$.

Proposition 5.4. *For* $i = 0, 1$*, the map* $S^f_Q : \widetilde{\mathbf{T}}^S_Q \to \mathbf{T}^S_Q$ *descends to homomorphisms*

$$\widetilde{\mathbf{T}}^S_Q(H^*(\widetilde{X}^P_{\widetilde{K}_i(Q)}, \mathcal{O})) \to \mathbf{T}^S_Q(H^*(X_{K_i(Q)}, \mathcal{O}))$$

$$\widetilde{\mathbf{T}}^S_Q(H^*(\partial \widetilde{X}_{\widetilde{K}_i(Q)}, \mathcal{O})_{\widetilde{\mathfrak{m}}}) \to \mathbf{T}^S_Q(H^*(X_{K_i(Q)}, \mathcal{O})_{\mathfrak{m}}).$$

Proof. To prove the first statement, we need to show that for $t \in \mathrm{Ann}_{\widetilde{\mathbf{T}}^S_Q}(H^*(\widetilde{X}^P_{\widetilde{K}_i(Q)}, \mathcal{O}))$, we have $S_Q(t) \in \mathrm{Ann}_{\mathbf{T}^S_Q}(H^*(X_{K_i(Q)}, \mathcal{O}))$. Let α be the right inverse of i^* and β be the left inverse of j^*. Take any $h \in H^*(X_{K_i(Q)}, \mathcal{O})$. Then we can write

$$\begin{aligned} S_Q(t)h &= r_G(r_P(t))h = \beta(j^*(r_G(r_P(t))h)) = \beta(r_P(t) j^*(h)) \\ &= \beta(r_P(t) i^*(\alpha(j^*(h)))) = \beta(i^*(t\alpha(j^*(h)))) = \beta(i^*(0)) = 0. \end{aligned}$$

To prove the second statement, it is enough to note that $H^*(\widetilde{X}^P_{\widetilde{K}_i(Q)}, \mathcal{O})_{\widetilde{\mathfrak{m}}} \cong H^*(\partial \widetilde{X}_{\widetilde{K}_i(Q)}, \mathcal{O})_{\widetilde{\mathfrak{m}}}$ by [ACC+18, Theorem 2.4.2]. □

6. Galois deformation theory

Let $E \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$, with valuation ring $\mathcal{O}$, uniformiser ϖ and residue field k. Given a complete Noetherian local $\mathcal{O}$-algebra Λ with residue field k, we let CNL_Λ denote the category of complete Noetherian local Λ-algebras with residue field k. We refer to an object in CNL_Λ as a CNL_Λ-algebra. We fix a number field F and let S_p be the set of places of F above p. We assume that E contains the images of all embeddings of F in $\overline{\mathbf{Q}}_p$. We also fix a continuous absolutely irreducible homomorphism $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$. We assume throughout that $p \nmid 2n$.

Following [ACC+18, Definition 6.2.2], we call a global deformation problem a tuple

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S}),$$

where

- S is a finite set of finite places of F containing S_p and all the places at which $\overline{\rho}$ is ramified.
- Λ_v is an object of $\mathrm{CNL}_{\mathcal{O}}$ for each $v \in S$.
- $\mathcal{D}_v$ is a local deformation problem ([ACC+18, Section 6.2.1]) for each $v \in S$.

Associated to this global deformation problem, we have a completed tensor product $\Lambda = \widehat{\otimes}_{v \in S} \Lambda_v$. A global deformation problem determines a representable functor $\mathcal{D}_{\mathcal{S}} : \mathrm{CNL}_\Lambda \to \mathbf{Set}$ which takes an object $A \in \mathrm{CNL}_\Lambda$ to the set of deformations $\rho : G_F \to \mathrm{GL}_n(A)$ of type $\mathcal{S}$.

Let v be a finite place of F, such that $v \notin S$ and $q_v \equiv 1 \pmod p$. We let $\mathcal{D}^1_v$ denote the local deformation problem consisting of all lifts which associate $A \in \mathrm{CNL}_{\Lambda_v}$ to the set of lifts which are $1 + M_n(\mathfrak{m}_A)$-conjugate to a lift of the form $s_v \oplus \psi_v$, where s_v is unramified and the image of ψ_v under

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inertia is contained in the set of scalar matrices. This is indeed a local deformation problem, as is shown in [Tho12, Lemma 4.2].

Lemma 6.1. *Let* $\overline{r} : G_{F_v} \to \mathrm{GL}_n(k)$ *be an unramified continuous representation and* A *is a complete Noetherian local* $\mathcal{O}$*-algebra with residue field* k *and a principal maximal ideal* $\mathfrak{m}_A$*. Suppose further that* $\overline{r}$ *may be written in the form* $\overline{r} = \overline{r}_1 \oplus \overline{r}_2$*, where* $\det(X - \overline{r}_1(\mathrm{Frob}_v))$ *and* $\det(X - \overline{r}_2(\mathrm{Frob}_v))$ *are relatively prime. Also suppose that* $q_v = 1$ *in* k*. Then any lift* $r : G_{F_v} \to \mathrm{GL}_n(A)$ *of* $\overline{r}$ *is* $1 + M_n(\mathfrak{m}_A)$*-conjugate to one of the form* $r = r_1 \oplus r_2$*, where* r_1 *and* r_2 *are lifts of* $\overline{r}_1$ *and* $\overline{r}_2$*, respectively.*

Proof. Let $n_i = \dim \overline{r}_i$. Suppose we have a lift $r_m : G_{F_v} \to \mathrm{GL}_n(A)$ of $\overline{r}$, such that $r_m \mod \mathfrak{m}_A^m$ can be written in the form $r_1 \oplus r_2$. We will show that there exists a matrix $X_m \in 1 + M_n(\mathfrak{m}_A^m)$, such that $r_{m+1} := X_m r_m X_m^{-1}$ satisfies the same condition mod $\mathfrak{m}_A^{m+1}$. Write

$$X_n = \begin{pmatrix} 1 & Y \\ Z & 1 \end{pmatrix} \qquad r_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $Y \in M_{n_1 \times n_2}(\mathfrak{m}_A^m)$ and $Z \in M_{n_2 \times n_1}(\mathfrak{m}_A^m)$. Then the condition on r_{m+1} transforms into

$$YD - AY + B = 0 \mod \mathfrak{m}_A^{m+1} \tag{6.2}$$

$$ZA - DZ + C = 0 \mod \mathfrak{m}_A^{m+1}. \tag{6.3}$$

We will focus on the first condition, the second is similar. We know that $r_m \mod \mathfrak{m}_A^m$ is block-diagonal, so we can consider $\overline{b}$, $\overline{y}$ to be the images of B and Y, respectively, in $\mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}$,

$$\overline{b}\,\overline{r}_2^{-1} = \overline{r}_1\,\overline{y}\,\overline{r}_2^{-1} - \overline{y} \tag{6.4}$$

in $M_n(\mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}) = M_n(k) \otimes_k \mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}$. Using the fact that r is a homomorphism, for $\sigma, \tau \in G_{F_v}$, we can write

$$A(\sigma)B(\tau) + B(\sigma)D(\tau) = B(\sigma\tau).$$

Rewriting and reducing mod $\mathfrak{m}_A^{n+1}$, we get

$$\overline{r}_1(\sigma)\,\overline{b}(\tau) + \overline{b}(\sigma)\,\overline{r}_2(\tau) = \overline{b}(\sigma\tau)$$

$$\overline{b}(\sigma\tau)\,\overline{r}_2^{-1}(\sigma\tau) = \overline{r}_1(\sigma)\,\overline{b}(\tau)\,\overline{r}_2^{-1}(\tau)\,\overline{r}_2^{-1}(\sigma) + \overline{b}(\sigma)\,\overline{r}_2^{-1}(\sigma). \tag{6.5}$$

Give $M_{n_1 \times n_2}(\mathfrak{m}_A^m/\mathfrak{m}_A^{m+1})$ the structure of a G_{F_v}-module via $\overline{r}_1(-)\,\overline{r}_2^{-1}$, and denote this module $\mathrm{ad}(\overline{r}_1, \overline{r}_2)$. Then the last equation implies that $\overline{b}\,\overline{r}_2^{-1}$ is in $Z^1(G_{F_v}, \mathrm{ad}(\overline{r}_1, \overline{r}_2))$. Since $\overline{r}_1, \overline{r}_2$ have coprime characteristic polynomials, we know that $H^1(G_{F_v}, \mathrm{ad}(\overline{r}_1, \overline{r}_2)) = 0$ by local Tate duality (here, we are using that $q_v = 1$ in k), which means $\overline{b}\,\overline{r}_2^{-1} \in B^1(G_{F_v}, \mathrm{ad}(\overline{r}_1, \overline{r}_2))$, and thus we can find y satisfying Eq. 6.4. □

Now we define our version of the Taylor-Wiles datum, analogous to the one appearing in [ACC+18, Section 6.2.27].

Definition 6.6. Let

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S})$$

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be a global deformation problem. A Taylor-Wiles datum of level $N \geq 1$ for $\mathcal{S}$ consists of a tuple $(Q, \alpha_{v\, v \in Q})$, where

- ∘ A finite set Q of places of F, disjoint from S, such that $q_v \equiv 1 \pmod{p^N}$ for each $v \in Q$.
- ∘ For each $v \in Q$, α_v is an eigenvalue of $\overline{\rho}(\mathrm{Frob}_v)$.

Given a Taylor-Wiles datum $(Q, (\alpha_v))$, we define a global deformation problem

$$\mathcal{S}_Q = (\overline{\rho}, S \cup Q, \{\Lambda_v\}_{v \in S} \cup \{\mathcal{O}_{F_v}\}_{v \in Q}, \{\mathcal{D}_v\}_{v \in S} \cup \{\mathcal{D}_v^1\}_{v \in Q}).$$

Define $\Delta_Q = \prod_{v \in Q} \Delta_v$. The representing object $R_{\mathcal{S}_Q}$ has a structure of a $\mathcal{O}[\Delta_Q]$-algebra satisfying $R_{\mathcal{S}_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} = R_{\mathcal{S}}$.

Proposition 6.7. *Take $T = S$, and let $q > h^1_{\mathcal{S}^\perp, T}(\mathrm{ad}\, \overline{\rho}(1))$. Assume that $F = F^+ F_0$, where F_0 is an imaginary quadratic field, that $\zeta_p \notin F$ and that $\overline{\rho}(G_{F(\zeta_p)})$ is adequate. Then for every $N \geq 1$, there exists a choice of Taylor-Wiles datum $(Q_N, (\alpha_v)_{v \in Q})$ of level N satisfying the following:*

1. $|Q_N| = q$.
2. *For each $v \in Q_N$, the rational prime below v splits in F_0 and $v^c \notin Q_N$.*
3. *Let $g = q - n^2[F^+ : \mathbf{Q}]$. Then there is a surjective morphism*

$$R_{\mathcal{S}}^{T, loc}[[X_1, \ldots, X_g]] \to R_{\mathcal{S}_Q}^T,$$

in CNL_Λ.

Proof. The proof is very similar to the proof of [ACC+18, Proposition 6.2.32] (cf. [Tho12, Proposition 4.4]), we omit the details. □

7. Representations into Hecke algebras

In this section, we construct the necessary Galois representations into the Hecke algebras associated to G. From Proposition 5.4, we know that we can create representations valued in the Hecke algebra acting on $H^*(X_{K_i(Q)}, \mathcal{O})_{\mathfrak{m}_Q}$ from representations valued in the Hecke algebra acting on $H^*(\partial \widetilde{X}_{\widetilde{K}_i(Q)}, \mathcal{O})_{\widetilde{\mathfrak{m}}_Q}$. The latter representations will be constructed by glueing together Galois representations associated to cuspidal cohomological automorphic representations of $\widetilde{G}(\mathbf{A}_{F^+}^\infty)$ as in [Sch15] and using the local computations of Section 3.

7.1. Hecke algebras for $\widetilde{G}$

Theorem 7.1. *Suppose that $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup which is decomposed with respect to P. Then there exists a $2n$-dimensional $\widetilde{\mathbf{T}}_Q^S(H_c^*(X_{\widetilde{K}_1(Q)}, \mathcal{O}))/I$-valued group determinant $D_{c,Q}$ of $G_{F,S}$ for some ideal I of nilpotence degree depending only on n and $[F : \mathbf{Q}]$, such that the following properties hold:*

1. *If $v \notin S$ is a place of F, then $D_{c,Q}(X - \mathrm{Frob}_v)$ is equal to the image of $\widetilde{P}_v(X)$ in $\widetilde{\mathbf{T}}_Q^S(H_c^*(X_{\widetilde{K}_1(Q)}, \mathcal{O}))/I[X]$.*
2. *If $v \in Q$, then for any $\sigma \in G_{F,S}$ and $\tau \in I_{F_v}$, we have the relation*

$$\mathrm{Tr}_{D_{c,Q}}\left(\sigma\, \mathrm{Res}_{q_v, \mu_v}^{(2n)!}\, \mathrm{Res}_{\mu_v}^{(2n)!}\left(\sum_{i=1}^{k-1} E_{\mu_v, i}(\varphi_v) + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu_v, k}(\varphi_v) - \mathrm{Res}_{\mu_v}\, \tau\right)\right) = 0.$$

Proof. This follows from Proposition 3.12 by using [ACC+18, Theorem 2.3.3] and [Sch15, Corollary 5.1.11] (see proof of [ACC+18, Proposition 3.2.2]). □

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Now we prove the version of the previous proposition for noncompactly supported cohomology:

Theorem 7.2. *Suppose that $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup which is decomposed with respect to P. Then there exists a 2n-dimensional $\widetilde{\mathbf{T}}_Q^S(H^*(X_{\widetilde{K}_1(Q)}, \mathcal{O}))/I$-valued group determinant D_Q of $G_{F,S}$ for some ideal I of nilpotence degree depending only on n and $[F : \mathbf{Q}]$, such that the following properties hold:*

1. *If $v \notin S$ is a place of F, then $D_Q(X - \mathrm{Frob}_v)$ is equal to the image of $\widetilde{P}_v(X)$ in $\widetilde{\mathbf{T}}_Q^S(H^*(X_{\widetilde{K}_1(Q)}, \mathcal{O}))/I[X]$.*
2. *If $v \in Q$, then for any $\sigma \in G_{F,S}$ and $\tau \in I_{F_v}$, we have the relation*

$$\mathrm{Tr}_{D_Q}\left(\sigma \,\mathrm{Res}_{q_v,\mu_v}^{(2n)!}\,\mathrm{Res}_{\mu_v}^{(2n)!}\left(\sum_{i=1}^{k-1} E_{\mu_v,i}(\varphi_v) + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu_v,k}(\varphi_v) - \mathrm{Res}_{\mu_v}\tau\right)\right) = 0.$$

Proof. Denote by $\widetilde{\mathbf{T}}_{Q,\iota}^S(H_c^*(X_{\widetilde{K}_1(Q)}, \mathcal{O}))$ the image of $\widetilde{\mathbf{T}}_Q^S$ under the homomorphism

$$\widetilde{\mathbf{T}}_Q^S \to \mathcal{H}_{\mathcal{O}}(\widetilde{G}(\mathbf{A}_{F^+}^\infty), \widetilde{K}_1(Q)) \xrightarrow{\iota_{\mathcal{H}}} \mathcal{H}_{\mathcal{O}}(\widetilde{G}(\mathbf{A}_{F^+}^\infty), \widetilde{K}_1(Q)) \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(H_c^*(X_{\widetilde{K}_1(Q)}, \mathcal{O})).$$

The same argument as in the proof of Theorem 7.1 shows that there exists a group determinant D_ι valued in $\widetilde{\mathbf{T}}_{Q,\iota}^S(H_c^*(X_{\widetilde{K}_1(Q)}, \mathcal{O}))/I$ satisfying the following properties:

1. If $v \notin S$ is a place of F, then $D_Q(X - \mathrm{Frob}_v)$ is equal to the image of $\widetilde{P}_v(X)$ in $\widetilde{\mathbf{T}}_{Q,\iota}^S(H_c^*(X_{\widetilde{K}_1(Q)}, \mathcal{O}))/I[X]$.
2. If $v \notin Q$, then for any $\sigma \in G_{F,S}$ and $\tau \in I_{F_v}$, we have the relation

$$\mathrm{Tr}_{D_\iota}\left(\sigma \,\mathrm{Res}_{q_v,\mu_v}^{(2n)!}\,\mathrm{Res}_{\mu_v}^{(2n)!}\left(\sum_{i=1}^{k-1} E_{\mu_v,i}(\varphi_v) + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu_v,k}(\varphi_v) - \mathrm{Res}_{\mu_v}\tau\right)\right) = 0.$$

By [NT16, Proposition 3.7], we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_{\mathcal{O}}(\widetilde{G}(\mathbf{A}_{F^+}^\infty), \widetilde{K}_1(Q)) & \longrightarrow & \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_{\widetilde{K}_1(Q)}, \mathcal{O})) \\ \downarrow{\scriptstyle \iota_{\mathcal{H}}} & & \downarrow{\scriptstyle \wr} \\ \mathcal{H}_{\mathcal{O}}(\widetilde{G}(\mathbf{A}_{F^+}^\infty), \widetilde{K}_1(Q)) & \longrightarrow & \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma_c(X_{\widetilde{K}_1(Q)}, \mathcal{O})), \end{array} \tag{7.3}$$

where the right vertical arrow is induced by Poincaré duality. Then we get an isomorphism

$$\widetilde{\mathbf{T}}_{Q,\iota}^S(H_c^*(X_{\widetilde{K}_1(Q)}, \mathcal{O}))/I_1 \xrightarrow{\sim} \widetilde{\mathbf{T}}_Q^S(H^*(X_{\widetilde{K}_1(Q)}, \mathcal{O}))/I_2$$

over $\widetilde{\mathbf{T}}_Q^S$ for some ideals $I_{1,2}$ of nilpotence degrees depending only on n and $[F : \mathbf{Q}]$. Moreover, we can choose I_1, such that it contains I. We can conclude by making D_Q the image of D_ι under this homomorphism. □

Lemma 7.4. *Let k be a field, and let $\overline{\rho}_1, \overline{\rho}_2 : G \to GL_n(k)$ be two nonisomorphic absolutely irreducible representations. Then the extended map $k[G] \to M_n(k) \oplus M_n(k)$ defined by $\overline{\rho}_1 \oplus \overline{\rho}_2$ is surjective.*

Proof. We may pass to the algebraic closure of k (which we still denote k). Let $\ell_i : k[G] \to M_n(k)$ be the linear extension of $\overline{\rho}_i$ for $i = 1, 2$. The two maps ℓ_i are surjective by Burnside's theorem. Let A be the image of $\ell_1 \oplus \ell_2$, and let $I_i = \ker(A \to M_n(k))$, where $i = 1, 2$ corresponds to projecting on the first and second factor. Since ℓ_i are surjective, I_i are in fact two-sided ideals of $M_n(k)$. Then $I_i = M_n(k)$ or $I_i = 0$. If $I_i = M_n(k)$ for some i, then $\ell_1 \oplus \ell_2$ is surjective. Suppose then that $I_1 = I_2 = 0$. Then we have an automorphism f of $M_n(k)$ defined by $(v, f(v)) \in A$ for all $v \in M_n(k)$. Since all the automorphisms

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of $M_n(k)$ are inner, we conclude that there exists $u \in \mathrm{GL}_n(k)$, such that $A = \{(v, uvu^{-1}) \mid v \in M_n(k)\}$. But this is impossible since $\overline{\rho}_1$ and $\overline{\rho}_2$ are nonisomorphic. □

Theorem 7.5. *Suppose that $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup which is decomposed with respect to P and that for each $v \in Q$, we have $\mathrm{Res}_{\mu_v} \notin \widetilde{\mathfrak{m}}_Q$. Then there exists a continuous representation*

$$\rho_{\mathfrak{m}_Q} : G_{F,S\cup Q} \to \mathrm{GL}_n(\mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})/I)$$

satisfying the conditions below for some ideal $I \subset \mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})$ of nilpotence degree depending only on n and $[F : \mathbf{Q}]$.

1. *If $v \notin S$ is a place of F, the characteristic polynomial of $\rho_{\mathfrak{m}_Q}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $\mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}})/I[X]$.*
2. *If $v \in Q$, then $\rho_{\mathfrak{m}_Q}|_{G_{F_{v^c}}}$ is unramified.*
3. *If $v \in Q$, then $\rho_{\mathfrak{m}_Q}|_{G_{F_v}} = s \oplus \psi$, where s is unramified and $\tau \in I_{F_v}$ acts on ψ as a scalar $\langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle$.*

Proof. Using Theorem 7.1 and Theorem 7.2, we can construct a $\widetilde{\mathbf{T}}_Q^S(H_c^*(X_{\widetilde{K}_1(Q)}, \mathcal{O})_{\widetilde{\mathfrak{m}}_Q} \oplus H^*(X_{\widetilde{K}_1(Q)}, \mathcal{O})_{\widetilde{\mathfrak{m}}_Q})/I$-valued group determinant D_Q of $G_{F,S\cup Q}$. Consider the long exact sequence

$$\ldots \to H_c^i(\widetilde{X}_{\widetilde{K}_1(Q)}, \mathcal{O}) \to H^i(\widetilde{X}_{\widetilde{K}_1(Q)}, \mathcal{O}) \to H^i(\partial \widetilde{X}_{\widetilde{K}_1(Q)}, \mathcal{O}) \to H_c^{i+1}(\widetilde{X}_{\widetilde{K}_1(Q)}, \mathcal{O}) \to .$$

Using this sequence and Proposition 5.4, we know that S_Q^f descends to a homomorphism

$$\widetilde{\mathbf{T}}_Q^S(H_c^*(\widetilde{X}_{\widetilde{K}_1(Q)}, \mathcal{O})_{\widetilde{\mathfrak{m}}_Q} \oplus H^*(\widetilde{X}_{\widetilde{K}_1(Q)}, \mathcal{O})_{\widetilde{\mathfrak{m}}_Q}) \to \mathbf{T}_Q^S(H^*(X_{K_1(Q),\mathcal{O}})_{\mathfrak{m}_Q})/I_0$$

for some ideal I_0 with square 0. We can use this to construct a $2n$-dimensional group determinant D_Q^0 valued in $\mathbf{T}_Q^S(H^*(X_{K_1(Q),\mathcal{O}})_{\mathfrak{m}_Q})/I$, such that:

1. For $v \notin S$, we have $D_Q^0(X - \mathrm{Frob}_v) = P_v(X) q_v^{n(2n-1)} P_{v^c}{}^{\vee}(q_v^{1-2n}X)$.
2. For $v \in Q$, we have

$$\mathrm{Tr}_{D_Q^0}\left(S_Q^f\left(\sigma\, \mathrm{Res}_{q_v,\mu_v}^{(2n)!}\, \mathrm{Res}_{\mu_v}^{(2n)!}\left(\sum_{i=1}^{k-1} E_{\mu_v,i}(\varphi_v) + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu_v,k}(\varphi_v) - \mathrm{Res}_{\mu_v} \tau\right)\right)\right) = 0,$$

and I has nilpotence degree depending only on n and $[F : \mathbf{Q}]$. By [ACC+18, Theorem 2.3.7], there also exists an n-dimensional group determinant D_Q^1 of $G_{F,S\cup Q}$ valued in $\mathbf{T}_Q^S(H^*(X_{K_1(Q),\mathcal{O}})_{\mathfrak{m}_Q})/I$, such that $D_Q^1(X - \mathrm{Frob}_v) = P_v(X)$ for $v \notin S$. Then the group determinants $D_Q^1 \oplus D_Q^{1\perp}$ and D_Q^0 are equal. Moreover, since $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, there exists a continuous representation

$$\rho_{\mathfrak{m}_Q} : G_{F,S\cup Q} \to \mathrm{GL}_n(\mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})/I),$$

such that the characteristic polynomial of $\rho_{\mathfrak{m}_Q}$ is associated to D_Q^1. Let $\rho'_{\mathfrak{m}_Q} := \rho_{\mathfrak{m}_Q} \oplus \rho_{\mathfrak{m}_Q}^\perp$. Writing out the relation at places $v \in Q$, we get

$$\mathrm{Tr}(\rho'_{\mathfrak{m}_Q}(\sigma) S_Q^f(\mathrm{Res}_{q_v,\mu_v}^{(2n)!}\, \mathrm{Res}_{\mu_v}^{(2n)!}(\sum_{i=1}^{k-1} E_{\mu_v,i}(\rho'_{\mathfrak{m}_Q}(\varphi_v))$$
$$+ \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu_v,k}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) - \mathrm{Res}_{\mu_v} \rho'_{\mathfrak{m}_Q}(\tau)))) = 0.$$

Since $\mathrm{Res}_{\mu_v} \notin \widetilde{\mathfrak{m}}_Q$, we know that $\overline{\rho}_{\mathfrak{m}}$ and $\overline{\rho}_{\mathfrak{m}}^{\perp}$ are not isomorphic. Applying Nakayama's lemma and Lemma 7.4, we see that the extended map

$$\mathbf{T}_Q^S[G_{F,S\cup Q}] \to M_n(\mathbf{T}_Q^S(H^*(X_{K_1(Q),\mathcal{O}})_{\mathfrak{m}_Q})/I) \oplus M_n(\mathbf{T}_Q^S(H^*(X_{K_1(Q),\mathcal{O}})_{\mathfrak{m}_Q})/I)$$

given by $\rho_{\mathfrak{m}_Q} \oplus \rho_{\mathfrak{m}_Q}^{\perp}$ is surjective. Considering the trace relation above with σ replaced by an arbitrary element of $\mathbf{T}_Q^S[G_{F,S\cup Q}]$, we conclude that

$$S_Q^f(\mathrm{Res}_{q_v,\mu_v}^{(2n)!}\,\mathrm{Res}_{\mu_v}^{(2n)!}(\sum_{i=1}^{k-1} E_{\mu_v,i}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) \\ + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu_v,k}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) - \mathrm{Res}_{\mu_v}\,\rho'_{\mathfrak{m}_Q}(\tau))) = 0.$$

Since $q_v \equiv 1 \mod p$, we know that $\mathrm{Res}_{q_v,\mu_v} \notin \widetilde{\mathfrak{m}}_Q$. Thus

$$S_Q^f\left(\sum_{i=1}^{k-1} E_{\mu_v,i}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle E_{\mu_v,k}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) - \mathrm{Res}_{\mu_v}\,\rho'_{\mathfrak{m}_Q}(\tau)\right) = 0.$$

This implies that

$$\rho_{\mathfrak{m}_Q}(\tau) = S_Q^f\left(\sum_{i=1}^{k-1} \mathrm{Res}_{\mu_v}^{-1} E_{\mu_v,i}(\rho_{\mathfrak{m}_Q}(\varphi_v))\right) + S_Q^f(\langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle\,\mathrm{Res}_{\mu_v}^{-1} E_{\mu_v,k}(\rho_{\mathfrak{m}_Q}(\varphi_v))).$$

Using Proposition 5.2, we can transform the equation above into

$$\rho_{\mathfrak{m}_Q}(\tau) = \mathrm{Res}_{\nu_v}^{-1} E_{\nu_v,1}(\rho_{\mathfrak{m}_Q}(\varphi_v)) + \langle \mathrm{Art}_{F_v}^{-1}(\tau)\rangle\,\mathrm{Res}_{\nu_v}^{-1} E_{\nu_v,2}(\rho_{\mathfrak{m}_Q}(\varphi_v)).$$

Let $\mathbf{T} := \mathbf{T}_Q^S(H^*(X_{K_1(Q),\mathcal{O}})_{\mathfrak{m}_Q})/I$. Consider the decomposition $\overline{\rho}_{\mathfrak{m}} = \overline{r}_1 \oplus \overline{r}_2$, corresponding to the Frobenius generalised eigenspaces of all eigenvalues not equal to α_v and α_v, respectively. Then

$$\mathbf{T}^n = \mathrm{Res}_{\nu_v}^{-1} E_{\nu_v,1}(\rho_{\mathfrak{m}_Q}(\varphi_v))\mathbf{T}^n \oplus \mathrm{Res}_{\nu_v}^{-1} E_{\nu_v,2}(\rho_{\mathfrak{m}_Q}(\varphi_v))\mathbf{T}^n$$

is the unique $\rho_{\mathfrak{m}_Q}(\varphi_v)$-invariant lift of $\overline{r}_1 \oplus \overline{r}_2$, and we are done by Lemma 6.1. □

7.2. Hecke algebras for G

Let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$. Further let S be a finite set of finite places of F containing the p-adic places and stable under complex conjugation satisfying the following condition:

1. Let l be a rational prime, such that there exists a place above l in S or l is ramified in F. Then there exists an imaginary quadratic subfield $F_0 \subset F$, such that l splits in F_0.

Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, such that for all $v \notin S$, we have $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$. Let $\mathfrak{m} \subset \mathbf{T}^S(K,\lambda)$ be a non-Eisenstein maximal ideal with residue field k. By [ACC+18, Theorem 2.3.5], there exists an associated residual representation $\overline{\rho}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K,\lambda)/\mathfrak{m})$. By [ACC+18, Theorem 2.3.7], there exists an ideal $I \subset \mathbf{T}^S(K,\lambda)$ of nilpotence degree depending only on n and $[F:\mathbf{Q}]$ and a continuous lift $\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}/I)$, such that for each $v \in S$, $\det(X - \rho_{\mathfrak{m}}(\mathrm{Frob}_v))$ is the image of $P_v(X)$ in $\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}/I[X]$. We consider the following Taylor-Wiles datum: a tuple $(Q,(\alpha_v)_{v\in Q})$ consisting of

- A finite set Q of places of F, disjoint from Q^c, such that $q_v \equiv 1 \pmod p$ for each $v \in Q$.
- Each $v \in Q$ is split in F^+, and there exists an imaginary quadratic subfield $F_0 \subset F$, such that v is split in F_0. Moreover, $\overline{\rho}_{\mathfrak{m}}$ is unramified at v and v^c.
- α_v is a root of $\det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v))$.

Consider the partition $\nu_v : n = d_v + (n - d_v)$, where d_v is the multiplicity of α_v as a root of $\det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v))$.

We define auxillary level subgroups $K_1(Q) \subset K_0(Q) \subset K$. They are good subgroups of $\mathrm{GL}_n(\mathbf{A}_F^\infty)$ defined by the following conditions:

- ◦ if $v \notin Q$, then $K_1(Q)_v = K_0(Q)_v = K_v$.
- ◦ if $v \in Q$, then $K_0(Q)_v = \mathfrak{p}_{\nu_v}$ and $K_1(Q)_v = \mathfrak{p}_{\nu_v,1}$.

We have a natural isomorphism $K_0(Q)/K_1(Q) \cong \Delta_Q = \prod_{v \in Q} \Delta_v$. Let $S' = S \cup Q \cup Q^c$. We define $\mathbf{T}_Q^{S'} = \mathbf{T}^{S \cup Q} \otimes_{\mathbf{Z}} \mathbf{Z}[\Xi_{v,1}]^{S_{\nu_v}}$. Let $\mathbf{T}_Q^{S'}(K_0(Q), \lambda)$ and $\mathbf{T}_Q^{S'}(K_0(Q)/K_1(Q), \lambda)$ be the images of $\mathbf{T}_Q^{S'}$ in $\mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_{K_0(Q)}, V_\lambda))$ and $\mathrm{End}_{\mathbf{D}(\mathcal{O}[\Delta_Q])}(R\Gamma(X_{K_1(Q)}, V_\lambda))$, respectively. Let $\mathfrak{m}_Q$ be the maximal ideal of $\mathbf{T}_Q^{S'}$ generated by $\mathfrak{m}$ and the kernels of the homomorphisms $\mathbf{Z}[\Xi_{v,1}]^{S_{\nu_v}} \to k$ given by the coefficients of polynomials $(X - \alpha_v)^{d_v}, \det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v))/(X - \alpha_v)^{d_v}$.

Theorem 7.6. *We have natural isomorphisms*

$$R\Gamma(X_K, V_\lambda)_{\mathfrak{m}} \simeq R\Gamma(X_{K_0(Q)}, V_\lambda)_{\mathfrak{m}_Q}$$

$$R\Gamma(X_{K_0(Q)}, V_\lambda)_{\mathfrak{m}_Q} \simeq R\Gamma(\Delta_Q, R\Gamma(X_{K_1(Q)}, V_\lambda))_{\mathfrak{m}_Q}$$

in $\mathbf{D}(\mathcal{O})$.

Proof. The second isomorphism is straightforward. For the first, we can check on the level of cohomology. It is enough to check that it is an isomorphism in $\mathbf{D}(k)$ after applying the functor $- \otimes^{\mathbf{L}} k$. Thus, we need to show that the map

$$H^*(X_K, V_\lambda/\varpi)_{\mathfrak{m}} \to H^*(X_{K_0(Q)}, V_\lambda/\varpi)_{\mathfrak{m}_Q}$$

is an isomorphism. We can do this one prime at a time, so we can assume $Q = \{v\}$. For each j, let

$$M_j := \lim_{m \to \infty} H^j(X_{K(v^m)}, V_\lambda/\varpi)_{\mathfrak{m}},$$

where $K(v^m)_w = K_w$ for places $w \neq v$ and $K(v^m)_v$ is the principal congruence subgroup of level v^m. We have two Hochschild-Serre spectral sequences:

$$H^i(\mathrm{GL}_n(\mathcal{O}_{F_v}), M_j) \Rightarrow H^{i+j}(X_K, V_\lambda/\varpi)_{\mathfrak{m}}$$

$$e_{\alpha_v} H^i(\mathfrak{p}_{\nu_v}, M_j) \Rightarrow e_{\alpha_v} H^{i+j}(X_{K_0(Q)}, V_\lambda/\varpi) = H^{i+j}(X_{K_0(Q)}, V_\lambda/\varpi)_{\mathfrak{m}_Q}.$$

There is a natural map ι^* between these spectral sequences, which arises from deriving the map

$$M_j^{\mathrm{GL}_n(\mathcal{O}_{F_v})} \to M_j^{\mathfrak{p}_{\nu_v}} \to e_{\alpha_v} M_j^{\mathfrak{p}_{\nu_v}}.$$

Thus, it is enough to show that ι^* is an isomorphism. M_j is admissible, and we can use [Vig98, Theorem III.6] to write M_j as a direct sum of $\mathrm{GL}_n(F_v)$-modules, each belonging to a single block. Let $N \subset M_j$ be a summand from a nonunipotent block. Let $T_p(k)$ be the p-power part of $T(k)$. We note that both $H^i(\mathrm{GL}_n(\mathcal{O}_{F_v}), N)$ and $H^i(\mathfrak{p}_{\nu_v}, N)$ inject into $H^i(\mathrm{Iw}, N)$, which in turn is equal to $H^i(T_p(k), N^{\mathrm{Iw}^p})$. Since N is a from a nonunipotent block, we know that $N^{\mathrm{Iw}^p} = 0$, and so

$$H^i(\mathrm{GL}_n(\mathcal{O}_{F_v}), N) = H^i(\mathfrak{p}_{\nu_v}, N) = 0.$$

Thus, we can restrict to the summand $M_j^1 \subset M_j$ from the unipotent block, and it is enough to prove that

$$\iota^* : H^i(\mathrm{GL}_n(\mathcal{O}_{F_v}), M_j^1) \to e_{\alpha_v} H^i(\mathfrak{p}_{\nu_v}, M_j^1)$$

is an isomorphism. By [CHT08, Theorem B.1], the unipotent block in our case consists of representations generated by their Iw^p-invariant vectors. Therefore, every irreducible subrepresentation $\pi \subset M_j^1$ has a Iw^p-invariant vector. It follows from the argument similar to the proof of Proposition 2.1 that

$$\pi \subset \mathrm{Ind}_B^{GL_n} \chi_1 \otimes \ldots \otimes \chi_n,$$

where χ_i are tamely ramified characters whose restriction to $\mathcal{O}_{F_v}/(1+\varpi\mathcal{O}_{F_v})$ has p-power order. But these characters are valued in $k^\times$ which has order coprime to p, which means χ_i are in fact unramified.

We can now select the smallest number $d > 0$, such that π embeds into $M_j[\mathfrak{m}^d]$. Since π is irreducible, it must then embed into $M_j[\mathfrak{m}^d]/M_j[\mathfrak{m}^{d-1}]$ and local-global compatibility for Iwahori level ([ACC⁺18, Theorem 3.1.1]) then implies that $\{\chi_i(\varpi)\}_{i=1,\ldots,n}$ is the set of eigenvalues of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$. Thus, we have shown that $M_j \in \mathcal{C}$, and we are done by Theorem 2.14. □

Theorem 7.7. *There exists an ideal $I \subset \mathbf{T}_Q^{S'}(K_0(Q)/K_1(Q),\lambda)_{\mathfrak{m}_Q}$ of nilpotence degree depending only on n and $[F:\mathbf{Q}]$, together with a continuous homomorphism*

$$\rho_{\mathfrak{m},Q} : G_{F,S\cup Q} \to \mathrm{GL}_n(\mathbf{T}_Q^{S'}(K_0(Q)/K_1(Q),\lambda)_{\mathfrak{m}_Q}/I)$$

lifting $\overline{\rho}_{\mathfrak{m}}$ and satisfying the following conditions:

1. *For a finite place $v \notin S \cup Q$ of F, $\det(X - \rho_{\mathfrak{m},Q}(\mathrm{Frob}_v))$ equals to the image of $P_v(X)$ in $\mathbf{T}_Q^{S'}(K_0(Q)/K_1(Q),\lambda)_{\mathfrak{m}_Q}/I[X]$.*
2. *For $v \in Q$, $\rho_{\mathfrak{m},Q}|_{G_{F_{v^c}}}$ is unramified and $\rho_{\mathfrak{m},Q}|_{G_{F_v}}$ is a lifting of type $\mathcal{D}_v$, and the induced map $\mathcal{O}[\Delta_Q] \to \mathbf{T}_Q^{S'}(K_0(Q)/K_1(Q),\lambda)_{\mathfrak{m}_Q}/I$ is a homomorphism of $\mathcal{O}[\Delta_Q]$-algebras.*

Proof. We first make a few reductions. Let us show that we can reduce to the situation where $\det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v))$ and $\det(X - \overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{v^c}))$ are coprime for each $v \in Q$. To achieve this, we will use twisting. Pick an odd prime $l \neq p$ and consider a character $\psi : G_F \to \mathcal{O}^\times$ of order ℓ, such that $\det(X-(\overline{\rho}_{\mathfrak{m}}\otimes\overline{\psi})(\mathrm{Frob}_v))$ and $\det(X-(\overline{\rho}_{\mathfrak{m}}\otimes\overline{\psi})(\mathrm{Frob}_{v^c}))$ are coprime. Let S_ψ denote the places of F at which ψ is ramified. We will further require that S_ψ is disjoint from S'. Define a good subgroup $K^\psi \subset K$ given by $K_v^\psi = K_v$ at places v at which ψ is unramified and $K_v^\psi = \ker(\mathrm{GL}_n(\mathcal{O}_{F_v}) \to k(v)^\times/(k(v)^\times)^l)$ at places v, where ψ is ramified. Following the discussion above [ACC⁺18, Proposition 2.2.22], we have a homomorphism $f_\psi : \mathbf{T}^{S'\cup S_\psi}(K^\psi,\lambda) \to \mathbf{T}^{S'\cup S_\psi}(K^\psi,\lambda)$ given by

$$f_\psi([K^{\psi\, S'\cup S_\psi} g K^{\psi\, S'\cup S_\psi}]) = \psi^{-1}(\mathrm{Art}(\det(g)))[K^{\psi\, S'\cup S_\psi} g K^{\psi\, S'\cup S_\psi}]. \tag{7.8}$$

We have a maximal ideal $\mathfrak{m}_\psi = f_\psi(\mathfrak{m})$ of $\mathbf{T}^{S'\cup S_\psi}(K^\psi,\lambda)$. [ACC⁺18, Proposition 2.2.22] implies an isomorphism $\overline{\rho}_{\mathfrak{m}} \otimes \overline{\psi} \cong \overline{\rho}_{\mathfrak{m}_\psi}$. Similarly to Eq. 7.8, we have an isomorphism

$$\mathbf{T}_Q^{S'\cup S_\psi}(K_0^\psi(Q)/K_1^\psi(Q),\lambda)_{\mathfrak{m}_{\psi Q}} \cong \mathbf{T}_Q^{S'\cup S_\psi}(K_0^\psi(Q)/K_1^\psi(Q),\lambda)_{\mathfrak{m}_Q},$$

where $\mathfrak{m}_{\psi Q}$ is the maximal ideal of $\mathbf{T}_Q^{S'\cup S_\psi}$ generated by $\mathfrak{m}_\psi$ and the kernels of the homomorphisms $\mathbf{Z}[\Xi_{v,1}]^{S_{v_v}} \to k$ given by the coefficients of polynomials $(X - \psi(\mathrm{Frob}_v)\alpha_v)^{d_v}, \det(X - \overline{\rho}_{\mathfrak{m}_\psi}(\mathrm{Frob}_v))/(X - \psi(\mathrm{Frob}_v)\alpha_v)^{d_v}$. We have a surjective map of $\mathbf{T}^{S'\cup S_\psi}$-algebras

$$\mathbf{T}_Q^{S'\cup S_\psi}(K_0^\psi(Q)/K_1^\psi(Q),\lambda)_{\mathfrak{m}_Q} \to \mathbf{T}_Q^{S'\cup S_\psi}(K_0(Q)/K_1(Q),\lambda)_{\mathfrak{m}_Q}.$$

Thus, if the theorem holds for representations into $\mathbf{T}_Q^{S'\cup S_\psi}(K_0^\psi(Q)/K_1^\psi(Q),\lambda)_{\mathfrak{m}_Q}$, it will hold for representations into $\mathbf{T}_Q^{S'\cup S_\psi}(K_0(Q)/K_1(Q),\lambda)_{\mathfrak{m}_Q}$. Since there are infinitely many ψ satisfying the conditions we require, we can vary them to conclude that the theorem holds for $\mathbf{T}_Q^{S'}(K_0(Q)/K_1(Q),\lambda)_{\mathfrak{m}_Q}$, which is our target Hecke algebra.

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Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup satisfying the following conditions:

1. $\widetilde{K}$ is decomposed with respect to P.
2. $\widetilde{K} \cap G(\mathbf{A}_{F^+}^\infty) \subset K$.
3. if $\overline{v}$ is a finite place of F^+, such that $\overline{v} \notin \overline{S}$, then $\widetilde{K}_{\overline{v}} = \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})$.

We can use the Hochschild-Serre spectral sequence to reduce to the case where $K = \widetilde{K} \cap G(\mathbf{A}_{F^+}^\infty)$. We can further reduce our theorem to the case $\lambda = 0$, by a standard use of the Hochschild-Serre spectral sequence to trivialise the weight modulo some power m at the expense of shrinking the level at p. Now the theorem follows from Theorem 7.5. □

8. Proof of Theorem 1.2 and Theorem 1.3

Let us recall the proof structure of [ACC+18, Theorem 6.1.1]. The theorem is reduced in [ACC+18] to [ACC+18, Corollary 6.5.5], which is proved using [ACC+18, Theorem 6.5.4]. The reduction does not use the 'enormous' assumption on the image of $\overline{\rho}$. Thus, it will be sufficient for us to prove an analog of [ACC+18, Theorem 6.5.4], replacing 'enormous' by 'adequate' in the hypotheses.

Let F be an imaginary CM number field, and fix the following data:

1. An integer $n \geq 2$ and a prime $p > n^2$.
2. A finite set S of finite places of F, including the places above p.
3. A (possibly empty) subset $R \subset S$ of places which are prime to p.
4. A cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$, which is regular algebraic of some weight λ.
5. A choice of isomorphism $\iota : \overline{\mathbf{Q}}_p \cong \mathbf{C}$.
   We assume that the following conditions are satisfied:
6. If l is a prime lying below an element of S, or which is ramified in F, then F contains an imaginary quadratic field in which l splits. In particular, each place of S is split over F^+ and the extension F/F^+ is everywhere unramified.
7. The prime p is unramified in F.
8. For each embedding $\tau : F \hookrightarrow \mathbf{C}$, we have
$$\lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau,n} - \lambda_{\tau c,n} < p - 2n.$$
9. For each $v \in S_p$, let $\overline{v}$ denote the place of F^+ lying below v. Then there exists a place $\overline{v}' \neq \overline{v}$ of F^+, such that $\overline{v}' \mid p$ and
$$\sum_{\overline{v}'' \neq \overline{v}, \overline{v}'} [F^+_{\overline{v}''} : \mathbf{Q}_p] > \frac{1}{2}[F^+ : \mathbf{Q}].$$
10. The residual representation $\overline{r_\iota(\pi)}$ is absolutely irreducible.
11. If v is a place of F lying above p, then π_v is unramified.
12. If $v \in R$, then $\pi_v^{\mathrm{Iw}_v} \neq 0$.
13. If $v \in S - (R \cup S_p)$, then π_v is unramified and $H^2(F_v, \mathrm{ad}\, \overline{r_\iota(\pi)}) = 0$. Moreover, v is absolutely unramified and of residue characteristic $q > 2$.
14. $S - (R \cup S_p)$ contains at least two places with distinct residue characteristics.
15. If $v \notin S$ is a finite place of F, then π_v is unramified.
16. If $v \in R$, then $q_v \equiv 1 \pmod p$ and $\overline{r_\iota(\pi)}|_{G_{F_v}}$ is trivial.
17. The representation $\overline{r_\iota(\pi)}$ is decomposed generic in the sense of [ACC+18, Definition 4.3.1] and the image of $\overline{r_\iota(\pi)}|_{G_{F(\zeta_p)}}$ is adequate.

We define an open compact subgroup $K = \prod_v K_v$ of $\mathrm{GL}_n(\widehat{\mathcal{O}}_F)$ as follows:

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- If $v \notin S$, or $v \in S_p$, then $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$.
- If $v \in R$, then $K_v = \mathrm{Iw}_v$.
- If $v \in S - (R \cup S_p)$, then $K_v = \mathrm{Iw}_{v,1}$.

By [ACC+18, Theorem 2.4.10], we can find a coefficient field $E \subset \overline{\mathbf{Q}}_p$ and a maximal ideal $\mathfrak{m} \subset \mathbf{T}^S(K, \mathcal{V}_\lambda)$, such that $\overline{\rho}_{\mathfrak{m}} \cong \overline{r_\iota(\pi)}$. After possibly enlarging E, we can and do assume that the residue field of $\mathfrak{m}$ is equal to k. For each tuple $(\chi_{v,i})_{v \in R, i=1,\dots,n}$ of characters $\chi_{v,i} : k(v)^\times \to \mathcal{O}^\times$ which are trivial modulo ϖ, we define a global deformation problem by the formula

$$\mathcal{S}_\chi = (\overline{\rho}_{\mathfrak{m}}, S, \{\mathcal{O}\}_{v \in S}, \{\mathcal{D}_v^{\mathrm{FL}}\}_{v \in S_p} \cup \{\mathcal{D}_v^\chi\}_{v \in R}) \cup \{\mathcal{D}_v^\square\}_{v \in S-(R \cup S_p)}).$$

We fix representatives $\rho_{\mathcal{S}_\chi}$ of the universal deformations which are identified modulo ϖ via the identifications $R_{\mathcal{S}_\chi}/\varpi \cong R_{\mathcal{S}_1}/\varpi$. We define an $\mathcal{O}[K_S]$-module $\mathcal{V}_\lambda(\chi^{-1}) = \mathcal{V}_\lambda \otimes_{\mathcal{O}} \mathcal{O}(\chi^{-1})$, where K_S acts on V_λ by projection to K_p and on $\mathcal{O}(\chi^{-1})$ by the projection $K_S \to K_R = \prod_{v \in R} \mathrm{Iw}_v \to \prod_{v \in R} (k(v)^\times)^n$.

Theorem 8.1. *Under assumptions (1)–(17) above, $H^*(X_K, \mathcal{V}_\lambda(1))_{\mathfrak{m}}$ is a nearly faithful $R_{\mathcal{S}_1}$-module. In other words, $\mathrm{Ann}_{R_{\mathcal{S}_1}}(H^*(X_K, \mathcal{V}_\lambda(1))_{\mathfrak{m}})$ is nilpotent.*

The rest of the paper is devoted to the proof of Theorem 8.1.

Consider the Taylor-Wiles datum $(Q, \{\alpha_v\}_{v \in Q})$ satisfying the following conditions:

- For each place $v \in Q$ of residue characteristic l, there exists an imaginary quadratic subfield $F_0 \subset F$, such that l splits in F_0.
- Q and Q^c are disjoint.

We have the following result, combining [ACC+18, Proposition 6.5.3] and Theorem 7.7:

Proposition 8.2. *There exists an integer $\delta \geq 1$ depending only on n and $[F : \mathbf{Q}]$, an ideal $J \subset \mathbf{T}_Q^{S'}(R\Gamma(X_{K_1(Q)}, V_\lambda(\chi^{-1}))_{\mathfrak{m}_Q})$, such that $J^\delta = 0$ and a continuous surjection of $\mathcal{O}[\Delta_Q]$-algebras $f_{\mathcal{S}_{\chi,Q}} : R_{\chi,Q} \to \mathbf{T}_Q^{S'}(R\Gamma(X_{K_1(Q)}, V_\lambda(\chi^{-1}))_{\mathfrak{m}_Q})/J$, such that for each finite place $v \notin S \cup Q$, the characteristic polynomial of $f_{\mathcal{S}_{\chi,Q}} \circ \rho_{\mathcal{S}_{\chi,Q}}$ equals the image of $P_v(X)$.*

Let

$$q = h^1(F_S/F, \mathrm{ad}\, \overline{\rho}_{\mathfrak{m}}(1)) \quad \text{and} \quad g = q - n^2[F^+ : \mathbf{Q}],$$

and set $\Delta_\infty = \mathbf{Z}_p^q$. Let $\mathcal{T}$ be a power series ring over $\mathcal{O}$ in $n^2|S| - 1$ variables, and let $S_\infty = \mathcal{T}[[\Delta_\infty]]$. Let $\mathfrak{a}_\infty$ be the augmentation ideal of S_∞ viewed as an augmented $\mathcal{O}$-algebra. Since $p > n$, for each $v \in R$, we can choose a tuple of pairwise distinct characters $\chi_v = (\chi_{v,1}, \dots, \chi_{v,n})$, with $\chi_{v,i} : \mathcal{O}_{F_v}^\times \to \mathcal{O}^\times$ trivial modulo ϖ. We write χ for the tuple $(\chi_v)_{v \in R}$ as well as for the induced character $\prod_{v \in R} I_v \to \mathcal{O}^\times$. Fix an imaginary quadratic subfield $F_0 \subset F$. Then for each $N \geq 1$, we fix a choice of Taylor-Wiles datum $(Q, \{\alpha_v\}_{v \in Q})$ for $\mathcal{S}_1$ of level N using Proposition 6.7. For $N = 0$, we set $Q_0 = \emptyset$. For each $N \geq 1$, we set $\Delta_N = \Delta_{Q_N}$ and fix a surjection $\Delta_\infty \to \Delta_N$. We let Δ_0 be the trivial group, viewed as a quotient of Δ_∞. For each $N \geq 0$, we set $R_N = R_{\mathcal{S}_1, Q_N}$ and $R'_N = R_{\mathcal{S}_\chi, Q_N}$. Let $R^{loc} = R_{\mathcal{S}_1}^{S,loc}$ and $R'^{loc} = R'^{S,loc}_{\mathcal{S}_\chi}$ denote the local deformation rings. We let R_∞ and R'_∞ be formal power series rings in g variables over R^{loc} and R'^{loc}, respectively. We also have canonical isomorphisms $R_N/\varpi \cong R'_N/\varpi$ and $R^{loc}/\varpi \cong R'^{loc}/\varpi$. Using [ACC+18, Proposition 6.2.24] and [ACC+18, Proposition 6.2.31], we have local $\mathcal{O}$-algebra surjections $R_\infty \to R_N$ and $R'_\infty \to R'_N$ for $N \geq 0$. We can and do assume that these are compatible with the fixed identifications modulo ϖ and with the isomorphisms $R_N \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} = R_0$ and $R'_N \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} = R'_0$.

Define $\mathcal{C}_0 = R\,\mathrm{Hom}_{\mathcal{O}}(R\Gamma(X_K, V_\lambda(1))_{\mathfrak{m}}, \mathcal{O})[-d] \in \mathbf{D}(\mathcal{O})$ and $T_0 = \mathbf{T}^S(\mathcal{C}_0)$. Similarly, we define $\mathcal{C}'_0 = R\,\mathrm{Hom}_{\mathcal{O}}(R\Gamma(X_K, V_\lambda(\chi^{-1}))_{\mathfrak{m}}$ and $T'_0 = \mathbf{T}^S(\mathcal{C}'_0)$. For any $N \geq 1$, we let

$$\mathcal{C}_N = R\,\mathrm{Hom}_{\mathcal{O}}(R\Gamma(X_{K_1(Q)}, V_\lambda(1))_{\mathfrak{m}_{Q_N}}, \mathcal{O})[-d],$$

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and

$$T_N = \mathbf{T}_Q^{S'}(\mathcal{C}_N).$$

Similarly, we let

$$\mathcal{C}'_N = R\operatorname{Hom}_{\mathcal{O}}(R\Gamma(X_{K_1(Q)}, V_\lambda(\chi^{-1}))_{\mathfrak{m}_{Q_N}}, \mathcal{O})[-d]$$

and

$$T'_N = \mathbf{T}_Q^{S'}(\mathcal{C}'_N).$$

For any $N \geq 0$, there are canonical isomorphisms

$$\mathcal{C}_N \otimes^{\mathbf{L}}_{\mathcal{O}[\Delta_N]} k[\Delta_N] \cong \mathcal{C}'_N \otimes^{\mathbf{L}}_{\mathcal{O}[\Delta_N]} k[\Delta_N]$$

in $\mathbf{D}(k[\Delta_N])$. These yield the identification

$$\operatorname{End}_{\mathbf{D}(\mathcal{O})}(\mathcal{C}_N \otimes^{\mathbf{L}}_{\mathcal{O}} k) \cong \operatorname{End}_{\mathbf{D}(\mathcal{O})}(\mathcal{C}'_N \otimes^{\mathbf{L}}_{\mathcal{O}} k).$$

Thus, we can write $\overline{T}_N$ for the image of both T_N and T'_N in the identified endomorphism algebras. By Theorem 7.6, there are canonical isomorphisms $\mathcal{C}_N \otimes^{\mathbf{L}}_{\mathcal{O}[\Delta_N]} \mathcal{O} \cong \mathcal{C}_0$ and $\mathcal{C}'_N \otimes^{\mathbf{L}}_{\mathcal{O}[\Delta_N]} \mathcal{O} \cong \mathcal{C}'_0$ in $\mathbf{D}(\mathcal{O})$, which are compatible with the reductions modulo ϖ. By Proposition 8.2, we can find an integer $\delta \geq 1$ and for each $N \geq 0$ ideals I_N of T_N and I'_N of T'_N of nilpotence degree $\leq \delta$, such that there exist local $\mathcal{O}[\Delta_N]$-algebra surjections $R_N \to T_N/I_N$ and $R'_N \to T'_N/I'_N$. Denoting by $\overline{I}_N$ and $\overline{I}'_N$ the images of I_N and I'_N, respectively, in $\overline{T}_N$, we get maps $R_N/\varpi \to \overline{T}_N/(\overline{I}_N + \overline{I}'_N)$ and $R'_N/\varpi \to \overline{T}_N/(\overline{I}_N + \overline{I}'_N)$ which are compatible with the identification $R_N/\varpi \cong R'_N/\varpi$. The objects constructed above satisfy the setup described in [ACC+18, Section 6.4.1]. Thus, we can apply the results of [ACC+18, Section 6.4.2] as in the second part of the proof of [ACC+18, Theorem 6.4.4] to conclude that $H^*(C_0)$ is a nearly faithful $R_{\mathcal{S}_1}$-module, which implies Theorem 8.1.

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