THE WEDDERBURN THEOREM

HARRY GOHEEN

Wedderburn, in 1905, proved that there are no finite skew-fields **(5)**. Wedderburn's result has also been proved by Dickson, Artin, Witt, and Zassenhaus **(2; 1; 6; 7)**; however, it seems to the author that the proofs so far given introduce concepts not obviously related to the theorem. It is the purpose of this note to use a result of Cartan, which was later proved in greater generality by Hua **(4)**, to give a simpler and more direct version of the proof of Zassenhaus.

LEMMA 1 (Hua). *Any invariant division subring of a skew field must be contained in the center of the skew field.*

The proof, being well known, is omitted.

Let $N(H)$ denote the normalizer of a subgroup H of a finite group G. We have

LEMMA 2 (Zassenhaus). *If all the elements of $N(H_i)$ commute with all the elements of H_i for all Abelian subgroups H_i of G, then G is Abelian.*

Proof.[1] Let the lemma be true for any finite group K in which the hypotheses hold, provided K has fewer elements than the finite group M. Suppose that the hypotheses hold for M. We distinguish two possible cases: (1) M has a non-trivial center Z and (2) the center of M is the identity. In case (1) the quotient group M/Z has fewer elements than M and the hypotheses hold for M/Z. Hence M/Z is Abelian, and if x and y are any two elements in M,

$$xy = yxz$$

for some element z in Z. Then y is in the normalizer of the Abelian group generated by x and Z. By hypothesis, y commutes with every element of this Abelian subgroup, and $yx = xy$. Since x and y are arbitrary, M is Abelian.

In case (2), let C_1 be a maximal subgroup in M of index $n > 1$. Then C_1 is Abelian. If C_1 is invariant, every element of M may be written as $t^i c_j$, for t an element of M not in C_1 and c_j an element of C_1. Then, if u and v are two elements of M,

$$uv = (t^i c_j)(t^k c_l).$$

Because of the commutativity of c_j and c_l, of t^i and t^k, and of t and any element of C_1, this may be written as $(t^k c_l)(t^i c_j)$. Hence $uv = vu$.

Received February 25, 1954.

[1]This is a modification of the proof given in **(7)**.

On the other hand, if in case (2) C_1 is its own normalizer, then C_1 has n conjugates in M, and M has a faithful representation as a permutation group. The permutation corresponding to x in M is

$$\sigma(x) = \begin{pmatrix} C_1 & C_2 & \ldots & C_n \\ x^{-1}C_1x & x^{-1}C_2x & \ldots & x^{-1}C_nx \end{pmatrix}.$$

In this representation, the subgroup fixing the letter C_1 is the subgroup C_1. The representation is of class $n - 1$; for if a permutation $\sigma(z)$ fixes two letters C_i and C_j, z must be in C_i and C_j. Then since both C_i and C_j are Abelian and together generate M, z is in the center of M and therefore is the identity. Since the representation is of class $n - 1$, then by a theorem of Frobenius on such groups **(3)**, M has an invariant subgroup A consisting of the elements corresponding to permutations of degree n in the representation. Furthermore, A is Abelian, and A and C_1 generate M. Thus every element of M may be written in the form a_ic_j with a_i from A and c_j from C_1. Then, if u and v are any two elements of M,

$$uv = (a_ic_j)(a_kc_l) = vu,$$

since A and C_1 are both Abelian and every element of C_1 is commutative with every element of A. Since u and v are arbitrary elements, M must be Abelian. Thus case (2) cannot occur, for we have the contradiction that Z is M itself. To complete the induction we note that the lemma is true for all groups of prime order.

THEOREM. *There is no finite skew field.*

Proof. Let D be a finite skew field, M its multiplicative group, H any Abelian subgroup of M, and Z the center of D. By an induction argument, it will be demonstrated that every element from the normalizer of H commutes with every element of H. By Lemma 2, it will follow that M is Abelian.

If H is in Z, its normalizer is M and, since H is in the center, every element of M commutes with every element of H. Let H be an Abelian subgroup of M which is not in Z. Then H and Z generate a commutative subfield X of D. It cannot be D itself since D is a skew field. However X is contained in a proper maximal division subring C of D.

Now we make the induction assumption that all finite division rings which have fewer elements than D must be fields. Then C is a field and its multiplicative group is in the normalizer of H in M. However, no element not in C can be in the normalizer of H; for if s is, then X is invariant under s and hence under the division ring generated by C and s. But this division ring is D and X is therefore an invariant division subring of D. Then, by Lemma 1, X is in Z. This contradicts the assumption that H is not in Z. Therefore, the multiplicative group of C is the normalizer of H in M. Since C is a field, every element of the normalizer of H in M commutes with every element of H. By Lemma 2, M is Abelian and D is a field, contrary to hypothesis. The hypothesis that D

is a skew field being contradictory, it follows that there are no finite skew fields with as many elements as D if there are none with fewer elements.

To complete the induction it suffices to note that any division ring with a prime number of elements must be a field.

References

1. E. Artin, *Ueber einen Satz von Herrn J. H. Maclagan–Wedderburn*, Abh. Math. Sem. Univ. Hamburg, *5* (1927), 245–250.

2. L. E. Dickson, *On finite algebras*, Gött. Nachr. (1905), 358–393.

3. G. Frobenius, *Ueber auflösbare Gruppen*, IV, Berl. Ber., (1901), 1223–1225.

4. L. K. Hua, *Some properties of a sfield*, Proc. Nat. Acad. Sci. U.S.A., *35* (1949), 533–537.

5. J. H. Maclagan-Wedderburn, *A theorem on finite algebras*, Trans. Amer. Math. Soc., *6* (1905), 349–352.

6. E. Witt, *Ueber die Kommutativität endlicher Schiefkörper*, Abh. Math. Sem. Univ. Hamburg, *8* (1930), 413.

7. H. J. Zassenhaus, *A group-theoretic proof of a theorem of Maclagan–Wedderburn*, Proc. Glasgow Math. Assoc., *1* (1952), 53–63.

Iowa State College