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On the Regularity of the *s*-Differential Metric

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Abstract. We show that the injective Kobayashi–Royden differential metric, as defined by Hahn, is upper semicontinous.

1 Introduction

Let M be a connected complex manifold and TM its holomorphic tangent bundle. A *differential metric* on M is a function $f_M\colon TM \to \mathbb{R}$ satisfying the following conditions:

(i) $f_M(X_p) \geq 0$,
(ii) $f_M(aX_p) = |a| f_M(X_p), \ \forall X_p \in T_pM, \ \forall a \in \mathbb{C}$.

If in addition f_M is upper semicontinuous on TM, then we call f_M a *Finsler-type metric*. A Finsler-type metric is called a *Finsler metric* if it satisfies the convexity condition

(iii) $f_M(X_p + Y_P) \leq f_M(X_p) + f_M(Y_p), \ \forall X_p, Y_p \in T_pM$.

For example, if h is a Hermitian metric on a complex manifold M, then the function $\tilde{h}\colon TM \to \mathbb{R}_{\geq 0}$ given by

$$\tilde{h}(X_x) := h(X_x, X_x)^{1/2}, \quad \forall X_x \in T_xM,$$

is a Finsler metric.

The indicatrix (at p) of a differential metric f_M is the set

$$I_{f_M}(p) = \{X_p \in T_pM : f_M(X_p) < 1\}.$$

If we consider the integrated form of a Finsler-type metric f, *i.e.*, the function $F\colon M \times$

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$M \to \mathbb{R}_{\geq 0}$ defined by

$$F(x, y) = \inf_{\gamma} \left\{ \int_0^1 f(\gamma'(t))\, dt \right\},$$

where the infimum is taken with respect to set of piecewise differentiable curves $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, then F is a *pseudo-distance*.

On a complex manifold M the Kobayashi–Royden metric is defined by

$$k_M(X_p) := \inf\{a > 0 : \exists \varphi \in \mathcal{H}(\mathbb{D}, M) \ni \varphi(0) = p \text{ and } \varphi'(a(\partial/\partial z)_0) = X_p\},$$

where $\mathcal{H}(\mathbb{D}, M)$ is the set of holomorphic mappings of the unit disc $\mathbb{D}$ to M. Royden showed that this metric is a Finsler-type metric [6]. A complex manifold M is called K-hyperbolic if the pseudo-distance K induced by k_M is a distance. Equivalently, M is K-hyperbolic if for every Hermitian metric h on TM and for every $p \in M$ there is a neighbourhood U of p and a constant $C > 0$, such that $k_M(X_q) \geq Ch(X_q)$, for all $X_q \in T_qM$ with $q \in U$. Kobayashi [4] originally defined the pseudo-distance K by considering the analytic chains of holomorphic mappings from the unit disc $\mathbb{D}$ to M. K-hyperbolic manifolds form a large class, *e.g.*, every compact Riemann surface of genus ≥ 2 and every bounded domain in $\mathbb{C}^n$ is K-hyperbolic.

Let M and N be complex manifolds and let $f\colon M \to N$ be holomorphic. Then, for each $X_p \in TM$, we have

$$k_N(f'(p)X_p) \leq k_M(X_p).$$

In particular, if f is biholomorphic, then $k_N(f'(p)X_p) = k_M(X_p)$.

K. T. Hahn considered the family $\mathcal{I}(\mathbb{D}, M)$ of injective holomorphic mappings of the unit disk $\mathbb{D}$ into M and, analogous to k_M, he defined the differential metric s_M on a complex manifold M [1]; S-hyperbolicity is defined similarly. This differential metric is invariant under biholomorphic maps, and

$$k_M(X_p) \leq s_M(X_p), \quad \forall X_p \in T_pM.$$

Vesentini showed that the domain $\mathbb{C}^* \times \Omega$ is not S-hyperbolic if Ω is a domain of dimension two or larger [8]. Moreover, Vigué proved that $G_1 \times G_2$ ($G_1, G_2 \subset \mathbb{C}$) is S-hyperbolic if and only if G_1 and G_2 are K-hyperbolic [9]. Finally, Overholt showed that $s_G = k_G$ on every domain $G \subset \mathbb{C}^n$, $n \geq 3$ [5].

Despite many similarities between s_M and k_M, they behave differently on certain domains. For example $\mathbb{C}^*$ is not K-hyperbolic, but Hahn proved that it is S-hyperbolic [1], which means $S_{\mathbb{C}^*} \neq k_{\mathbb{C}^*}$.

The validity of $s_G = k_G$ on domains $G \subset \mathbb{C}^2$ is still an *open* problem. If equality holds s_G would be upper semicontinuous. In this paper we show that s_G is upper semicontinuous on the tangent bundle of each domain G in $\mathbb{C}^n$, $n \geq 1$. This is a positive answer to the question raised in [7].

2 Regularity

The following result is a step forward toward proving that $s_G = k_G$ on domains $G \subset \mathbb{C}^2$.

Theorem 2.1 *Let $n \geq 1$ and $G \subset \mathbb{C}^n$ be a domain. Then the differential metric s_G on G is upper semicontinuous.*

Proof Without loss of generality, instead of $\mathcal{I}(\mathbb{D}, M)$ in the definition of s_G-differential metric, we will consider $\mathcal{I}(\overline{\mathbb{D}}, M)$. For $n = 1$, by applying condition (i) we can ignore the tangent vector ξ. In this case it is sufficient to prove the function $s_G\colon G \to \mathbb{R}_{\geq 0}$, defined by

$$s_G(z) = \inf\Big\{ \frac{1}{|\varphi'(0)|} : \varphi \in \mathcal{I}(\overline{\mathbb{D}}, G), \varphi(0) = z \Big\},$$

is upper semicontinuous. For $z^0 \in G$, let $s_G(z^0) < A$, then there exists $\varphi \in \mathcal{I}(\overline{\mathbb{D}}, G)$ with $\varphi(0) = z^0$ and $1/|\varphi'(0)| < A$. Since $\varphi \in \mathcal{I}(\overline{\mathbb{D}}, G)$, an ϵ-neighbourhood of $\varphi(\overline{\mathbb{D}})$ remains inside G. For $z \in G$ such that $|z - z^0| < \epsilon/2$, we consider the function $\psi : \mathbb{D} \to \mathbb{C}$, defined by

$$\psi(\xi) := \varphi(\xi) + (z - z^0).$$

The function ψ is injective, its image is inside G, $\psi(0) = z$ and $\psi'(0) = \varphi'(0)$, which implies that

$$s_G(z) \leq \frac{1}{|\psi'(0)|} < A,$$

and shows that s_G is upper semicontinuous at z^0.

As we mentioned, Overholt [5] proved that for $n \geq 3$ the Kobayashi–Royden differential metric k_G coincides with s_G and since k_G is upper semicontinuous [6]. It remains to consider $n = 2$. However our proof is different from Royden's proof and works for $n \geq 2$.

Let $z^0 \in G, 0 \neq X^0 \in \mathbb{C}^n, \alpha > 0$ and let $\varphi\colon \overline{\mathbb{D}} \to G$ be an injective holomorphic mapping with $\varphi(0) = z^0$, $\alpha\varphi'(0) = X^0$. Also, let (z_n, X_n) be a sequence in $G \times \mathbb{C}^n$ which converges to (z^0, X^0). We can choose $v_2, \dots, v_n \in \mathbb{C}^n$ such that $\{X^0, v_2, \dots, v_n\}$ is a basis of $\mathbb{C}^n$. For sufficiently large m so that $\{X_m, v_2, \dots, v_n\}$ is still a basis of $\mathbb{C}^n$, we define the mapping $\Phi_{(z_m,X_m)}\colon \mathbb{C}^n \to \mathbb{C}^n$ by,

$$\Phi_{(z_m,X_m)}(z^0 + \zeta_1 X^0 + \zeta_2 v_2 + \cdots + \zeta_n v_n) := z_m + \zeta_1 X_m + \zeta_2 v_2 + \cdots + \zeta_n v_n.$$

The mapping $\Phi_{(z_m,X_m)}$ is biholomorphic and converges uniformly to $\mathrm{id}_{\mathbb{C}^n}$ when $m \to \infty$. Moreover,

$$\Phi_{(z_m,X_m)}(z^0) = z_m \quad \text{and} \quad \Phi'_{(z_m,X_m)}(z^0)(X^0) = X_m.$$

If we define $\varphi_{(z_m,X_m)} := \Phi_{(z_m,X_m)} \circ \varphi$, then $\varphi_{(z_m,X_m)}$ is an injective holomorphic mapping,

$$\varphi_{(z_m,X_m)}(0) = \Phi_{(z_m,X_m)}(z_0) = z_m,$$

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and

$$\alpha\varphi'_{(z_m,X_m)}(0) = \alpha\Phi'_{(z_m,X_m)}(z^0)\varphi'(0) = X_m.$$

Since $\Phi_{(z_m,X_m)}$ converges uniformly to $\mathrm{id}_{\mathbb{C}^n}$, for sufficiently large m, the mapping $\varphi_{(z_m,X_m)}$ maps $\overline{\mathbb{D}}$ to G. This shows that

$$\limsup_{n\to\infty} s_G(z_n, X_n) \le s_G(z^0, X^0).$$

Hence, s_G is upper semicontinouos on $G \times \big(\mathbb{C}^n \setminus \{0\}\big)$.

Let $\overline{\mathbb{B}}_r(z^0) \subset G$ where $\mathbb{B}_r(z^0)$ denotes the Euclidean ball with center z^0 and radius r, and let

$$K := \max_{\overline{\mathbb{B}}_r(z^0)\times\partial\mathbb{B}_1(0)} s_G.$$

Since s_G is upper semicontinuous on $G \times \big(\mathbb{C}^n \setminus \{0\}\big)$, K is finite. It follows that $s_G(z, X) \le \varepsilon K$ for $(z, X) \in \mathbb{B}_r(z^0) \times \mathbb{B}_\varepsilon(0)$, which shows that S_G is continuous at $(z^0, 0)$. ■

3 The Metric $\hat{s}_M$

Let M be a complex manifold and let

$$\hat{s}_M(\zeta) := \inf\{t > 0 : t^{-1}\zeta \in \hat{I}_{s_M}(p)\}, \quad \forall\zeta \in T_pM,$$

where $\hat{I}_{s_M}(p)$ is the convex hull of $I_{s_M}(p)$. The following result shows that $\hat{s}_M$ behaves better than s_M on a complex manifold M.

Theorem 3.1 *Let M be a complex manifold. Then $\hat{s}_M$ is a differential metric and satisfies the convexity condition* (iii). *In particular, when $M \subset \mathbb{C}^n$ is a domain, then $\hat{s}_M$ is a Finsler metric on TM.*

Proof Following Kobayashi [3], we define a function s^*_M on the cotangent space T^*_pM. We set

$$s^*_M(\lambda) := \sup \|f^*\lambda\|, \quad \forall\lambda \in T^*_pM,$$

where supremum is taken over all $f \in \mathcal{I}(\mathbb{D}, M)$ with $f(0) = p$ and

$$\|f^*\lambda\| = \sup\{|(f^*\lambda)(\zeta)| : \zeta \in T_0\mathbb{D}, \|\zeta\| < 1\},$$

and where $\|\zeta\|$ denotes the Poincaré norm of $\zeta \in T_0\mathbb{D}$. We have

$$s^*_M(a\lambda) = |a|s^*_M(a) \quad \forall\lambda \in T^*_pM,\ \forall a \in \mathbb{C},$$

$$s^*_M(\lambda + \mu) \le s^*_M(\lambda) + s^*_M(\mu), \quad \forall\lambda, \mu \in T^*_pM.$$

This means that s^*_M is a semi-norm on T^*_pM. Dual to s^*_M, we consider

$$\hat{s}_M(\zeta) = \sup_{\lambda\in I_{s^*_M(p)}} |\lambda(\zeta)|, \quad \forall\zeta \in T_pM,$$

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where,

$$I_{s_M^*(p)} = \{\lambda \in T_p^*M : s_M^*(\lambda) < 1\}.$$

Let $\mathcal{I}_p(\mathbb{D}, M)$ denote the subset of $\mathcal{I}(\mathbb{D}, M)$ consisting of mappings f with $f(0) = p$, then we have,

$$I_{s_M}(p) = \{f_*\zeta : \zeta \in T_0\mathbb{D}, \|\zeta\| < 1, f \in \mathcal{I}_p(\mathbb{D}, M)\}.$$

Therefore, for each $\lambda \in T_p^*M$,

$$s_M^*(\lambda) = \sup_f \|f^*\lambda\| = \sup_{f,\ \|\zeta\|<1} |\lambda(f_*(\zeta))| = \sup_{s_M(\xi)<1} |\lambda(\xi)|.$$

Hence the first part of the proof is complete. By Theorem 2.1, when $M \subset \mathbb{C}^n$ is a domain, s_M is upper semicontinuous, and by the same technique as [2, Proposition 3.6.2] we can complete the proof of second part. ■

Now, the two pseudo-distances S_G and $\hat{S}_G$ induced by s_G and $\hat{s}_G$, respectively, can be considered on G. Since s_G is upper semicontinuous, for any compact subset K and any domain G of $\mathbb{C}^n$, there exists a constant $C > 0$ such that

$$s_G(z, X) \leq C||X|| \quad \forall z \in K, \quad \forall X \in \mathbb{C}^n.$$

Thus by the same argument as [2, Theorem 3.6.4], the following theorem can be proved.

Theorem 3.2 *Let $G \subset \mathbb{C}^n$ be a domain. The pseudo-distance S_G coincides with $\hat{S}_G$.*

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