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ON RINGS WHOSE SIMPLE MODULES ARE FLAT

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ABSTRACT. A ring R is called a *right* SF-*ring* if all of its simple right R-modules are flat. It is well known that a von Neumann regular ring is a right SF-ring. In this paper we study conditions under which the converse holds.

In this paper all rings are rings with unity and all modules are unital. A homomorphism is written on the side opposite to the operation of the ring. Let R be a ring, and M a right R-module. Then we write M_R in order to indicate the ring which is involved. The socle of M is denoted by $\mathrm{Soc}(M)$ and the annihilator of M in R is denoted by $\mathrm{Ann}(M)$.

A ring R is called a *right (left)* SF-*ring* if all of its simple right (left) R-modules are flat. It is well known that a ring R is von Neumann regular if and only if every right (left) R-module is flat (*cf.* [7, Proposition 5.4.4]). Hence a von Neumann regular ring is a right and left SF-ring. Ramamurthi [9] raised a question whether a right SF-ring is necessarily von Neumann regular, and several authors (*e.g.* [3], [10], [13], [14]) studied this question. In this paper we find a class of rings containing the PI-rings, in which the two conditions of being von Neumann regular and right SF-ring are equivalent.

We begin with the following lemma.

LEMMA 1. *Let R be a ring, and I an ideal of R such that R/I is a simple artinian ring. Then R/I_R is flat if and only if ${}_RR/I$ is injective.*

PROOF. Let Γ denote an irredundant set of representatives of the simple left R-modules and let E denote the injective envelope of $\oplus_{T\in\Gamma} T$. Then there exists a unique simple left R-module M in Γ such that ${}_RR/I \simeq M^{(n)}$ for some positive integer n. Let us write $R/I = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ with $T_i \simeq M$ for each i. Consider the mapping $\varphi\colon \mathrm{Hom}_R(R/I, E) \longrightarrow E$ defined by $(f)\varphi = (1+I)f$ for all $f \in \mathrm{Hom}_R(R/I, E)$. Clearly φ is an R-monomorphism. Let f be an element of $\mathrm{Hom}_R(R/I, E)$. Then, for each i, $(T_i)f$ is either M or 0, so that $(1+I)f \in M$. Since M is simple, we have $\mathrm{Im}\,\varphi = M$ and hence ${}_R\mathrm{Hom}_R(R/I, E) \simeq {}_RM$. Therefore ${}_RR/I \simeq {}_R\mathrm{Hom}_R(R/I, E)^{(n)}$. Now, by virtue of [12, Proposition 1.10.4], ${}_R\mathrm{Hom}_R(R/I, E)$ is injective if and only if R/I_R is flat. This completes the proof.

THEOREM 1. *Let R be a ring all of whose left primitive factor rings are artinian. Then the following conditions are equivalent:*

(1) R is a right SF-*ring.*

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(2) R is von Neumann regular.

PROOF. Assume that R is a right SF-ring and let M be a simple left R-module. Then $\bar{R} = R/\operatorname{Ann}(M)$ is left primitive, so that $\bar{R}$ is a simple artinian ring by hypothesis. Since $\bar{R}_R$ is a finite direct sum of simple right R-modules, $\bar{R}_R$ is flat. Then ${}_R\bar{R}$ is injective by Lemma 1. Since ${}_R\bar{R} \simeq M^{(n)}$ for some positive integer n, M is an injective left R-module. This proves that R is a left V-ring. By virtue of [1, Theorem], we conclude that R is von Neumann regular.

REMARK 1. Let R be a ring all of whose left primitive factor rings are artinian. Let $\operatorname{fd}(M)$ denote the flat dimension of the right R-module M and set $s(R) = \sup\{\operatorname{fd}(T) \mid T$ is a simple right R-module$\}$. Theorem 1 asserts that the weak global dimension $\operatorname{wgld}(R)$ of R equals 0 if and only if $s(R) = 0$. Hence it may be suspected that $\operatorname{wgld}(R) = s(R)$. However K. L. Fields [4, p. 348] constructed a right noetherian local ring S with $\operatorname{wgld}(S)$ $(= \operatorname{rt.gld}(R)) = 2$ and $s(S) = 1$ (*cf.* [11, Theorem 9.22]).

Let R be a PI-ring. Then all right or left primitive factor rings of R are artinian by Kaplansky [5, Theorem]. Hence we have the following corollary.

COROLLARY 1. *Let R be a* PI-*ring. Then the following conditions are equivalent:*
(1) R is a right SF-*ring.*
(2) R is a left SF-*ring.*
(3) R is von Neumann regular.

Let R be a ring, and G a group. Then RG denotes the group ring of G over R.

COROLLARY 2. *Let R be a ring all of whose left primitive factor rings are artinian, and G be a group. Then the following conditions are equivalent:*
(1) RG is a right SF-*ring.*
(2) RG is von Neumann regular.

PROOF. Assume that RG is a right SF-ring. Let ωG denote the augmentation ideal of RG. Since $R \simeq RG/\omega G$, R is also a right SF-ring. Hence R is von Neumann regular by Theorem 1. Let P be a left primitive ideal of R, and consider the factor ring $\bar{R} = R/P$. Since $\bar{R}G \simeq RG/PG$, $\bar{R}G$ is also a right SF-ring. Since $\bar{R}$ is a finite direct sum of simple right $\bar{R}G$-modules, $\bar{R}$ is a flat right $\bar{R}G$-module. By [8, Lemma 6.5], G is locally finite and the order of every element in G is a unit of $\bar{R}$. Let g be an element of G, and n the order of g. Since R is von Neumann regular, there exists an $x \in R$ such that $n = n^2x$. Since n is a unit in $\bar{R} = R/P$, we have $nx - 1 \in P$. Since P is an arbitrary left primitive ideal of R and since the Jacobson radical of the von Neumann regular ring R is 0, we obtain $nx - 1 = 0$. Thus we proved that R is von Neumann regular, G is locally finite and the order of every element in G is a unit of R. Then RG is von Neumann regular by [7, Proposition 2, p. 155].

We try to extend Theorem 1 and we consider the following condition:

$(*)$ For any singular simple left R-module M, $R/\operatorname{Ann}(M)$ is artinian.

EXAMPLE. Let $L = \mathrm{End}_F V$ be the full right linear ring over an infinite dimensional vector space V over a field F, let S be the ideal consisting of linear transformations of finite rank, and let $R = S + F$ be the subring generated by S and the subring F consisting of scalar transformations. Then R is a left primitive von Neumann regular ring with $\mathrm{Soc}(_RR) = S$. Since R is not artinian, R does not satisfy the hypothesis of Theorem 1. Now let M be a singular simple left R-module. Then we can easily see that $\mathrm{Ann}(M) = S$ and $R/\mathrm{Ann}(M) \simeq F$. Therefore R satisfies the condition $(*)$.

THEOREM 2. *Let R be a ring satisfying the condition $(*)$. Then the following conditions are equivalent:*

(1) R is a right SF-*ring.*

(2) R is von Neumann regular.

PROOF. Suppose that R is a right SF-ring. We first claim that every minimal left ideal of R is generated by an idempotent. Let K be a minimal left ideal of R. If K is non-singular, then we can easily see that K is projective. By the proof of [3, Theorem 2] it follows that the right annihilator of a finitely generated proper left ideal is always nonzero. Hence, by [2, Theorem 4.5], K is a direct summand of $_RR$. Next, assume that K is singular. Then $\bar{R} = R/\mathrm{Ann}(K)$ is artinian by the condition $(*)$. Hence $\bar{R}_R$ is a finite direct sum of simple right R-modules, so that $\bar{R}_R$ is flat. Therefore $_R\bar{R}$ is injective by Lemma 1. Since $_R\bar{R} \simeq K^{(n)}$ for some positive integer n, K is injective, and hence K is a direct summand of $_RR$. This contradicts the singularity of K. Next, we claim that, for any $a \in \mathrm{Soc}(_RR)$, there exists an idempotent $e \in R$ such that $Ra = Re$. We prove this by induction on the composition length $c(Ra)$ of $_RRa$. By the previous claim, we may assume $n = c(Ra) > 1$. Then we can write $Ra = K_1 \oplus \cdots \oplus K_n$ for some minimal left ideals $K_1, \ldots, K_n$. By the previous claim, there exists an idempotent $f \in R$ such that $K_1 = Rf$. Then $Ra = Rf \oplus R(a - af)$ and $c\big(R(a - af)\big) = n - 1$. By induction hypothesis there exists an idempotent $g \in R$ such that $R(a - af) = Rg$. Note that $gf = 0$. Hence, if we set $e = f + g - fg$, then e is an idempotent and $Ra = Re$. Now we can show that $\mathrm{Soc}(_RR)$ is von Neumann regular. Let $a \in \mathrm{Soc}(_RR)$ and take an idempotent $e \in R$ such that $Ra = Re$. Then $e = ra$ for some $r \in R$ and $a = ae = ara$. Finally we claim that $R/\mathrm{Soc}(_RR)$ is von Neumann regular. If M is a non-singular simple left R-module, then $M \simeq Re$ for some idempotent $e \in R$, so that $\mathrm{Soc}(_RR)M \neq 0$. Therefore the condition $(*)$ implies that all left primitive factor rings of $R/\mathrm{Soc}(_RR)$ are artinian. Thus $R/\mathrm{Soc}(_RR)$ is von Neumann regular by Theorem 1. Since both $\mathrm{Soc}(_RR)$ and $R/\mathrm{Soc}(_RR)$ are von Neumann regular, R is von Neumann regular by [6, Theorem 22, p. 112].

REMARK 2. A ring is called a MELT *ring* if every maximal essential left ideal is two sided. Clearly a MELT ring satisfies the condition $(*)$. Hence Theorem 2 improves [14, Proposition 9].

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