K. Tahara
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ON THE HYPERELLIPTIC RIEMANN SURFACES OF INFINITE GENUS WITH ABSOLUTELY CONVERGENT RIEMANN'S THETA FUNCTIONS

KENICHI TAHARA

Introduction

The Riemann's theta functions associated with a closed Riemann surface are absolutely convergent. In the present paper, we shall show an example of an hyperelliptic Riemann surface $\mathfrak{R}$ of infinite genus such that the Riemann's theta functions associated with $\mathfrak{R}$ are absolutely convergent.

In §1, we shall formally define theta functions of countably many variables with rational characteristics in the same way as the usual theta functions of finite variables, and show the sufficient conditions under which these theta functions are absolutely convergent.

In §2, using the condition we shall really construct an hyperelliptic Riemann surface $\mathfrak{R}$ of infinite genus such that the Riemann's theta functions associated with $\mathfrak{R}$ are absolutely convergent.

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We shall freely use the following notations and conventions throughout the present paper;

Ω: the coordinate vector space consisting of all vectors with countably many components in the rational number field $\boldsymbol{Q}$, of which almost all components are zero,

Γ: the subgroup of Ω consisting of all the integral vectors,

$\boldsymbol{A} = \Omega/\Gamma$: the residue group of Ω by Γ,

$[\boldsymbol{a}] = [a_1, a_2, \cdots\cdot]$: the class of a vector $\boldsymbol{a} = (a_1, a_2, \cdots\cdot)$ in the residue group $\boldsymbol{A}$.

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§1. The sufficient conditions for absolute convergence of the theta functions of countably many variables

1. 1 We shall formally define the theta functions of countably many variables with rational characteristics in the same way as usual theta functions of finite variables.

Let $\tau_{i,j}(i, j = 1, 2, \cdots\cdots)$ be complex numbers such that $\tau_{i,j} = \tau_{j,i}$, and $z_i(i = 1, 2, \cdots\cdots)$ be complex variables. For the sake of simplicity we shall use the matrices notations; $\boldsymbol{\tau} = (\tau_{i,j})$ and $\boldsymbol{z} = (z_1, z_2, \cdots\cdots)$. For each element $[\boldsymbol{a}] = [a_1, a_2, \cdots\cdots]$ in $\boldsymbol{A}$, we shall formally define *the theta function of variables* $z_1, z_2, \cdots\cdots$ *with characteristic* $[\boldsymbol{a}]$ by the formal series

$$(1.\ 1) \qquad \vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{z}) = \sum_{(m_1, m_2, \cdots)\in \boldsymbol{z}^\infty} e^{\pi\sqrt{-1}\left\{\sum_{i,j=1}^{\infty}\tau_{i,j}(m_i+a_i)(m_j+a_j)+2\sum_{i=1}^{\infty}(m_i+a_i)z_i\right\}}.$$

The function $\vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{z})$ does not depend on the choice of the representative $\boldsymbol{a} = (a_1, a_2, \cdots\cdots)$ of the class $[\boldsymbol{a}]$, and generally it does not converge. *The theta zero-value* is defined by

$$(1.\ 2) \qquad \vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}) = \sum_{(m_1, m_2, \cdots)\in \boldsymbol{z}^\infty} e^{\pi\sqrt{-1}\left\{\sum_{i,j=1}^{\infty}\tau_{i,j}(m_i+a_i)(m_j+a_j)\right\}}.$$

From the definitions (1. 1) and (1. 2), we have the following formula in the same way as the usual theta functions;

$$(1.\ 3) \qquad \vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{l\tau}+\boldsymbol{z}) = e^{-\pi\sqrt{-1}\left(\sum_{i,j=1}^{\infty}\tau_{i,j}l_il_j+2\sum_{i=1}^{\infty}l_iz_i\right)}\vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{z})$$

$$(\boldsymbol{l} = (l_1, l_2, \cdots\cdots) \in \boldsymbol{z}^\infty)$$

$$(1.\ 4) \qquad \vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|-\boldsymbol{z}) = \vartheta_{-[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{z})$$

$$(1.\ 5) \qquad \vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{b\tau}+\boldsymbol{z}) = e^{-\pi\sqrt{-1}\left(\sum_{i,j=1}^{\infty}\tau_{i,j}b_ib_j+2\sum_{i=1}^{\infty}b_iz_i\right)}\vartheta_{[\boldsymbol{a}]+[\boldsymbol{b}]}(\boldsymbol{\tau}|\boldsymbol{z})$$

$$([\boldsymbol{b}] \in \boldsymbol{A})$$

$$(1.\ 6) \qquad \vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}) = \vartheta_{-[\boldsymbol{a}]}(\boldsymbol{\tau})$$

$$(1.\ 7) \qquad \vartheta_{[\boldsymbol{a}]+[\boldsymbol{b}]}(\boldsymbol{\tau}) = e^{\pi\sqrt{-1}\left(\sum_{i,j=1}^{\infty}\tau_{i,j}b_ib_j\right)}\vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{b\tau})$$

$$([\boldsymbol{b}] \in \boldsymbol{A}).$$

1. 2 We shall first be concerned with the special case: the infinite products of the elliptic theta functions with rational characteristics.

Let τ be a complex number of which imaginary part is positive, and z be a complex variable. For each element $[a]$ in $\boldsymbol{Q}/\boldsymbol{Z}$, the elliptic theta function with characteristic $[a]$ is defined by

$$\vartheta_{[a]}(\tau\,|\,z)=\sum_{m\in\boldsymbol{Z}} e^{\pi\sqrt{-1}\{\tau(m+a)^2+2z(m+a)\}}.$$

Then these functions $\vartheta_{[a]}(\tau\,|\,z)$ $([a]\in\boldsymbol{Q}/\boldsymbol{Z})$ are absolutely convergent in any bounded domain of values of z.

We shall recall the estimations of the elliptic theta functions $\vartheta_{[a]}(\tau\,|\,z)$.

LEMMA 1. *Let s be the imaginary part of τ, being positive, and x be the imaginary part of z. Then*

$$|\vartheta_{[a]}(\tau\,|\,z)|\leqq e^{-\pi a(as+2x)}+\frac{1}{\sqrt{s}}e^{\frac{\pi x^2}{s}} \qquad ([a]\in\boldsymbol{Q}|\boldsymbol{Z})$$

and

$$\left|\vartheta_{[a]}(\tau\,|\,z)-1\right|\leqq\left|1-e^{-\pi a(as+2x)}\right|+\frac{1}{\sqrt{s}}e^{\frac{\pi x^2}{s}} \qquad ([a]\in\boldsymbol{Q}/\boldsymbol{Z}).$$

Proof. From the definition of the functions $\vartheta_{[a]}(\tau\,|\,z)$ it follows that

$$\begin{aligned}\left|\vartheta_{[a]}(\tau\,|\,z)\right| &\leqq \sum_{m\in\boldsymbol{Z}} e^{-\pi\{s(m+a)^2+2x(m+a)\}}\\ &= e^{\frac{\pi x^2}{s}}\sum_{m\in\boldsymbol{Z}} e^{-\pi s\left(m+a+\frac{x}{s}\right)^2}\\ &\leqq e^{\frac{\pi x^2}{s}}\left\{e^{-\pi s\left(a+\frac{x}{s}\right)^2}+\int_{-\infty}^{\infty}e^{-\pi s\left(y+a+\frac{x}{s}\right)^2}dy\right\}\\ &= e^{-\pi a(as+2x)}+\frac{1}{\sqrt{s}}e^{\frac{\pi x^2}{s}}.^{1)}\end{aligned}$$

Similarly we have the last inequalities, Q.E.D.

Putting $[a]=[0]$, we have

COROLLARY.

$$\left|\vartheta_{[0]}(\tau\,|\,z)\right|\leqq 1+\frac{1}{\sqrt{s}}e^{\frac{\pi x^2}{s}}$$

1) See p. 10, [1].

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and

$$\left|\vartheta_{[0]}(\tau\,|\,z)-1\right|\leqq\frac{1}{\sqrt{s}}e^{\frac{\pi x^2}{s}}.$$

Let $\tau_i (i=1,2,\cdots\cdots)$ be complex numbers of which imaginary parts s_i are positive, and $z_i (i=1,2,\cdots\cdots)$ be complex variables. For each element $[\boldsymbol{a}]=[a_1,a_2,\cdots\cdots]$ in $\boldsymbol{A}$, consider the infinite product

$$\prod_{i=1}^{\infty}\vartheta_{[a_i]}(\tau_i\,|\,z_i)$$

of the elliptic theta functions $\vartheta_{[a_i]}(\tau_i\,|\,z_i)$.

PROPOSITION 1. *Let s_i $(i=1,2,\cdots\cdots)$ be the imaginary parts of τ_i, being positive for all i. If the infinite series*

$$\sum_{i=1}^{\infty}\frac{1}{\sqrt{s_i}}$$

is convergent, then the infinite products of the elliptic theta functions $\vartheta_{[a_i]}(\tau_i\,|\,z_i)$

$$\prod_{i=1}^{\infty}\vartheta_{[a_i]}(\tau_i\,|\,z_i)\qquad([\boldsymbol{a}]=[a_1,a_2,\cdots\cdots]\in\boldsymbol{A})$$

are absolutely convergent in any bounded domain of values of each variable z_i [2]*.*

Proof. The infinite product $\prod_{i=1}^{\infty}\vartheta_{[a_i]}(\tau_i\,|\,z_i)$ is absolutely convergent in any bounded domain $\boldsymbol{D}$ of values of each variable z_i if and only if the infinite series $\sum_{i=1}|\vartheta_{[a_i]}(\tau_i\,|\,z_i)-1|$ is convergent for each $\boldsymbol{z}=(z_1,z_2,\cdots\cdots)$ such that all z_i are in $\boldsymbol{D}$. Since $[\boldsymbol{a}]$ belongs to $\boldsymbol{A}$, there exists a natural number N such that $a_i=0$ for all $i>N$. From Lemma 1 and it's corollary it follows that

$$\sum_{i=1}^{\infty}\left|\vartheta_{[a_i]}(\tau_i\,|\,z_i)-1\right|\leqq\sum_{i=1}^{N}\left|e^{-\pi a_i(a_i s_i+2x_i)}-1\right|+\sum_{i=1}^{\infty}\frac{1}{\sqrt{s_i}}e^{\frac{\pi x_i^2}{s_i}}$$

where x_i mean the imaginary parts of z_i. If the infinite series $\sum_{i=1}^{\infty}\frac{1}{\sqrt{s_i}}$ is convergent, then the infinite series

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2) "Variables $z=(z_1,z_2,\cdots)$ are in a bounded domain of values of each variable z_i" means that each variable z_i is in one and the same bounded domain in the complex plane.

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$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{s_i}} e^{\frac{\pi x_i^2}{s_i}}$$

converges for each $z = (z_1, z_2, \cdots\cdots)$ such that all z_i are in $\boldsymbol{D}$, Q.E.D.

1. 3 Let $\tau_{i,j}(i, j = 1, 2, \cdots\cdots)$ be complex numbers such that $\tau_{i,j} = \tau_{j,i}$, and z_i $(i = 1, 2, \cdots\cdots)$ be complex variables. We shall give the sufficient conditions such that the theta functions $\vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{z})$ $([\boldsymbol{a}] \in \boldsymbol{A})$ are absolutely convergent in any bounded domain of values of each variable z_i.

Proposition 2. *Let $s_{i,j}(i, j = 1, 2, \cdots\cdots)$ be the imaginary parts of $\tau_{i,j}$. If the following conditions are satisfied;*

$$(*) \qquad s_{i,i} - \sum_{\substack{j=1 \\ j \neq i}}^{\infty} |s_{i,j}| \text{ are positive for all } i$$

and

$$(**) \qquad \sum_{i=1}^{\infty} \frac{1}{\sqrt{s_{i,i} - \sum_{\substack{j=1 \\ j \neq i}}^{\infty} |s_{i,j}|}} < \infty,$$

then the theta functions $\vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{z})$ $([\boldsymbol{a}] \in \boldsymbol{A})$ are absolutely convergent in any bounded domain of values of each variable z_i.

Proof. Assume that the conditions $(*)$ and $(**)$ are satisfied. Denote by x_i $(i = 1, 2, \cdots\cdots)$ the imaginary parts of z_i. From the inequalities

$$2\,|s_{i,j}(m_i + a_i)(m_j + a_j)| \leqq |s_{i,j}| \{(m_i + a_i)^2 + (m_j + a_j)^2\},$$

it follows that

$$|\vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{z})|$$

$$\leqq \sum_{(m_1, m_2, \cdots) \in \boldsymbol{Z}^\infty} e^{-\pi \sum_{i=1}^{\infty} s_{i,i}(m_i+a_i)^2 + 2\pi \sum_{j>i}^{\infty} |s_{i,j}(m_i+a_i)(m_j+a_j)| - 2\pi \sum_{i=1}^{\infty} x_i(m_i+a_i)}$$

$$\leqq \sum_{(m_1, m_2, \cdots) \in \boldsymbol{Z}^\infty} e^{-\pi \sum_{i=1}^{\infty} (s_{i,i} - \sum_{\substack{j=1 \\ j \neq i}}^{\infty} |s_{i,j}|)(m_i+a_i)^2 - 2\pi \sum_{i=1}^{\infty} x_i(m_i+a_i)}$$

Putting $s_i = s_{i,i} - \sum_{\substack{j=1 \\ j \neq i}}^{\infty} |s_{i,j}|$, then s_i are positive for all i. If the infinite series

$$(1.\ 8) \qquad \sum_{i=1}^{\infty} \Big| \sum_{m_i \in \boldsymbol{Z}} e^{-\pi s_i(m_i+a_i)^2 - 2\pi x_i(m_i+a_i)} - 1 \Big|$$

is convergent, then the infinite product

$$(1.9) \qquad \prod_{i=1}^{\infty} \sum_{m_i \in \boldsymbol{Z}} e^{-\pi s_i (m_i + a_i)^2 - 2\pi x_i (m_i + a_i)}$$

is bounded for each $\boldsymbol{z} = (z_1, z_2, \cdots\cdots)$ such that all z_i are in any bounded domain $\boldsymbol{D}$. Since $[\boldsymbol{a}] = [a_1, a_2, \cdots\cdots]$ is in $\boldsymbol{A}$, there exists a natural number N such that $a_i = 0$ for $i > N$. We have the following inequalities in the same way as the proof of Lemma 1,

$$\sum_{i=1}^{\infty} \Big| \sum_{m_i \in Z} e^{-\pi s_i (m_i + a_i)^2 - 2\pi x_i (m_i + a_i)} - 1 \Big|$$

$$\leqq \sum_{i=1}^{N} \big| e^{-\pi a_i (a_i s_i + 2x_i)} - 1 \big| + \sum_{i=1}^{\infty} \frac{1}{\sqrt{s_i}} e^{\frac{\pi x_i^2}{s_i}}.$$

Similarly as the proof of Proposition 1, if the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{s_i}}$$

is convergent, then the infinite series (1. 8) hence the infinite product (1. 9) are bounded, which completes the proof of Proposition 2.

PROPOSITION 3. *Let $s_{i,j} (i, j = 1, 2, \cdots\cdots)$ be the imaginary parts of $\tau_{i,j}$. If the following conditions are satisfied;*

$$(*)' \qquad s_{i,i} - \sum_{j>i}^{\infty} s_{i,j}^2 - (i-1) \text{ are positive for all } i,$$

and

$$(**)' \qquad \sum_{i=1}^{\infty} \frac{1}{\sqrt{s_{i,i} - \sum_{j>i}^{\infty} s_{i,j}^2 - (i-1)}} < \infty,$$

then the theta functions $\vartheta_{[\boldsymbol{a}]}(\tau|\boldsymbol{z})$ ($[\boldsymbol{a}] \in \boldsymbol{A}$) are absolutely convergent in any bounded domain of values of each variable z_i.

Proof. Denote by x_i $(i = 1, 2, \cdots\cdots)$ the imaginary parts of z_i. From the inequalities

$$2|s_{i,j}(m_i + a_i)(m_j + a_j)| \leqq \{|s_{i,j}|^2 (m_i + a_i)^2 + (m_j + a_j)^2\},$$

it follows that

$$|\vartheta_{[\boldsymbol{a}]}(\boldsymbol{\tau}|\boldsymbol{z})|$$

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$$\leqq \sum_{(m_1, m_2, \cdots) \in Z^\infty} e^{-\pi \sum_{i=1}^{\infty} s_{i,i}(m_i+a_i)^2 + 2\pi \sum_{j>i}^{\infty} |s_{i,j}(m_i+a_i)(m_j+a_j)| - 2\pi \sum_{i=1}^{\infty} x_i(m_i+a_i)}$$

$$\leqq \sum_{(m_1, m_2, \cdots) \in Z^\infty} e^{-\pi \sum_{i=1}^{\infty} \{s_{i,i} - \sum_{j>i}^{\infty} s_{i,j}{}^2 - (i-1)\}(m_i+a_i)^2 - 2\pi \sum_{i=1}^{\infty} x_i(m_i+a_i)}.$$

Therefore, similarly as the proof of Proposition 2, we have Proposition 3, Q.E.D.

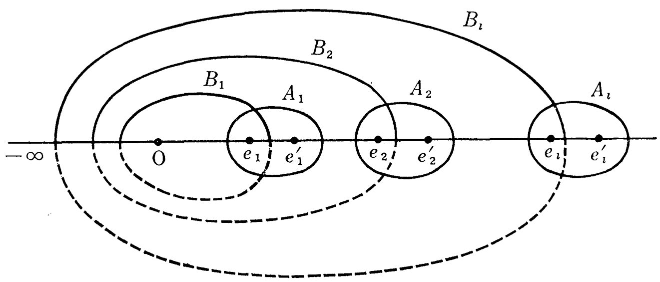
§2. An hyperelliptic Riemann surface of infinite genus with absolutely convergent Riemann's theta functions

2. 1 We shall show an example of an hyperelliptic Riemann surface $\mathfrak{R}$ of infinite genus such that the Riemann's theta functions associated with $\mathfrak{R}$ are absolutely convergent in any bounded domain of values of each variable.

Let e_1, e'_1, e_2, e'_2, $\cdots\cdot$ be a set of countably many number of successively increasing points on the real axis of the complex plane, which are the candidates of branch points of an hyperelliptic Riemann's branch covering of infinite genus over the Riemann sphere. Let C_p be an hyperelliptic curve of genus p defined by

$$y_{(p)}^2 = x \prod_{i=1}^{p} \left(1 - \frac{x}{e_i}\right)\left(1 - \frac{x}{e'_i}\right).$$

We construct the two-sheeted Riemann surface $\mathfrak{R}_p$ of C_p by joining the sheets along $(p+1)$ non-intersecting cuts; $-\infty 0$; $e_1e'_1$; $e_2e'_2$; $\cdots$; $e_pe'_p$. We define on $\mathfrak{R}_p$ a set of $2p$ retrosections A_i, B_i in the usual way: Let A_i be a circuit in the first (upper) sheet surrounding the cut $e_ie'_i$, and B_i be a circuit which crosses the only cuts $-\infty 0$, $e_ie'_i$. Then these circuits $A_1, A_2, \cdots, A_p$; $B_1, B_2, \cdots, B_p$ are a canonical system of $\mathfrak{R}_p$, from which a canonical system of an hyperelliptic Riemann surface $\mathfrak{R} = \lim_{p\to\infty} \mathfrak{R}_p$ is obtained by the limit $p \to \infty$ in the usual sense.



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Therefore $\{A_i, B_i\}_{i=1,2,\cdots}$ is a canonical homology basis on the Riemann surface $\mathfrak{R}$; namely; let (C, C') denote the intersection number of two cycles C, C' on $\mathfrak{R}$, then the intersection numbers of cycles on $\mathfrak{R}$ are characterized by

$$(A_i, A_j) = 0, \qquad (B_i, B_j) = 0 \qquad (i, j = 1, 2, \cdots\cdots)$$
$$(A_i, B_j) = 0 \quad (i \neq j), \qquad (A_i, B_i) = 1.$$

We shall construct a system of elementary normal integrals of the first kind on $\mathfrak{R}$.

Proposition 4 (Myrberg). *There exists a system of linearly independent integrals Ψ_i $(i = 1, 2, \cdots\cdots)$ of the first kind on $\mathfrak{R}$ with the following periodsystem with respect to the canonical homology basis $\{A_i, B_i\}_{i=1,2,\cdots\cdots}$;*

$$(2.1)\qquad \left\|\begin{matrix} -\pi & 0 & 0 & \cdots\cdots & ; & \tau_{1,1} & \tau_{1,2} & \tau_{1,3} & \cdots\cdots \\ 0 & -\pi & 0 & \cdots\cdots & ; & \tau_{2,1} & \tau_{2,2} & \tau_{2,3} & \cdots\cdots \\ 0 & 0 & -\pi & \cdots\cdots & ; & \tau_{3,1} & \tau_{3,2} & \tau_{3,3} & \cdots\cdots \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \end{matrix}\right\|$$

such that

$$(2.2)\qquad \tau_{i,j} = \tau_{j,i} = \sqrt{-1}\, s_{j,i} \quad (i, j = 1, 2, \cdots\cdots)$$

are pure imaginary numbers, and

$$(2.3)\qquad s_{i,j} > 0 \quad (i, j = 1, 2, \cdots\cdots).$$

Moreover Ψ_i $(i = 1, 2, \cdots\cdots)$ are uniquely determined by the initial conditions $\Psi_i(0) = 0$.

Proof. From the results in [3], there exists a system of linearly independent integrals φ_i $(i = 1, 2, \cdots\cdots)$ of the first kind on $\mathfrak{R}$ with the following periodsystem with respect to the canonical homology basis $\{A_i, B_i\}_{i=1,2,\cdots\cdots}$;

$$\left\|\begin{matrix} 2\pi\sqrt{-1} & 0 & 0 & \cdots\cdots & ; & t_{1,1} & t_{1,2} & t_{1,3} & \cdots\cdots \\ 0 & 2\pi\sqrt{-1} & 0 & \cdots\cdots & ; & t_{2,1} & t_{2,2} & t_{2,3} & \cdots\cdots \\ 0 & 0 & 2\pi\sqrt{-1} & \cdots\cdots & ; & t_{3,1} & t_{3,2} & t_{3,3} & \cdots\cdots \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \end{matrix}\right\|$$

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where $t_{i,j}$ are real numbers and $t_{i,j} > 0$. Put for each natural numbers i

$$\Psi_i = \frac{\sqrt{-1}}{2} \varphi_i.$$

Then Ψ_i $(i = 1, 2, \cdots\cdots)$ are linearly independent integrals of the first kind on $\mathfrak{R}$, and it follows that

$$\Psi_i(A_i) = -\pi, \quad \Psi_i(A_j) = 0 \quad (i \neq j),$$

and

$$\Psi_i(B_j) = \sqrt{-1}\left(\frac{t_{i,j}}{2}\right).$$

Therefore putting $s_{i,j} = \frac{t_{i,j}}{2}$, we have the system of linearly independent integrals Ψ_i $(i = 1, 2, \cdots\cdots)$ of the first kind on $\mathfrak{R}$ with the required periodsystem (2. 1). Moreover, since the integrals of the first kind on $\mathfrak{R}$ are uniquely determined except constants by A-periods whenever the canonical homology basis choosen,[3] Ψ_i $(i = 1, 2, \cdots\cdots)$ are uniquely determined by the initial conditions $\Psi_i(0) = 0$, Q.E.D.

Denoting

$$y^2 = x \prod_{i=1}^{\infty} \left(1 - \frac{x}{e_i}\right)\left(1 - \frac{x}{e'_i}\right),$$

we have the explicite expressions of the integrals Ψ_i;

(2. 4) $$\Psi_i = \int \frac{h_i(x)}{y} dx \quad (i = 1, 2, \cdots\cdots),$$

where $h_i(x) = k_i \prod_{\substack{j=1 \\ j \neq i}}^{\infty} \left(1 - \frac{x}{a_j}\right)$ such that only one point a_j belongs to the open interval (e_j, e'_j) on the real axis and k_i is constant[4].

2. 2 To construct a nice example for our purpose, choose countably many real numbers e_1, e'_1, e_2, e'_2, $\cdots\cdots$ as the following;

(2. 5) $$e'_i = e_i + 1 \quad (i = 1, 2, \cdots\cdots)$$

and

(2. 6) $$\sum_{i=1}^{\infty} \frac{1}{e_i} < \infty.$$

3) See Satz 1 in [5].

4) See [5].

We shall start with the estimations of the lower bounds of the absolute values $s_{i,i}$ of the diagonal elements $\tau_{i,i}$ in the B-period matrix of (2. 1).

We choose countably many real numbers Δ_1, Δ_2, $\cdots$ such that

$$0 < \Delta_i < e_{i+1} - e_i'. \tag{2. 7}$$

Put

$$\Delta_{j,i} = |e_j - e_i| - \Delta_i - 1 \quad (j \neq i), \tag{2. 8}$$

and

$$\rho_i = \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{1}{\Delta_{j,i}} \qquad (i = 1, 2, \cdots). \tag{2. 9}$$

In the following we shall make the assumptions

$$\Delta_{j,i} > 0 \quad (j \neq i) \tag{2. 10}$$

and

$$\rho_i < 1 \quad (i = 1, 2, \cdots). \tag{2. 11}$$

Then it follows that

$$0 < \frac{1}{\Delta_{j,i}} < 1 \quad (j \neq i). \tag{2. 12}$$

Under the above assumptions (2. 5), (2. 6), (2. 10) and (2. 11), we have the following;

LEMMA 2.

$$s_{i,i} > (1 - \rho_i) e^{-\rho_i} \sqrt{\frac{e_i}{e_i' + \Delta_i}} \log \Delta_i \quad (i = 1, 2, \cdots).$$

Proof. Since $\Psi_i(A_i) = -\pi$ and $\Psi_i(B_i) = \sqrt{-1}\, s_{i,i}$, it follows that

$$\pi = \int_{e_i}^{e_i'} \left| \frac{h_i(x)}{y} \right| dx$$

$$< |k_i| \left[\frac{1}{\sqrt{x}} \prod_{\substack{j=1 \\ j \neq i}}^{\infty} \left| \frac{1 - \dfrac{x}{a_j}}{\sqrt{\left(1 - \dfrac{x}{e_j}\right)\left(1 - \dfrac{x}{e_j'}\right)}} \right| \right]_{\max} \int_{e_i}^{e_i'} \frac{dx}{\sqrt{\left(\dfrac{x}{e_i} - 1\right)\left(1 - \dfrac{x}{e_i'}\right)}} \tag{2. 13}$$

where x runs over the closed interval $[e_i, e_i']$, and

$$s_{i,i} = \int_{e_i'}^{e_i'+\Delta_i} \left| \frac{h_i(x)}{y} \right| dx$$

$$(2.14) \quad < |k_i| \left[\frac{1}{\sqrt{x'}} \prod_{\substack{j=1 \\ j \neq i}}^{\infty} \left| \frac{1 - \frac{x'}{a_j}}{\sqrt{\left(1 - \frac{x'}{e_i}\right)\left(1 - \frac{x'}{e_i'}\right)}} \right| \right]_{\min} \int_{e_i'}^{e_i'+\Delta_i} \frac{dx'}{\sqrt{\left(\frac{x'}{e_i} - 1\right)\left(\frac{x'}{e_i'} - 1\right)}}$$

where $0 < \Delta_i < e_{i+1} - e_i'$ and x' runs over the closed interval $[e_i', e_i' + \Delta_i]$. Since only one point a_j belongs to the open interval (e_j, e_j') $(j \neq i)$, we have

$$\left| \frac{x - a_j}{x - e_j} \right| < 1 + \frac{1}{\Delta_{j,i}}, \quad \left| \frac{x - a_j}{x - e_j'} \right| < 1 + \frac{1}{\Delta_{j,i}}$$

and

$$\left| \frac{x' - a_j}{x' - e_j} \right| > 1 - \frac{1}{\Delta_{j,i}}, \quad \left| \frac{x' - a_j}{x' - e_j'} \right| > 1 - \frac{1}{\Delta_{j,i}}$$

where $\Delta_{j,i} = |e_j - e_i| - \Delta_i - 1$. From (2.12), we have

$$\prod_{\substack{j=1 \\ j \neq i}}^{\infty} \left(1 + \frac{1}{\Delta_{i,j}}\right) = e^{\sum_{\substack{j=1 \\ j \neq i}}^{\infty} \log\left(1 + \frac{1}{\Delta_{j,i}}\right)} < e^{\sum_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{1}{\Delta_{j,i}}} = e^{\rho_i},$$

$$\prod_{\substack{j=1 \\ j \neq i}}^{\infty} \left(1 - \frac{1}{\Delta_{j,i}}\right) > 1 - \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{1}{\Delta_{j,i}} = 1 - \rho_i,$$

and

$$\int_{e_i'}^{e_i'+\Delta_i} \frac{dx'}{\sqrt{\left(\frac{x'}{e_i} - 1\right)\left(\frac{x'}{e_i'} - 1\right)}} = 2\sqrt{e_i e_i'} \log\left(\sqrt{\Delta_i + 1} + \sqrt{\Delta_i}\right)$$

$$> \sqrt{e_i e_i'} \log \Delta_i.$$

Therefore by virtue of inequalities (2.13) and (2.14), it follows that

$$\pi < M \frac{|k_i|}{\sqrt{e_i}} e^{\rho_i}$$

and

$$s_{i,i} > M \frac{|k_i|}{\sqrt{e_i' + \Delta_i}} (1 - \rho_i) \log \Delta_i \quad (i = 1, 2, \cdots)$$

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where $M=(\prod_{i=1}^{\infty}\sqrt{e_i e_j'})(\prod_{\substack{j=1\\j\neq i}}^{\infty} a_j)^{-1}$. Hence we have

$$s_{i,i}>(1-\rho_i)e^{-\rho_i}\sqrt{\frac{e_i}{e_i'+\varDelta_i}}\log \varDelta_i \quad (i=1,2,\cdots\cdot), \quad \text{Q.E.D.}$$

We shall estimate the upper bounds of the absolute values $s_{i,j}$ of non-diagonal elements $\tau_{i,j}$ in the B-period matirix of (2. 1).

Under the assumptions (2. 5), (2. 6), (2. 10) and (2. 11), we have the following;

LEMMA 3.

$$s_{i,j}<\frac{2e^{\rho_i}}{(1-\rho_i)}\sqrt{\frac{e_i'}{e_j'}}\Big(\frac{\varDelta_j+1}{|e_j+\varDelta_j-e_i|-1}\Big)\varDelta_j^{\frac{1}{2}} \quad (j\neq i)$$

Furthermore

$$s_{i,j}<\frac{4e^{\rho_i}}{(1-\rho_i)}\frac{\sqrt{e_i'}}{(e_i-e_{i-1}-\varDelta_{i-1}-1)}\varDelta_j\Big(\frac{\varDelta_j}{e_j'}\Big)^{\frac{1}{2}} \quad (j<i)$$

and

$$s_{i,j}<\frac{4e^{\rho_i}}{(1-\rho_i)}\frac{\sqrt{e_i'}}{(e_{i+1}+\varDelta_{i+1}-e_i-1)}\varDelta_j\Big(\frac{\varDelta_j}{e_j'}\Big)^{\frac{1}{2}} \quad (j>i).$$

Proof. By the similar method as the diagonal elements $\tau_{i,i}$, we have the following estimations;

$$\pi=\int_{e_i}^{e_i'}\Big|\frac{h_i(x)}{y}\Big|dx$$

$$(2.15)\qquad >|k_i|\left[\frac{1}{\sqrt{x}}\prod_{\substack{j=1\\j\neq i}}^{\infty}\left|\frac{1-\dfrac{x}{a_j}}{\sqrt{\Big(1-\dfrac{x}{e_j}\Big)\Big(1-\dfrac{x}{e_j'}\Big)}}\right|\right]_{\min}\int_{e_i}^{e_i'}\frac{dx}{\sqrt{\Big(\dfrac{x}{e_i}-1\Big)\Big(1-\dfrac{x}{e_i'}\Big)}}$$

where x runs over the closed interval $[e_i, e_j']$, and

$$s_{i,j}=\int_{e_j'}^{e_j'+\varDelta_j}\Big|\frac{h_i(x'')}{y}\Big|dx'' \quad (j\neq i)$$

$$(2.16)\qquad <|k_i|\left[\frac{1}{\sqrt{x''}}\left|\frac{1-\dfrac{x''}{a_j}}{\sqrt{\Big(1-\dfrac{x''}{e_i}\Big)\Big(1-\dfrac{x''}{e_i'}\Big)}}\right|\prod_{\substack{n=1\\n\neq i,j}}^{\infty}\left|\frac{1-\dfrac{x''}{a_n}}{\sqrt{\Big(1-\dfrac{x''}{e_n}\Big)\Big(1-\dfrac{x''}{e_n'}\Big)}}\right|\right]_{\max}$$

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$$\times \int_{e_j'}^{e_j'+\Delta_j} \frac{dx''}{\sqrt{\left(\frac{x''}{e_j}-1\right)\left(\frac{x''}{e_j'}-1\right)}}$$

where x'' runs over the closed interval $[e_j', e_j' + \Delta_j]$. Similarly as the proof of Lemma 2 we have

$$\left|\frac{x-a_n}{x-e_n}\right| > 1 - \frac{1}{\Delta_{n,i}}, \quad \left|\frac{x-a_n}{x-e_n'}\right| > 1 - \frac{1}{\Delta_{n,i}}$$

$$\left|\frac{x''-a_n}{x''-e_n}\right| < 1 + \frac{1}{\Delta_{n,j}}, \quad \left|\frac{x''-a_n}{x''-e_n'}\right| < 1 + \frac{1}{\Delta_{n,j}}$$

and

$$\left|\frac{x''-a_j}{x''-e_i}\right| < \frac{\Delta_j + 1}{|e_j + \Delta_j - e_i| - 1}, \quad \left|\frac{x''-a_j}{x''-e_i'}\right| < \frac{\Delta_j + 1}{|e_j + \Delta_j - e_i| - 1}.$$

Moreover

$$\int_{e_j'}^{e_j'+\Delta_j} \frac{dx''}{\sqrt{\left(\frac{x''}{e_j}-1\right)\left(\frac{x''}{e_j'}-1\right)}} < \sqrt{e_j e_j'} \int_{e_j'}^{e_j'+\Delta_j} \frac{dx''}{\sqrt{x''-e_j'}} = 2\sqrt{e_j e_j'}\, \Delta_j^{\frac{1}{2}}.$$

From the inequalities (2. 15) and (2. 16), it follows that

$$\pi > M \frac{|k_i|}{\sqrt{e_i'}} (1-\rho_i)\pi$$

and

$$s_{i,j} < 2M \frac{|k_i|}{\sqrt{e_j'}} \left(\frac{\Delta_j + 1}{|e_j + \Delta_j - e_i| - 1}\right) e^{\rho_j} \Delta_j^{\frac{1}{2}}.$$

Hence we have

$$s_{i,j} < \frac{2e^{\rho_j}}{(1-\rho_i)} \sqrt{\frac{e_i'}{e_j'}} \left(\frac{\Delta_j + 1}{|e_j + \Delta_j - e_i| - 1}\right) \Delta_j^{\frac{1}{2}}.$$

Furthermore

$$s_{i,j} < \frac{4e^{\rho_j}}{(1-\rho_i)} \sqrt{\frac{e_i'}{e_j'}} \left(\frac{\Delta_j + 1}{e_i - e_j - \Delta_j - 1}\right) \Delta_j^{\frac{1}{2}} \quad (j < i)$$

$$< \frac{4e^{\rho_j}}{(1-\rho_i)} \frac{\sqrt{e_i'}}{(e_i - e_{i-1} - \Delta_{i-1} - 1)} \Delta_j \left(\frac{\Delta_j}{e_j'}\right)^{\frac{1}{2}},$$

and

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$$s_{i,j} < \frac{2e^{\rho_j}}{(1-\rho_i)}\sqrt{\frac{e'_i}{e'_j}}\left(\frac{\Delta_j+1}{e_j+\Delta_j-e_i-1}\right)\Delta_j^{\frac{1}{2}} \quad (j>i)$$

$$< \frac{4e^{\rho_i}}{(1-\rho_i)}\frac{\sqrt{e'_i}}{(e_{i+1}+\Delta_{i+1}-e_i-1)}\Big)\Delta_j\left(\frac{\Delta_j}{e'_j}\right)^{\frac{1}{2}}, \qquad \text{Q.E.D.}$$

2.3 Finally we shall construct an example for our purpose. Put for all national numbers i

$$e_i = e^{i^6}, \quad \Delta_i = e^{i^3}. \tag{2.17}$$

Then we shall consider the assumptions (2.10) and (2.11) in this case. From (2.17) we have the inequalities

$$\begin{aligned}\Delta_{j,i} &= e_i - e_j - \Delta_j - 1 \quad (j<i)\\ &\geqq e_i - e_{i-1} - \Delta_{i-1} - 1\\ &\geqq e^{2^6} - e - e - 1 > 0,\\ \Delta_{j,i} &= e_j + \Delta_j - e_i - 1 \quad (j>i)\\ &\geqq e_{i+1} + \Delta_{i+1} - e_i - 1\\ &\geqq e^{2^6} + e^{2^3} - e - 1 > 0.\end{aligned}$$

Thus the assumption (2.10) are satisfied in this case.

From the inequalities

$$\frac{e_j}{(e_i - e_j - \Delta_j - 1)} < \frac{e_{i-1}}{(e_i - e_{i-1} - \Delta_{i-1} - 1)} < 1 \quad (j<i,\ i \geqq 2)$$

and

$$\frac{e_j}{(e_j + \Delta_j - e_i - 1)} < \frac{e_{i+1}}{(e_{i+1} - e_i - 1)} \quad (j<i),$$

we have

$$\rho_i = \sum_{j=1}^{i-1}\frac{1}{(e_i - e_j - \Delta_j - 1)} + \sum_{j=i+1}^{\infty}\frac{1}{(e_j + \Delta_j - e_i - 1)}$$

$$< \frac{e_{i+1}}{(e_{i+1} - e_i - 1)}\sum_{j=1}^{\infty}\frac{1}{e^{j^6}} < \frac{e^{2^6}}{(e^{j^6} - e - 1)}\sum_{j=1}^{\infty}\frac{1}{e^{j^6}} < 1.$$

Thus the assumption (2.11) are satisfied in this case.

Put

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(2. 18) $$\rho = \frac{e^{2^6}}{(e^{2^6} - e - 1)} \sum_{j=1}^{\infty} \frac{1}{e^{j^6}},$$

and

(2. 19) $$\sigma = (1-\rho)\, e^{-\rho} \sqrt{\frac{e}{2e+1}}\,.$$

By virtue of the inequalities

$$\frac{e_i}{e_i' + \Delta_i} > \frac{e_1}{e_1' + 1} = \frac{e}{2e+1}\,.$$

and by Lemma 2 and Lemma 3, we have

$$s_{i,i} > \frac{(1-\rho)}{e^{\rho}} \sqrt{\frac{e}{2e+1}}\, i^3 = \sigma i^3 \quad (i = 1, 2, \cdots\cdots)$$

and

$$s_{i,j} < \frac{4e^{\rho}}{(1-\rho)} \frac{\sqrt{e_i'}}{(e_i - e_{i-1} - \Delta_{i-1} - 1)} \Delta_j \left(\frac{\Delta_j}{e_j'}\right)^{\frac{1}{2}} \quad (j < i),$$

$$s_{i,j} < \frac{4e^{\rho}}{(1-\rho)} \frac{\sqrt{e_i'}}{(e_{i+1} + \Delta_{i+1} - e_i - 1)} \Delta_j \left(\frac{\Delta_j}{e_j'}\right)^{\frac{1}{2}} \quad (j > i).$$

Therefore it follows that

$$\sum_{j=2}^{\infty} s_{i,j} < \frac{4e^{\rho}\sqrt{e+1}}{(1-\rho)(e^{2^6} + e^{2^3} - e - 1)} \sum_{j=2}^{\infty} \frac{\Delta_j^{\frac{3}{2}}}{\sqrt{e_j}}$$

$$< \frac{4e^{\rho}\sqrt{e+1}}{(1-\rho)(e^{2^6} + e^{2^3} - e - 1)} \sum_{j=2}^{\infty} e^{-\frac{5}{2} j^3} < \sigma.$$

Since

$$e_{i+1} + \Delta_{i+1} - e_i - 1 > e_i - e_{i-1} - \Delta_{i-1} - 1 > e^{2^6} - 2e - 1 \quad (i \geqq 2),$$

it follows that

$$\sum_{\substack{j=1 \\ j \neq i}}^{\infty} s_{i,j} < \frac{4e^{\rho}\sqrt{e^{2^6}+1}}{(1-\rho)(e^{2^6} - 2e - 1)} \sum_{j=1}^{\infty} \frac{\Delta_j^{\frac{3}{2}}}{\sqrt{e_j}}$$

$$< \frac{4e^{\rho}\sqrt{e^{2^6}+1}}{(1-\rho)(e^{2^6} - 2e - 1)} \sum_{j=1}^{\infty} e^{-\frac{5}{2} j^3} < \sigma (i = 2, 3, \cdots\cdots).$$

These mean the gool for our purpose;

$$s_{i,i} - \sum_{\substack{j=1 \\ j \neq i}}^{\infty} s_{i,j} > 0 \quad (i = 1, 2, \cdots\cdot)$$

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{s_{i,i} - \sum_{\substack{j=1 \\ j \neq i}}^{\infty} s_{i,j}}} < \infty.$$

Hence by virtue of Proposition 2 we have got the example for our purpose.

THEOREM. *Let $\mathfrak{R}$ be the hyperelliptic Riemann surface of infinite genus defined by*

$$y^2 = x \prod_{i=1}^{\infty} \left(1 - \frac{x}{e_i}\right)\left(1 - \frac{x}{e_i'}\right)$$

where $e_i = e^{i^6}$ *and* $e_i' = e_i + 1$ $(i = 1, 2, \cdots\cdot)$. *Let* $\{A_i, B_i\}_{i=1,2,\cdots\cdot}$ *be the canonical homology basis on* $\mathfrak{R}$ *and* Ψ_i $(i = 1, 2, \cdots\cdot)$ *be the system of linearly independent integrals of the first kind on* $\mathfrak{R}$, *which are uniquely determined by the initial conditions* $\Psi_i(0) = 0$ *and have the following periodsystem with respect to the canonical homology basis* $\{A_i, B_i\}_{i=1,2,\cdots\cdot}$;

$$\left\| \begin{matrix} -\pi & 0 & 0 & \cdots\cdot & ; & \tau_{1,1} & \tau_{1,2} & \tau_{1,3} & \cdots\cdot \\ 0 & -\pi & 0 & \cdots\cdot & ; & \tau_{2,1} & \tau_{2,2} & \tau_{2,3} & \cdots\cdot \\ 0 & 0 & -\pi & \cdots\cdot & ; & \tau_{3,1} & \tau_{3,2} & \tau_{3,3} & \cdots\cdot \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \end{matrix} \right\|$$

where $\tau_{i,j} = \sqrt{-1}\, s_{i,j}$ *are pure imaginary numbers and* $\tau_{i,j} = \tau_{j,i}$. *Then the Riemann's theta functions* $\vartheta_{[a]}(\tau | z)$ $([a] \in A)$ *are absolutely convergent in any bounded domain of values of each variable* z_i.

Remark. We can also construct an example of an hyperelliptic Riemann surface $\mathfrak{R}'$ of infinite genus such that the imaginary parts of B-periods of a normal integrals of the first kind on $\mathfrak{R}'$ with respect to a canonical homology basis satisfy the conditions $(*)'$ and $(**)'$ of Proposition 3.

Open problems. It is well-known that Jacobian varieties are very usefull to study the closed Riemann surfaces. The following natural question then arises concerning the open Riemann surfaces; What kind of the open Riemann

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surfaces have something like Jacobian varieties which shall be usefull to study the open Riemann surfaces? Similarly as the finite case, we can define an infinite dimensional variety by the Riemann's theta functions associated with the open Riemann surface $\mathfrak{R}$ which is given in Theorem. The infinite dimensional variety is a kind of such a variety. Furthermore the natural question arises concerning the open Riemann surface $\mathfrak{R}$ which is given in Theorem; Are Abel's and Jacobi's theorems realized for the open Riemann surface $\mathfrak{R}$.

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Nagoya University