PARTITIONING AN ARITHMETIC INTERVAL

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The purpose of this paper is to characterize all ways in which an initial interval of natural numbers can be partitioned into a unique arithmetic sum of certain of its subsets.

Preliminaries. We set forth below certain explanations, conventions, and definitions pertinent to our subject.

Numbers and sets. By number we mean a natural (that is to say, an arithmetic or finite ordinal) number 0, 1, 2, ... Every set of numbers is required to contain 0. A set containing a number other than 0 will be called proper. The improper set comprising solely 0 will be written 0. The set consisting of all numbers dx as x ranges over a set X is denoted by dX.

Intervals. An (initial arithmetic) interval is a set of numbers containing every predecessor of each number in it. A finite interval containing d numbers thus consists of the first d numbers $r < d$ where $d \geqslant 1$. An infinite interval contains all natural numbers, that is, all numbers $r < \omega$ where ω is the first transfinite ordinal. Denote by $I(d)$ the interval consisting of all numbers $r < d$ where $1 \leqslant d \leqslant \omega$. $I(d)$ is finite or infinite according as $d < \omega$ or $d = \omega$, proper or improper according as $d > 1$ or $d = 1$. For this reason we call an ordinal d proper if $1 < d \leqslant \omega$.

Proper sequence. A finite (terminating) or infinite (non-terminating) numerically indexed[1] sequence $d_1, d_2, d_3, \ldots$ of ordinals will be called proper if not only is each ordinal in it proper but its ordinal product is also. Thus $1 < d_m \leqslant \omega$ for each index m and $1 < d_1 d_2 d_3 \ldots \leqslant \omega$. Consequently, if there is an index m such that $d_m = \omega$, the sequence terminates at this index m.

Partitions. We shall say that a set X is partitioned into (a unique arithmetic sum of) finitely or infinitely many numerically indexed[2] sets X^μ if every number x in X can be uniquely expressed in the form $x = \sum x^\mu$ where the unique coordinate x^μ of x belongs to X^μ for each index μ; and if furthermore every number x of this form belongs to X. To indicate this we write $X = \sum X^\mu$. It is easily seen that order and grouping of the terms X^μ in such a sum are not significant and that any two differently indexed terms X^μ have 0, and only 0, in common.

Division algorithm. As an apposite example of partitioning an interval consider the division algorithm. Let d_1, d_2 be a proper sequence of two ordinals. According to the division algorithm every number $r < d_1 d_2$ can be uniquely expressed in the form

Received April 14, 1951; in revised form August 14, 1952. This work was done under a grant from ONR.

[1]Subscript indices indicate that the order of indexing is significant.

[2]Superscript indices indicate that the order of indexing is not significant.

$$r = r_1 + d_1 r_2,$$

where $r_1 < d_1$ and $r_2 < d_2$; and furthermore every number r of this form is $< d_1 d_2$. That is,

$$I(d_1 d_2) = I(d_1) + d_1 I(d_2).$$

Statement of results. Our first result concerning partitions of an interval merely extends the division algorithm to any proper sequence and provides a recipe for constructing interval partitions. Our second result asserts that every interval partition can be constructed, and in a certain sense uniquely, by following this recipe.

THEOREM I. *Let $d_1, d_2, d_3, \ldots$ be a proper sequence of ordinals. Then*

$$I(d_1 d_2 d_3 \ldots) = I(d_1) + d_1 I(d_2) + d_1 d_2 I(d_3) + \ldots .$$

Coarser repartitions of I $(d_1 d_2 d_3 \ldots)$ can be formed from this partition by grouping the above terms together in arbitrary fashion.

THEOREM II. *Let $I = \sum X^\mu$ be a partition of a proper interval I by indexed sets X^μ. There then exists a unique proper sequence of ordinals $d_1, d_2, d_3, \ldots$ generating the following finer repartition of I:*

$$I = I(d_1 d_2 d_3 \ldots) = I(d_1) + d_1 I(d_2) + d_1 d_2 I(d_3) + \ldots ,$$

the sets X^μ of the given partition of I being formed by appropriately grouping together the terms of this finer repartition of I, with consecutive terms allocated to different X^μ sets.

Proofs. We shall establish the results stated above somewhat formally by using three lemmas. The first lemma is used in proving the first theorem, all three lemmas in proving the second.

LEMMA 1. *Let there be given a proper sequence of ordinals d_m, and sequences of sets A_m and X_m, all sequences of the same length, such that:*

$$X_m = \begin{cases} A_m & (m \text{ terminal}), \\ A_m + d_m X_{m+1} & (m \text{ non-terminal}). \end{cases}$$

Then

$$X_1 = A_1 + d_1 A_2 + d_1 d_2 A_3 + \ldots .$$

Proof. For each x_1 in X_1 the sequences x_m in X_m, a_m in A_m are uniquely determined by recursion as follows:

$$x_m = \begin{cases} a_m & (m \text{ terminal}), \\ a_m + d_m x_{m+1} & (m \text{ non-terminal}), \end{cases}$$

so for each m,

$$x_1 = a_1 + d_1 a_2 + \ldots + d_1 d_2 \ldots d_{m-1} x_m.$$

https://doi.org/10.4153/CJM-1953-009-4 Published online by Cambridge University Press

If m is terminal $x_m = a_m$. If, on the other hand, the sequence d_m does not terminate, then by choosing m so large that $d_1 d_2 \ldots d_m > x_1$ we find $x_n = 0$ and $a_n = 0$ for all $n \geqslant m$. Therefore, whether the sequences terminate or not,

$$x_1 = a_1 + d_1 a_2 + d_1 d_2 a_3 + \ldots,$$

this representation being unique, as was to be shown.

Proof of Theorem I. Let d_m be a proper sequence of ordinals, terminating or not. According to the division algorithm

$$I(d_m d_{m+1} \ldots) = I(d_m) + d_m I(d_{m+1} \ldots)$$

for all nonterminal m; so the hypotheses of Lemma 1 are satisfied by taking $A_m = I(d_m)$ and $X_m = I(d_m d_{m+1} \ldots)$, whereupon

$$I(d_1 d_2 d_3 \ldots) = I(d_1) + d_1 I(d_2) + d_1 d_2 I(d_3) + \ldots,$$

as was to be shown.

LEMMA 2. *Let $I = A + B$ be a partition of a proper interval I and d a proper ordinal such that A contains $I(d)$ but not d. There then exist: an interval $\bar{I}$ with $\bar{I} = 0$ if $d = \omega$, and sets $\bar{A}$ and $\bar{B}$ such that*

(1) $$A = I(d) + d\bar{A},$$

(2) $$B = d\bar{B},$$

(3) $$\bar{I} = \bar{A} + \bar{B}.$$

Proof. If $d = \omega$ it is evident that $A = I = I(\omega)$ and $B = 0$, so the lemma is verified by taking $\bar{I}$, $\bar{A}$, $\bar{B}$ all equal to 0. Furthermore, propositions (1) and (2) imply the remainder of the lemma. For since $A + B$ is a unique sum, $\bar{A} + \bar{B}$ is also a unique sum. Define $\bar{I} = \bar{A} + \bar{B}$. Thus $\bar{I}$ satisfies (3) and in addition

$$I = I(d) + d\bar{I}.$$

From this it follows that $\bar{I}$ is an interval. For if $u \leqslant v$ with v in $\bar{I}$, then $du \leqslant dv$ with dv in I, whence du is in I and hence u is in $\bar{I}$.

It therefore remains to prove (1) and (2) for $d < \omega$. Now (1) and (2) together are equivalent to affirming for each $q = 0, 1, 2, \ldots$ the following proposition.

$\mathfrak{P}_q$: *if $r < d$ and $r + dq$ belongs to A or B, then $\rho + dq$ is in A for all $\rho < d$ in case $r + dq$ is in A, and $r = 0$ in case $r + dq$ is in B.*

We establish these propositions $\mathfrak{P}_q$ by induction. Obviously $\mathfrak{P}_0$ is true. Assume then that $\mathfrak{P}_n$ holds for all $n < q$ with $q \geqslant 1$: to prove $\mathfrak{P}_q$. We do this by first establishing from the induction hypothesis the following weaker proposition $\mathfrak{p}_q$ obtained from $\mathfrak{P}_q$ by putting $r = 0$:

$\mathfrak{p}_q$: *$\rho + dq$ is in A for all $\rho < d$ in case dq is in A.*

Then from $\mathfrak{p}_q$ and the induction hypothesis we prove $\mathfrak{P}_q$.

To prove $\mathfrak{p}_q$, let dq be in A: we are to show that $\rho + dq$ is in A for all $\rho < d$. Because $q \geqslant 1$, $d \leqslant dq$ and dq is in I, so d is in I. Since A contains $I(d)$ but not d, the B-coordinate of d is > 0 and hence $\geqslant d$, so must equal d. Therefore d is in B. This, together with dq in A, shows that $dq + d$ belongs to $A + B = I$.

Consequently $\rho + dq$, being $< dq + d$, is also in I. Let $\rho + dq$ have coordinates a in A, b in B. We wish to show that $\rho + dq$ is in A, or, what amounts to the same, that $b = 0$. Suppose, to the contrary, that $b > 0$. Write b in the form $b = r + d\beta$ with $r < d$. Hence $\beta \leqslant q$. If $\beta = q$, then $r > 0$, for otherwise we would have $dq = b$ in B as well as dq in A. The number $dq + d$ with coordinates dq in A, d in B thus has alternative coordinates $d - r$ in A, $r + dq = b$ in B; which is impossible. On the other hand, if $\beta < q$, then $r = 0$ by the induction hypothesis, so $b = d\beta$. Thus a has the form $a = \rho + d\alpha$ where $\alpha + \beta = q$ and $0 < \alpha < q$ since $0 < \beta < q$. Therefore $d\alpha$ is in A by the induction hypothesis: so dq in A has positive coordinates $d\alpha$ in A, $d\beta$ in B; which is impossible. This dilemma proves our contention that $\rho + dq$ is in A. Thus $\mathfrak{p}_q$ is proved.

To prove $\mathfrak{P}_q$, let $r < d$ and $r + dq$ belong to A or B: we are to show that $\rho + dq$ is in A for all $\rho < d$ in case $r + dq$ is in A, and that $r = 0$ in case $r + dq$ is in B. Now $r + dq$ is in I, so dq is in I also. Let dq have coordinates a in A, b in B. Then b has the form $b = d\beta$ with $\beta \leqslant q$: obviously in case $b = dq$, by the induction hypothesis in case $b < dq$. Hence a has the form $a = d\alpha$ with $\alpha \leqslant q$. Consequently $r + d\alpha$ is in A: by $\mathfrak{p}_q$ in case $\alpha = q$, by the induction hypothesis in case $\alpha < q$. The number $r + dq$ then has the coordinates $r + d\alpha$ in A, $d\beta$ in B. Therefore if $r + dq$ is in A, $d\beta = 0$ so $dq = d\alpha$ is in A; whence $\rho + dq$ is in A for all $\rho < d$ by $\mathfrak{p}_q$. And if $r + dq$ is in B, $r + d\alpha = 0$; whence $r = 0$. This completes the proof of $\mathfrak{P}_q$ and hence, by induction, the proof of the lemma.

LEMMA 3. *Let $I = \sum X^\mu$ be a partition of a proper interval I. There then exist: a proper ordinal d, an interval $\bar{I}$ with $\bar{I} = 0$ if $d = \omega$, an index α with d not in X^α, and sets $\bar{X}^\mu$, all of these unique, such that*

$$X^\mu = \delta^{\mu\alpha} I(d) + d\bar{X}^\mu, \tag{4}$$

$$\bar{I} = \sum \bar{X}^\mu, \tag{5}$$

where δ is Kronecker's delta: $\delta^{\mu\alpha} = 1$ or 0 according as $\mu = \alpha$ or not.

Proof. Since $I \neq 0$, 1 is in I. Therefore $1 = \sum x^\mu$ with x^μ in X^μ. Clearly a unique index α exists such that $x^\alpha = 1$ and $x^\beta = 0$ for all remaining indices $\beta \neq \alpha$. Define $A = X^\alpha$ and $B = \sum X^\beta$. Then $I = A + B$. Let d be the smallest ordinal not in $A = X^\alpha$. Since 1 is in A, d is a proper ordinal. Now A contains $I(d)$ but not d, so the hypothesis of Lemma 2 is satisfied, and thus the conclusion, whereby the unique sets $\bar{I}, \bar{A}, \bar{B}$ are furnished. Therefore $\bar{I} = 0$ if $d = \omega$. Define $\bar{X}^\alpha = \bar{A}$; then by (1),

$$X^\alpha = I(d) + d\bar{X}^\alpha.$$

By (2), d divides every number in B and hence every number in each X^β, so X^β is of the form

$$X^\beta = d\bar{X}^\beta,$$

with $\bar{B} = \sum X^\beta$. Evidently (4) summarizes the two formulae above. As for (5),

$$\bar{I} = \bar{A} + \bar{B} = \bar{X}^\alpha + \sum \bar{X}^\beta = \sum \bar{X}^\mu$$

by virtue of (3). This completes proof of the lemma.

https://doi.org/10.4153/CJM-1953-009-4 Published online by Cambridge University Press

Proof of Theorem II. Let $I = \sum X^{\mu}$ be a partition of a proper interval I. By recursive use of Lemma 3 we construct: a proper sequence of ordinals d_m, a sequence of indices α_m, consecutive ones being different, and sequences of sets X_m^{μ} with $X_1^{\mu} = X^{\mu}$, all these sequences unique and of the same length, such that

$$X_m^{\mu} = \begin{cases} \delta^{\mu\alpha_m} I(d_m) & (m \text{ terminal}), \\ \delta^{\mu\alpha_m} I(d_m) + d_m X_{m+1}^{\mu} & (m \text{ non-terminal}). \end{cases}$$

Therefore, by Lemma 1,

$$X^{\mu} = \delta^{\mu\alpha_1} I(d_1) + \delta^{\mu\alpha_2} d_1\, I(d_2) + \delta^{\mu\alpha_3} d_1\, d_2\, I(d_3) + \ldots,$$

whereupon

$$I = \sum X^{\mu} = I(d_1) + d_1\, I(d_2) + d_1\, d_2\, I(d_3) + \ldots,$$

as was to be shown.

This finer repartition of the given partition of I has evidently been constructed in a unique fashion; it will be called the resolution of the given partition. Any two consecutive terms of this resolution lie in different X^{μ} sets, or, as we shall say, are separated.

Number of partitions. Consider for given positive numbers m and n a partition of the interval $I(n)$ into m sets:

$$I(n) = X_1 + \ldots + X_m.$$

Call the ordered sequence of sets $X_1, \ldots, X_m$ an ordered m-partition of $I(n)$, and let $p_m(n)$ be the number of such partitions.

The values of this partition counting function[3] can be obtained recursively as follows. Let $n > 1$. The (terminating) resolution of an ordered m-partition of $I(n)$ may be formed from the resolution of an ordered m-partition of $I(d)$, where $d < n$ is a divisor of n, by adding the term $T = dI(n/d)$ as separated terminal term to one of the partitioning sets of $I(d)$. If $d = 1$, all m partitioning sets of $I(d)$ are 0, so T can be added to any one of these m sets. If $d > 1$, T can be added to any one of the m partitioning sets of $I(d)$ except that one which contains the terminal term of the resolution of the partition of $I(d)$. This procedure uniquely delivers all ordered m-partitions of $I(n)$, so

$$p_m(n) = 1 + (m - 1) \sum p_m(d),$$

the summation extending over all divisors $d < n$ of n. Though derived for $n > 1$, this formula is also valid for $n = 1$, since obviously $p_m(1) = 1$.

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[3]Two explicit formulae for this function are derived from the recursion relation developed here in Proc. Amer. Math. Soc., *3* (1952), 31–35.

https://doi.org/10.4153/CJM-1953-009-4 Published online by Cambridge University Press