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Best approximation and intersections of balls in Banach spaces

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Let E be a Banach space, M a closed subspace of E with the 3-ball property. It is known that M is proximinal in E, and that its metric projection admits a continuous selection. This means that there is a continuous (generally non-linear) map $\pi : E \to M$ satisfying $\|x-\pi(x)\| = d(x, M)$ for all x in E. Here it is shown that the same conclusion holds under a much weaker hypothesis on M, which we call the $1\frac{1}{2}$-ball property. We also establish that if M has the $1\frac{1}{2}$-ball property in E, then there is a continuous Hahn-Banach extension map from M^* to E^*.

Introduction

Let M be a closed subspace of a Banach space E. This paper clarifies the relationship between approximative properties of M, and intersection properties of balls pertaining to M. Recall that M is said to be an L-summand (respectively, an M-summand) of E if there is a linear projection Q from E onto M such that $\|x\| = \|Qx\| + \|x-Qx\|$ (respectively, $\|x\| = \max\{\|Qx\|, \|x-Qx\|\}$) for all $x \in E$. If M^0, the polar of M, is an L-summand of E^*, then M is said to be an M-ideal in E. We say that M has the n-ball property in E if given n closed balls $B(a_i, r_i)$ such that $M \cap B(a_i, r_i)$ is non-empty for each i, and $\bigcap_{i=1}^{n} B(a_i, r_i)$ has non-empty interior, then $M \cap \bigcap_{i=1}^{n} B(a_i, r_i)$ is

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non-empty. These notions were introduced by Alfsen and Effros [1], who showed that an M-ideal has the n-ball property for every n and, conversely, that any subspace with the 3-ball property is already an M-ideal.

Let $H(\cdot)$ denote the family of all closed, bounded, convex, and non-empty subsets of a given Banach space. The metric projection $P = P_M : E \to H(M) \cup \{\emptyset\}$ is the set-valued map defined by $P(a) = M \cap B(a, d(a, M))$. Thus $P(a)$ is the set of points in M which are nearest to a . M is said to be proximinal in E if $P(a) \neq \emptyset$, for all $a \in E$. Then a proximity map $\pi : E \to M$ is any (not necessarily continuous) selection for P . Note that $P(a+x) = P(a) + x$ whenever $x \in M$. We say that a selection π is quasi-additive if $\pi(a+x) = \pi(a) + x$ whenever $x \in M$.

Alfsen and Effros [1, Corollary 5.6] and Ando [2, Theorem 2.1] independently showed that every M-ideal is proximinal. Holmes, Scranton, and Ward [6, Theorem 2.2] improved this by showing that the metric projection onto an M-ideal admits a continuous, homogeneous selection.

We will say that M has the $1\frac{1}{2}$-ball property in E if the conditions $a_1 \in M$, $M \cap B(a_2, r_2) \neq \emptyset$, and $\|a_1-a_2\| < r_1 + r_2$ imply that $M \cap B(a_1, r_1) \cap B(a_2, r_2) \neq \emptyset$. After translating and scaling it is evident that this is equivalent to requiring $M \cap B(0, 1) \cap B(a, r) \neq \emptyset$ whenever $M \cap B(a, r) \neq \emptyset$ and $\|a\| < r + 1$. Our main result is that every subspace with the $1\frac{1}{2}$-ball property is proximinal, and that its metric projection admits a continuous, homogeneous, quasi-additive selection. In Section 2 we give examples of closed subspaces of Banach spaces which possess the $1\frac{1}{2}$-ball property. Not all of these subspaces are M-ideals, so our result has wider applicability than that of [6]. We also show that if M has the $1\frac{1}{2}$-ball property in E , then there is a continuous, homogeneous map $\psi : M^* \to E^*$ such that each $\psi(f)$ is a norm preserving extension of f . Under additional hypotheses, we are able to establish the Lipschitz continuity and linearity of certain proximity maps and Hahn-Banach extension maps.

Except when specific mention is made to the contrary, scalars may be real or complex. By $C(X, E)$ we denote the Banach space of continuous functions from the compact, Hausdorff space X into the Banach space E. If S is a sequence space, then $S(E)$ will denote the Banach space of all sequences (x_n) from E such that the sequence $(\|x_n\|)$ is in S. $B(E, F)$ is the space of bounded, linear operators from E to F, and $K(E, F)$ is the subspace of compact operators. We use d_H for the Hausdorff metric on $H(E)$,

$$d_H(A, B) = \sup(\{d(x, A) : x \in B\} \cup \{d(x, B) : x \in A\}) .$$

By Michael's Selection Theorem we mean [*11*, Theorem 3.2"].

1. Existence of continuous selections

We establish the results stated in the abstract.

LEMMA 1.1. *Suppose M has the $1\frac{1}{2}$-ball property in E. Then*

(i) M is proximinal in E,

(ii) for all $a, b \in E$ we have $d_H(a-P(a), b-P(b)) \leq 3d(a-b, M)$.

The constant 3 is, in general, best possible.

Proof. *(i)* Let $a \in E$, $\delta = d(a, M)$. We inductively construct a sequence $(x_n) \subset M$ satisfying

(1) $$\|x_n-x_{n+1}\| \leq 2^{-n}$$

and

(2) $$\|x_n-a\| \leq \delta + 2^{-n} .$$

Obviously a suitable x_1 exists. Suppose x_n is given, and satisfies (2). Then we have $x_n \in M$, $M \cap B(a, \delta+2^{-n-1}) \neq \emptyset$ and $\|x_n-a\| < \delta + 2^{-n-1} + 2^{-n}$. Since M has the $1\frac{1}{2}$-ball property, $M \cap B(x_n, 2^{-n}) \cap B(a, \delta+2^{-n-1}) \neq \emptyset$. Any point x_{n+1} in this intersection will satisfy (1) and (2).

The induction completed, (1) implies that (x_n) is Cauchy, and hence converges to some $x \in M$. Then (2) yields $\|x-a\| = \delta$. Thus $P(a) \neq \emptyset$.

(ii) Let $a, b \in E$ with $d(a-b, M) < \varepsilon$. It suffices to show that, given $x \in P(a)$, we can find $y \in P(b)$ with $\|(a-x)-(b-y)\| < 3\varepsilon$. If $b \in M$, then $P(b) = \{b\}$ and we must take $y = b$. Then $\|a-x-b+y\| = \|a-x\| = d(a, M) = d(a-b, M) < \varepsilon$ as required. If $b \notin M$, then $\delta = d(b, M) > 0$. Choose $z \in M$ with $\|a-b+z\| < \varepsilon$. Then $z+x \in M$, $M \cap B(b, \delta) \neq \emptyset$ by *(i)*, and

$$\|z+x-b\| \leq \|a-b+z\| + \|x-a\| < \varepsilon + d(a, M) < 2\varepsilon + \delta .$$

Since M has the $1\frac{1}{2}$-ball property, we can find

$$y \in M \cap B(b, \delta) \cap B(z+x, 2\varepsilon) .$$

Clearly $y \in P(b)$. Finally

$$\|a-x-b+y\| \leq \|y-(x+z)\| + \|a-b+z\| < 2\varepsilon + \varepsilon .$$

To show that this estimate is sharp, consider the real Banach space $E = l_\infty(3)$ (that is, $E = \mathbf{R}^3$, with the sup norm), with M the one-dimensional subspace spanned by $(1, 1, 0)$. It is elementary to check that M has the $1\frac{1}{2}$-ball property in E. Let $a = (0, 0, 3)$, $b = (1, -1, 2)$, and $x = (-3, -3, 0)$. Then

$$P(b) = \{(\lambda, \lambda, 0) : -1 \leq \lambda \leq 1\}$$

and so $d\big(a-x, b-P(b)\big) = 3$. Now $x \in P(a)$, so $d_H\big(a-P(a), b-P(b)\big) \geq 3$. But $d(a-b, M) \leq \|a-b\| = 1$. //

We remark that if M has the 2-ball property in E, then the estimate of Lemma 1.1 can be sharpened to $d_H\big(a-P(a), b-P(b)\big) \leq d(a-b, M)$. The preceding example then shows that the $1\frac{1}{2}$-ball property is strictly weaker than the 2-ball property.

THEOREM 1.2. *If M has the $1\frac{1}{2}$-ball property in E, then*

(i) *there is a continuous, homogeneous map $\psi : E/M \to E$ satisfying $\psi(a+M) \in a+M$ and $\|\psi(a+M)\| = \|a+M\|$ for all $a \in E$,*

(ii) *there is a continuous, homogeneous, quasi-additive proximity map $\pi : E \to M$,*

(iii) *there is a continuous, homogeneous Hahn-Banach extension map $\psi : M^* \to E^*$.*

Proof. *(i)* Define $\eta : E/M \to H(E)$ by $\eta(a+M) = a - P_M(a)$. Since M is proximinal, η is well-defined. By Lemma 1.1, η is continuous with respect to the Hausdorff metric on $H(E)$, and is therefore lower semicontinuous. Michael's selection theorem ensures the existence of ψ, a continuous selection for η. An argument of Kadison [see *11*, p. 376] shows that ψ can be chosen homogeneous. Clearly ψ has the stated properties.

(ii) Let ψ be given by *(i)*, and define π by $\pi(a) = a - \psi(a+M)$. Then π is continuous, homogeneous, quasi-additive, and satisfies $\pi(a) \in P(a)$ for all $a \in E$.

(iii) We claim that M^0 has the $1\frac{1}{2}$-ball property in E^*. So let $M^0 \cap B(f, r) \neq \emptyset$, $\|f\| \leq r + 1$. To show that $M^0 \cap B(0, 1) \cap B(f, r) \neq \emptyset$ it suffices, by [7, Theorem 1.2], to show that $|f(a_2)| \leq \|a_1\| + r\|a_2\|$ whenever $a_1+a_2 \in M$. If $\|a_2\| \leq \|a_1\|$ then

$$|f(a_2)| \leq (r+1)\|a_2\| \leq \|a_1\| + r\|a_2\| .$$

So assume $\|a_2\| > \|a_1\|$ and fix $\varepsilon > 0$. Since $a_1+a_2 \in M \cap B(a_2, \|a_1\|+\varepsilon)$, the $1\frac{1}{2}$-ball property gives us some

$$a \in M \cap B(0, \|a_2\|-\|a_1\|) \cap B(a_2, \|a_1\|+\varepsilon) .$$

Now $\|f|M\| = d(f, M^0) < r$, so

$$|f(a_2)| = |f(a)-f(a-a_2)| \leq r\|a\| + (r+1)\|a-a_2\|$$
$$\leq r(\|a_2\|-\|a_1\|) + (r+1)(\|a_1\|+\varepsilon) .$$

Letting $\varepsilon \to 0$ establishes the claim.

From *(i)* we obtain a continuous, homogeneous map $\psi : E^*/M^0 \to E^*$ satisfying $\psi(f+M^0) \in f+M^0$ and $\|\psi(f+M^0)\| = d(f, M^0) = \|f|M\|$ for all $f \in E^*$. Identifying E^*/M^0 with M^* completes the proof. //

If $P_M(a)$ is a singleton for each $a \in E$, then M is said to be a Chebyshev subspace of E. In this case the proximity map is unique and is usually referred to as the metric projection. Let us say that M is a semi-L-summand in E [7, Section 5] if M is Chebyshev in E and the metric projection $\pi : E \to M$ satisfies $\|x\| = \|\pi(x)\| + \|x-\pi(x)\|$ for all $x \in E$. It is routine to check that every semi-L-summand (*a fortiori*, every L-summand) has the $1\frac{1}{2}$-ball property.

THEOREM 1.3. *Let* M *be a semi-L-summand in* E. *Then*

(i) *the metric projection* $\pi : E \to M$ *is a contraction,*

(ii) *there is a linear Hahn-Banach extension map* $\psi : M^* \to E^*$ *and a linear proximity map* $P : E^* \to M^0$,

(iii) M^{00} *is the range of a norm one projection on* E^{**}.

Proof. *(i)* Fix $a, b \in E$ and assume without loss of generality that $\|\pi(a)-a\| \le \|\pi(b)-b\|$. Since M is Chebyshev, π must be quasi-additive. Thus $\pi(\pi(a)-b) = \pi(a) - \pi(b)$ and so

$$\begin{aligned}\|\pi(a)-\pi(b)\| &= \|\pi(a)-b\| - \|\pi(b)-b\| \\ &\le \|\pi(a)-a\| + \|a-b\| - \|\pi(b)-b\| \\ &\le \|a-b\| .\end{aligned}$$

(ii) We have just shown the existence of a Lipschitz continuous retraction of E onto M with Lipschitz constant 1. The existence of ψ follows from [9, Theorem 3 (a)]. If $Pf = f - \psi(f|M)$ then P is linear and $\|f-Pf\| = \|f|M\| = d(f, M^0)$ for all $f \in E^*$.

(iii) Define $Q : E^{**} \to M^{00}$ by $QF = F \circ (I-P)$. //

Lima [7, Section 6] calls M a semi-M-ideal in E if M^0 is a semi-L-summand in E^*, and shows this is equivalent to M having what he calls the 2-ball property. The reader is warned that the definition of the 2-ball property used in [7] is, formally at least, weaker than that which we employ.

COROLLARY 1.4. *Let* M *be a semi-M-ideal in* E.

(i) The Hahn-Banach extension map $\psi : M^* \to E^*$ *is uniquely determined and satisfies* $\|\psi(f)-\psi(g)\| \le 2\|f-g\|$ *for all* $f, g \in E^*$. *The Lipschitz constant* 2 *can not, in general, be decreased.*

(ii) M^0 *is the range of a norm one projection on* E^*.

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Proof. *(i)* Again we identify E^*/M^0 and M^*. If $\pi : E^* \to M^0$ is the (unique) metric projection, then $\psi : E^*/M^0 \to E^*$ satisfies $\psi(f+M^0) = f - \pi(f)$. Fix $f+M^0, g+M^0 \in E^*/M^0$. Adding a suitable element of M^0, we may assume that $\pi(f-g) = 0$. Then

$$\|\psi(f+M^0)-\psi(g+M^0)\| = \|f-g-\pi(f)+\pi(g)\| \le 2\|f-g\|$$
$$= 2d(f-g, M^0) = 2\|(f+M^0)-(g+M^0)\| .$$

To show that the estimate is sharp, let E be the real Banach space $l_1(3)$ and take $M = \{(x, y, z) : x+y+z = 0\}$. Then $E^* = l_\infty(3)$ and $M^0 = \mathbf{R}1$. It is easy to verify that M^0 is a semi-L-summand. In E^*/M^0, let $f = (0, 2, 2) + \mathbf{R}1$ and $g = (-2, 0, -2) + \mathbf{R}1$. Then $\|f-g\| = 1$. Routine checking gives $\pi(0, 2, 2) = (1, 1, 1)$ and $\pi(-2, 0, -2) = (-1, -1, -1)$. Thus $\psi(f) = (-1, 1, 1)$, $\psi(g) = (-1, 1, -1)$ and so $\|\psi(f)-\psi(g)\| = 2$.

(ii) By Theorem 1.3 *(iii)* there is a norm one projection $Q : E^{***} \to M^{000}$. Let $f \mapsto \hat{f}$ denote the canonical embedding of E^* into E^{***}. Since $\hat{f} \in M^{000}$ whenever $f \in M^0$, the required projection is given by $f \mapsto Q(\hat{f})|E$. //

2. Examples

We give examples, mostly in spaces of operators and spaces of continuous functions, of subspaces which have the $1\frac{1}{2}$-ball property but are not M-ideals. For some of these examples, previous authors ([3, Corollary 3.19] and [12, 7.5.6]) have used *ad hoc* methods to establish the existence of continuous proximity maps, or simply to establish proximinality. The existence of continuous Hahn-Banach extension maps seems to have gone unnoticed. Checking that these subspaces have the $1\frac{1}{2}$-ball property provides a uniform, and often easier, method of establishing such results. We also give some new example of M-ideals. Lastly we consider the relationship between the n-ball property and algebraic structure in subspaces of Banach algebras.

Let us say that a real Banach space E is a (real) Lindenstrauss space if every collection of pairwise intersecting closed balls in E, whose centres form a compact set, has non-empty intersection. Lindenstrauss [8, p. 62] showed that a real Banach E has this property if and only if $E^* = L_1(\mu)$ for some measure μ.

THEOREM 2.1. *Let E be a real Lindenstrauss space, X and Y compact Hausdorff spaces, and $\psi : X \to Y$ a continuous surjection. Let $\psi^* : C(Y, E) \to C(X, E)$ denote the natural isometric embedding, $\psi^* f = f \circ \psi$. Then $M = \psi^* C(Y, E)$ has the $1\frac{1}{2}$-ball property in $C(X, E)$.*

Proof. Suppose we are given $f \in C(X, E)$ and $r > 0$ with $M \cap B(f, r) \neq \emptyset$ and $\|f\| \le r+1$. Define $\eta : Y \to H(E)$ by

$$\begin{aligned}\eta(y) &= B(0, 1) \cap \bigcap_{x \in \psi^{-1}(y)} B\big(f(x), r\big) \\ &= B(0, 1) \cap \big\{a \in E : f\big(\psi^{-1}(y)\big) \subset B(a, r)\big\} .\end{aligned}$$

Clearly each $\eta(y)$ is closed and convex. We must check that $\eta(y)$ is non-empty. Let $\psi^* g \in M \cap B(f, r)$. If $x_1, x_2 \in \psi^{-1}(y)$ then

$$\begin{aligned}\|f(x_1)-f(x_2)\| &\le \|f(x_1)-g(y)\| + \|g(y)-f(x_2)\| \\ &\le 2\|f-\psi^* g\| \le 2r ,\end{aligned}$$

and so $B\big(f(x_1), r\big)$ meets $B\big(f(x_2), r\big)$. Since $\|f\| \le r+1$, $B(0, 1)$ must meet each $B\big(f(x), r\big)$. Thus the family of balls defining $\eta(y)$ intersect pairwise. Since the collection of centres $\{0\} \cup f\big(\psi^{-1}(y)\big)$ is compact, we have $\eta(y) \neq \emptyset$. We claim that η is lower semicontinuous.

So let $G \subset E$ be open. Let $y_0 \in \{y : \eta(y) \text{ meets } G\}$ be given, and choose $a \in \eta(y_0) \cap G$. Then $\|a\| \le 1$, $f\big[\psi^{-1}(y_0)\big] \subset B(a, r)$ and $B(a, \varepsilon) \subset G$ for some $\varepsilon > 0$. It follows from the compactness of X that the map $y \mapsto \psi^{-1}(y)$ is upper semicontinuous. Hence $N = \big\{y : f\big(\psi^{-1}(y)\big) \subset \operatorname{int} B(a, r+\varepsilon)\big\}$ is an open set containing y_0. If $y \in N$, then $B(a, \varepsilon)$ meets $B\big(f(x), r\big)$ for all $x \in \psi^{-1}(y)$. Clearly $B(a, \varepsilon)$ meets $B(0, 1)$. Since E is a real Lindenstrauss space, we deduce that $\eta(y)$ meets $B(a, \varepsilon)$, whenever $y \in N$. Thus $N \subset \{y : \eta(y) \text{ meets } G\}$. It follows that $\{y : \eta(y) \text{ meets } G\}$ is open, and this proves η is lower semicontinuous.

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By Michael's selection theorem, there is a continuous function $h : Y \to E$ satisfying $h(y) \in \eta(y)$ for all y. It is routine to verify that $\psi^*h \in M \cap B(0, 1) \cap B(f, r)$. //

COROLLARY 2.2. *Let* X, Y, ψ, E *be as in Theorem* 2.1. *Fix* $y_0 \in Y$ *and let* $M = \{\psi^*f : f \in C(Y, E) \text{ and } f(y_0) = 0\}$. *Then* M *has the* $1\frac{1}{2}$*-ball property in* $C(X, E)$.

Proof. Let f, r, η be as in the previous proof. If $\psi^*g \in M \cap B(f, r)$, then $\|f(x)\| = \|f(x)-(\psi^*g)(x)\| \le r$ whenever $x \in \psi^{-1}(y_0)$. Thus $0 \in \eta(y_0)$. If we define $\eta_0 : Y \to H(E)$ by $\eta_0(y) = \eta(y)$ for $y \ne y_0$, and $\eta_0(y_0) = \{0\}$, then η_0 will be lower semicontinuous. The existence of a continuous selection for η_0 shows that $M \cap B(0, 1) \cap B(f, r) \ne \emptyset$. //

COROLLARY 2.3. *Any closed subalgebra of* $C(X, \mathbb{R})$ *has the* $1\frac{1}{2}$*-ball property.*

Proof. This follows from the Stone-Weierstrass Theorem and Theorem 2.1 (for subalgebras containing the constant functions) or Corollary 2.2 (for subalgebras not containing the constants). //

It follows from [7, Theorem 7.6] that any closed subspace of $C(X, \mathbb{R})$ with the 2-ball property must be an ideal. Thus the examples given by the preceding results will not, in general, be M-ideals.

PROPOSITION 2.4. *Let* E *be any Banach space,* X *a compact Hausdorff space,* Y *a closed subset of* X, $n \in \mathbb{N}$. *Then* $M = \{f \in C(X, E) : f|Y = 0\}$ *has the* n*-ball property in* $C(X, E)$.

Proof. Suppose that we have $M \cap B(f_i, r_i) \ne \emptyset$ for $i \le n$, and $\operatorname{int} \bigcap_{i=1}^{n} B(f_i, r_i) \ne \emptyset$. Define $\psi : X \to H(E)$ by $\psi(x) = \bigcap_{i=1}^{n} B(f_i(x), r_i)$. Clearly each $\psi(x)$ is closed, convex, and has non-empty interior. Hence $\psi(x) = \overline{\operatorname{int} \psi(x)}$ for all $x \in X$. Now let G be any open subset of E, and let $x_0 \in \{x : \psi(x) \text{ meets } G\}$. Then $\overline{\operatorname{int} \psi(x_0)}$ meets G, so we can find $a \in \operatorname{int} \psi(x_0) \cap G$. Then $\|a-f_i(x_0)\| < r_i$ for each i. By continuity, x_0 has a neighbourhood N such that $x \in N \Rightarrow \|a-f_i(x)\| < r_i$,

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for all i . Then $a \in \psi(x)$ whenever $x \in N$, so $N \subset \{x : \psi(x) \text{ meets } G\}$. This proves that ψ is lower semicontinuous.

Fix $x \in Y$. If $g_i \in M \cap B(f_i, r_i)$; then

$$\|f_i(x)\| = \|f_i(x)-g_i(x)\| \le \|f_i-g_i\| \le r_i .$$

This proves that $0 \in \psi(x)$.

Now define $\eta : X \to H(E)$ by $\eta(x) = \psi(x)$ for $x \notin Y$, and $\eta(x) = \{0\}$ for $x \in Y$. Since Y is closed, it is easily shown that η is lower semicontinuous. Let $f \in C(X, E)$ be a continuous selection for η . Then $f \in M \cap \bigcap_{i=1}^{n} B(f_i, r_i)$. //

We note that Corollary 2.3 fails in spaces of complex-valued functions.

PROPOSITION 2.5. *A closed *subalgebra A in $C(X, \mathbb{C})$ has the $1\frac{1}{2}$-ball property if and only if it is an ideal.*

Proof. That ideals have the $1\frac{1}{2}$-ball property is immediate from Proposition 2.4, with $E = \mathbb{C}$. Suppose now that A is not an ideal. We assume that A does not contain the constant functions. (If $1 \in A$, the result follows from a simplification of the following argument.) By the Stone-Weierstrass Theorem, there is a compact Hausdorff space Y , a continuous surjection $\psi : X \to Y$ and a point $y_0 \in Y$ such that $A = \{\psi^*f : f \in C(Y, \mathbb{C}) \text{ and } f(y_0) = 0\}$. If the restriction of ψ to $X \backslash \psi^{-1}(y_0)$ is injective, it can readily be shown that A is an ideal. Thus we may find distinct $x_0, x_1 \in X$ such that $\psi(x_0) = \psi(x_1) \ne y_0$. Let $y_1 = \psi(x_1)$, and construct continuous functions $a : X \to \mathbb{R}$, $b : Y \to \mathbb{R}$ satisfying $-1 \le a \le 1$, $0 \le b \le 1$, $a(x_n) = (-1)^n$ and $b(y_n) = n$ $(n = 0, 1)$. Then $\|a-i\psi^*b\| \le \sqrt{2} < 1 + \frac{1}{2}$, $i\psi^*b \in A$, and $A \cap B(a, 1) \ne \emptyset$. However $A \cap B(a, 1) \cap B(i\psi^*b, \frac{1}{2}) = \emptyset$, which shows that A does not have the $1\frac{1}{2}$-ball property. For suppose $\psi^*f \in A \cap B(a, 1)$. Then, for $n = 0, 1$, $|f(y_1) \pm 1| = |(\psi^*f)(x_n)-a(x_n)| \le \|\psi^*f-a\| \le 1$. Hence $f(y_1) = 0$. But then $\|\psi^*f-i\psi^*b\| \ge |f(y_1)-ib(y_1)| = 1 > \frac{1}{2}$. //

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By Proposition 2.4, $c_0(E)$ is an M-ideal in $l_\infty(E)$ if E is finite dimensional. It is useful to know that this is true for arbitrary E.

LEMMA 2.6. *For any Banach space E, $c_0(E)$ is an M-ideal in $l_\infty(E)$.*

Proof. If $x = (x(n)) \in l_\infty(E)$ and $c_0(E) \cap B(x, r) \neq \emptyset$, then $\limsup\|x(n)\| \leq r$. Suppose $\bigcap_{i=1}^{3} B(x_i, r_i) \neq \emptyset$ and $c_0(E) \cap B(x_i, r_i) \neq \emptyset$ for each i. Then for all $\varepsilon > 0$, $\bigcap_{i=1}^{3} B(x_i, r_i+\varepsilon)$ contains a sequence with only finitely many non-zero terms and so meets $c_0(E)$. Although formally weaker than the 3-ball property, the property just established does characterize M-ideals [7, Theorem 6.9]. //

COROLLARY 2.7. *For any Banach space E, $K(E, c_0)$ is an M-ideal in $B(E, c_0)$.*

Proof. This follows from the natural identifications $K(E, c_0) = c_0(E^*)$ and $B(E, c_0) = \{(f_n) \in l_\infty(E^*) : f_n \to 0 \text{ weak*}\}$. //

PROPOSITION 2.8. *$K(l_1)$ has the $1\frac{1}{2}$-ball property in $B(l_1)$.*

Proof. Recall that for any operator matrix $a = (a_{ij}) \in B(l_1)$ we have $\|a\| = \sup_{j=1}^{\infty} \sum_{i=1}^{\infty} |a_{ij}|$ and $a \in K(l_1) \Leftrightarrow \lim_{n\to\infty} \sup_{j=1}^{\infty} \sum_{i=n}^{\infty} |a_{ij}| = 0$. Fix $a \in B(l_1)$ with $\|a\| \leq r+1$, and $K(l_1) \cap B(a, r) \neq \emptyset$. We assume that $\|a\| > r$, otherwise $0 \in K(l_1) \cap B(0, 1) \cap B(a, r)$.

Fix $j \in \mathbb{N}$. If $\sum_{i=1}^{\infty} |a_{ij}| \leq r$, put $x_{ij} = 0$ for all i. Otherwise choose $n = n(j)$ and $0 \leq \lambda \leq 1$ so that $\lambda|a_{nj}| + \sum_{i=n+1}^{\infty} |a_{ij}| = r$. Putting $x_{ij} = a_{ij}$ for $i < n$, $x_{nj} = (1-\lambda)a_{nj}$ and $x_{ij} = 0$ for $i > n$, we have $\sum_{i=1}^{\infty} |x_{ij}| = \sum_{i=1}^{\infty} |a_{ij}| - r$.

It follows that $x \in B(l_1)$ with $\|x\| \le \|a\| - r \le 1$. For each j, either $x_{ij} = 0$ for all i, or $\sum_{i=1}^{\infty} |a_{ij}-x_{ij}| = r$. Hence $\|a-x\| \le r$.

We must show $x \in K(l_1)$. Fix $\varepsilon > 0$. Since $K(l_1)$ meets $B(a, r)$, there is a finite rank operator in $B(a, r+\varepsilon)$. Thus, for some N, $\sup_{j=1}^{\infty} \sum_{i=N}^{\infty} |a_{ij}| < r + \varepsilon$. Fix j. If $\sum_{i=1}^{\infty} |a_{ij}| \le r$, or if $N > n(j)$, then $\sum_{i=N}^{\infty} |x_{ij}| = 0$. If $N \le n(j)$ then $\sum_{i=N}^{\infty} |x_{ij}| = \sum_{i=N}^{\infty} |a_{ij}| - r < \varepsilon$. Thus $\sup_{j=1}^{\infty} \sum_{i=N}^{\infty} |x_{ij}| < \varepsilon$, as desired. //

If E and F are separable sequence spaces (that is, c_0 or l_p, $1 \le p < \infty$), what is the largest value of n such that $K(E, F)$ has the n-ball property in $B(E, F)$? Hennefeld [5] showed that $K(l_p)$ is an M-ideal in $B(l_p)$ if $1 < p < \infty$. Minor modifications to his argument yield that $K(l_p, l_q)$ is an M-ideal in $B(l_p, l_q)$ if $1 < p < q < \infty$. By [13, Theorem 6.2] $K(l_1)$ fails the 2-ball property in $B(l_1)$. We show that $K(l_1, l_p)$ fails the $1\frac{1}{2}$-ball property in $B(l_1, l_p)$ if $1 < p < \infty$. Since $K(E, F) = B(E, F)$ in all the remaining cases [10, Proposition 2.c.3], this completely answers the question.

For any matrix $a = (a_{ij}) \in B(l_1, l_p)$ we have

$$\|a\| = \sup_{j=1}^{\infty} \left\{ \sum_{i=1}^{\infty} |a_{ij}|^p \right\}^{1/p} \quad \text{and} \quad a \in K(l_1, l_p) \Leftrightarrow \lim_{n\to\infty} \sup_{j=1}^{\infty} \sum_{i=n}^{\infty} |a_{ij}|^p = 0 .$$

Choose λ so that $1 < \lambda^p < 2^p-1$ and put $a_{1j} = \lambda$ for all j, $a_{jj} = 1$ for $j \ne 1$, and $a_{ij} = 0$ for all other (i, j). It is easy to verify that $\|a\| < 2$ and that $K(l_1, l_p) \cap B(a, 1) \ne \emptyset$. However $K(l_1, l_p) \cap B(0, 1) \cap B(a, 1) = \emptyset$. To see this, let $x \in K(l_1, l_p) \cap B(a, 1)$. Then $x_{jj} \to 0$ as $j \to \infty$, and

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$$|\lambda - x_{1j}|^p + |1 - x_{jj}|^p \le \sum_{i=1}^{\infty} |a_{ij} - x_{ij}|^p \le 1 \quad \text{for all} \quad j \; .$$

Thus $x_{1j} \to \lambda$, so $\|x\| \ge \lambda > 1$.

We finish by considering subspaces with the n-ball property in Banach algebras. It is known [13, Theorem 5.3] that the M-ideals in a C^* algebra are precisely the closed two-sided ideals. We give a short proof of this fact. For elementary C^* algebra theory, we refer the reader to [4, Chapter 5].

LEMMA 2.9. *Let J be an M-summand in a unital C^* algebra A. Then J is an ideal in A.*

Proof. Let $Q = I - P$, where P is the M-projection onto J . We first note that if $f \in A^*$ is positive, then so are P^*f and Q^*f . For

$$\begin{aligned} |(P^*f)(1)| + |(Q^*f)(1)| &\le \|P^*f\| + \|Q^*f\| = \|f\| \\ &= f(1) = (P^*f)(1) + (Q^*f)(1) \; . \end{aligned}$$

Hence $(P^*f)(1) = \|P^*f\|$ and $(Q^*f)(1) = \|Q^*f\|$.

Now let $p = P(1)$. If $f \in A^*$ is positive, then $f(p) = (P^*f)(1) \ge 0$. Hence p is positve. We show that $ap^{\frac{1}{2}} \in J$ for all $a \in A$.

Let $f \in A^*$ be positive. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left|f\left(Q\left(ap^{\frac{1}{2}}\right)\right)\right|^2 = \left|(Q^*f)\left(ap^{\frac{1}{2}}\right)\right|^2 &\le (Q^*f)(aa^*)(Q^*f)\left(p^{\frac{1}{2}}p^{\frac{1}{2}}\right) \\ &= 0 \; , \end{aligned}$$

since $(Q^*f)(p) = f(Qp) = 0$. Thus $Q\left(ap^{\frac{1}{2}}\right)$ lies in the kernel of every positive functional on A . It follows that $Q\left(ap^{\frac{1}{2}}\right) = 0$, so $ap^{\frac{1}{2}} \in J$.

Thus $ap \in J = P(A)$ for all $a \in A$. Similarly $a(1-p) \in Q(A)$ for all a . It follows that $Pa = ap$ for all a , so $J = P(A) = Ap$ is a left ideal. A similar argument shows that J is a right ideal. //

PROPOSITION 2.10. *Let A be a C^* algebra, J a closed subspace of A. Then J is an M-ideal if and only if J is an ideal.*

Proof (ONLY IF). If J^0 is an L-summand in A^* , then J^{00} is an

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M-summand in the unital C^* algebra A^{**}. By Lemma 2.9, J^{00} is an ideal in A^{**}. Hence $J = J^{00} \cap A$ is an ideal in A.

(IF) If J is an ideal in A, then J^{00} is a weak* closed ideal in the W^* algebra A^{**}. Thus $J^{00} = A^{**}p$ for some central projection p. Straightforward calculations show that $A^{**} = J^{00} \oplus A^{**}(1-p)$, and that the two subspaces are weak* closed complementary M-summands. Taking polars, we deduce that J^0 is an L-summand in A^*. //

It is natural to ask to what extent the previous result can be generalized to Banach algebras. Smith and Ward [*13*, Theorem 3.8] showed that in a commutative, unital Banach algebra, every M-ideal is an ideal. By showing that $K(l_1)$ fails the 2-ball property in $B(l_1)$, they gave a non-commutative counterexample to the converse problem. Commutative examples are easily obtained by giving a suitable Banach space the zero product, then adjoining an identity. We give a less trivial counter-example.

Let A be the disc algebra [*4*, p. 6] and take $J = \{f \in A : f(0) = 0\}$. Clearly J is an ideal in A. Using the maximum modulus principle, it is easily shown that $P_J(f) = \{f-f(0)\}$, for all $f \in A$. Consideration of the balls $B(0, 2)$ and $B(f, 1)$, where $f(z) = z^2 + 2z - 1$, shows that J fails the $1\frac{1}{2}$-ball property.

In fact, the disc algebra even contains a non-proximinal ideal. This time, take $J = \{f \in A : f(0) = f(1) = 0\}$. Obviously J is an ideal in A. Let $f(z) = 1 - z$. For any $g \in J$ we have, by the maximum modulus principle, $\|f-g\| > |f(0)-g(0)| = 1$. Fix $\varepsilon > 0$, and let $g(z) = z(z-1)/(1+\varepsilon-z)$. Then $g \in J$ and $\|f-g\| = (1+\varepsilon)/\big(1+(\varepsilon/2)\big)$. Thus $d(f, J) = 1$, but $P(f) = J \cap B(f, 1)$ is empty.

Smith and Ward [*13*, Theorem 3.6] also showed that every M-ideal in a unital Banach algebra is a subalgebra. This is not so for subspaces with the $1\frac{1}{2}$-ball property, even in commutative Banach algebras. Let $\mathbb{T}$ denote the circle group, and let $S = \{z \in \mathbb{T} : 0 < \arg z < \pi\}$. With convolution as multiplication, $L_1(\mathbb{T})$ is a commutative Banach algebra. Now $M = \{f \in L_1(\mathbb{T}) : f|S = 0\}$ is an L-summand, and so has the $1\frac{1}{2}$-ball

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property in $L_1(\Pi)$. If $a \in M$ is defined by $a(S) = \{0\}$ and $a(\Pi \backslash S) = \{1\}$ then $a^2 \notin M$. Thus M is not a subalgebra. Although $L_1(\Pi)$ is not a unital Banach algebra, a unital example is easily obtained via the adjunction of an identity.

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