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First Ruelle resonance for an Anosov flow with smooth potential

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Abstract. We combine methods from microlocal analysis and dimension theory to study resonances with largest real part for an Anosov flow with smooth real valued potential. We show that the resonant states are closely related to special systems of measures supported on the stable manifolds introduced by Climenhaga [SRB and equilibrium measures via dimension theory. *A Vision for Dynamics in the 21st Century: The Legacy of Anatole Katok*. Cambridge University Press, Cambridge, 2024, pp. 94–138]. As a result, we relate the presence of the resonances on the critical axis to mixing properties of the flow with respect to certain equilibrium measures and show that these equilibrium measures can be reconstructed from the spectral theory of the Anosov flow.

Key words: Ruelle resonances, first resonance, anisotropic spaces, leaf measures
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1. *Introduction*

Let $(\mathcal{M}, g)$ be a smooth closed connected Riemannian manifold of dimension $n \geq 3$. We consider a smooth flow φ_t on $\mathcal{M}$ and its generating vector field

$$X(x) := \frac{d}{dt}\varphi_t(x)|_{t=0}, \quad x \in \mathcal{M}.$$

Assumption 1. We suppose that the flow is *Anosov*, topologically transitive, and that the stable and unstable bundles are orientable and of dimension d_s and d_u, respectively. Furthermore, we consider a smooth, real valued potential V.

1.1. *Leading resonant state.* We study the operator $\mathbf{P} := -X + V$ acting on specially designed *anisotropic Sobolev spaces*. The sets of eigenvalues of $\mathbf{P}$ on these spaces are called the *Ruelle resonances*. Their set, which we will denote by Res, is intrinsic to the Anosov flow and contains valuable dynamical meaning. Its understanding is essential to estimate the speed of decay of correlations, see for instance [**29**, **37**]. More concretely, Ruelle resonances are defined by (see for instance [**16**, Lemma 5.1])

$$\lambda \in \mathrm{Res} \iff \text{there exists } u \in \mathcal{D}'(\mathcal{M}) \setminus \{0\}, \quad \mathrm{WF}(u) \subset E_u^*, \quad (\mathbf{P} - \lambda)u = 0. \tag{1}$$

Here, $\mathrm{WF}(u)$ denotes the wavefront set of the distribution u and E_u^* is the dual counterpart of the unstable bundle (see (20) for a precise definition).

It is actually useful to study the operator $\mathbf{P}$ not only on functions but more generally on the space of k-forms in the kernel of the contraction by the flow:

$$\begin{cases} \mathbf{P} : \mathscr{E}_0^k := \{u \in C^\infty(\mathcal{M}; \Lambda^k T^*\mathcal{M}) \mid \iota_X u = 0\} \to \mathscr{E}_0^k, \\ \mathbf{P}\omega := (-\mathcal{L}_X + V)\omega = -\dfrac{d}{dt}\varphi_t^*\omega|_{t=0} + V\omega. \end{cases}$$

We can also define resonances for k-forms by the equivalence (1) and we denote their set by Res_k. (In this case, u should be a k-current, more precisely, in the dual of $\mathscr{E}_0^k$.) The decay rate of correlations is dictated by the resonances with large real part and hence of special interest is the study of resonances on the *critical axis*:

$$C_k := \{\lambda \in \mathrm{Res}_k \mid \mathrm{Re}(\lambda) = \sup_{\mu \in \mathrm{Res}_k} \mathrm{Re}(\mu)\}. \tag{2}$$

Starting from the relation

$$\text{for all } \omega \in \mathscr{E}_0^k, \text{ for all } \lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \gg 1, \quad (\mathbf{P}_{|\mathscr{E}_0^k} - \lambda)^{-1}\omega = \int_0^\infty e^{t(\mathbf{P}-\lambda)}\omega\, dt,$$

we see that the position of C_k should be linked to the exponential growth of the norm of the propagator $e^{t\mathbf{P}}$ on relevant functional spaces (the so-called anisotropic Sobolev spaces that will be introduced in §2.3). A form in $\mathscr{E}_0^k$ is a linear combination of k-wegdes of elements of E_u^* and E_s^* (see (12) for the exact definition of these bundles). However, an element in E_s^* is contracted exponentially fast while an element of E_u^* is expanded exponentially fast by the Anosov property. This means that to maximize the exponential growth of the norm, one should study the resolvent on d_s-forms.

Moreover, the exact location of C_0 (respectively C_{d_s}) is given by the $P(V + J^u)$ (respectively $P(V)$), where P denotes the topological pressure and $J^u := -(d/dt)\det(d\varphi_t(x)_{|E_u(x)})|_{t=0}$ is the unstable Jacobian. The corresponding eigenvectors (referred to as *resonant states*) also bear dynamical significance. More precisely, the resonant states at the first resonance $P(V + J^u)$ (respectively $P(V)$) are linked to the system of *leaf measures* $m^s_{V+J^u}$ (respectively m^s_V). Leaf measures are systems of reference measures on stable and unstable leaves which are used to obtain the equilibrium state via a product construction. Their introduction goes back to Sinai [36] for maps and Margulis [30] for flows (when $V = 0$). The measure of maximal entropy was obtained using leaf measures by Hamenstädt [22] for geodesic flows in negative curvature and by Hasselblatt [24] for Anosov flows, see also [23] for an extension to non-zero potentials. Recently, Climenhaga, Pesin, and Zelerowicz [10–12] gave a new construction of leaf measures using dimension theory. Their construction extends to certain classes of partially hyperbolic flows, see also the related works of Carrasco and Rodriguez-Hertz [7, 8].

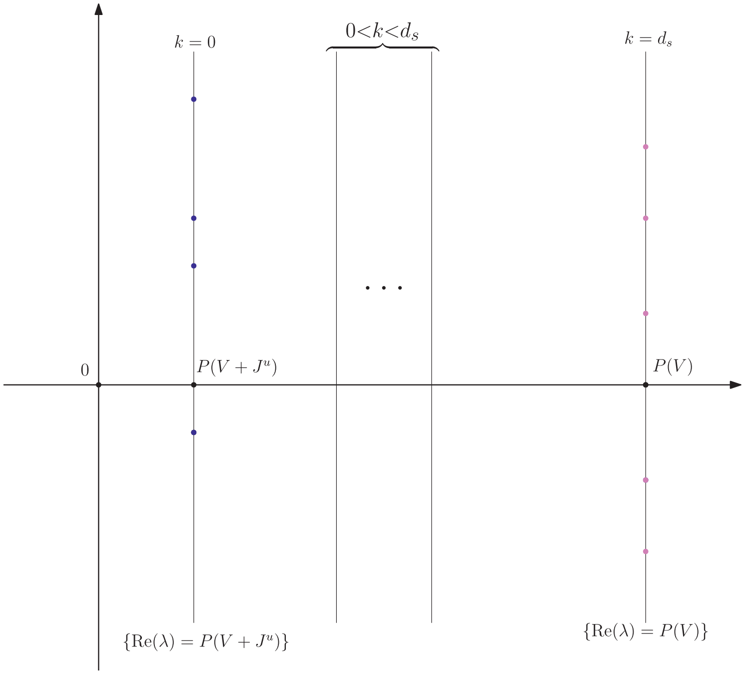

FIGURE 1. Critical axes for different values of k. According to Theorem 1.1, the resonances in purple cannot exist if the flow is weakly mixing with respect to μ_V and the resonances in blue cannot exist if the flow is weakly mixing with respect to μ_{V+J^u}. The position of the critical axes for intermediate values of k should be linked to the pressure on the span of largest Lyapunov exponents.

THEOREM 1.1. *Under Assumption 1, the critical axes for the action on* 0*-forms and* d_s*-forms are given by*

$$C_0 = \{\lambda \mid \mathrm{Re}(\lambda) = P(V + J^u)\}, \quad C_{d_s} = \{\lambda \mid \mathrm{Re}(\lambda) = P(V)\}.$$

Moreover, $P(V + J^u)$ *(respectively* $P(V)$*) is a resonance called the first resonance for the action on* 0*-forms (respectively* d_s*-forms).*

There is $\delta > 0$ *such that for any* $k \neq d_s$*, we have* $C_k \subset \{\lambda \mid \mathrm{Re}(\lambda) \leq P(V) - \delta\}$*, i.e., all other critical axes are to the left of* C_{d_s}*, see Figure 1. (For* $1 \leq k < l \leq d_s$*, the proof actually gives that* C_l *is to the right of* C_k*.)*

Moreover, all $\lambda \in C_0$ *(respectively* $\lambda \in C_{d_s}$*) have no Jordan block and the first resonance* $P(V + J^u)$ *(respectively* $P(V)$*) is simple:*

$$\{u \in \mathcal{D}'(\mathcal{M}) \mid (\mathbf{P} - P(V + J^u))u = 0, \mathrm{WF}(u) \subset E_u^*\} = \mathrm{Span}(\eta), \tag{3}$$

where η *is a measure constructed in Theorem 3.1 from the system of leaf measures* $m^s_{V+J^u}$*.*

$$\{u \in \mathcal{D}'(\mathcal{M}; \Lambda^{d_s}(E_u^* \oplus E_s^*)) \mid (\mathbf{P} - P(V))u = 0, \mathrm{WF}(u) \subset E_u^*\} = \mathrm{Span}(m^s_V). \tag{4}$$

We note that for hyperbolic maps, similar results were already obtained by various authors, using the formalism of anisotropic Banach spaces. The study for the action of 0-forms can be found in [2, Theorem 7.5] and for d_s-forms, it can be found in

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[21, Theorem 5.1]. We also remark that Adam and Baladi used anisotropic techniques to study the first resonant state for a contact Anosov flows in dimension 3 for the potential $V = -J^u$ in [1]. (Because the unstable Jacobian is not smooth, this is actually, strictly speaking, out of the range of Theorem 1.1.)

1.2. *Equilibrium states.* The leaf measures constructed in [10–12] can be used to reconstruct the *equilibrium state* (i.e., the unique invariant probability measure that maximizes the variational principle recalled in (17)). In our context, this means that the equilibrium state can be reconstructed from the spectral theory of the Anosov vector field X. Define the divergence of X by the relation $\mathcal{L}_X \mathrm{vol} = \mathrm{div}_{\mathrm{vol}}(X)\mathrm{vol}$ for the Riemannian volume vol. Then, the L^2-adjoint of $\mathbf{P}$ acting on 0-forms is $\mathbf{P}^* = X + V + \mathrm{div}_{\mathrm{vol}}(X)$. Applying Theorem 1.1 to the adjoint gives two co-resonant states ν and m_V^u, and one has

$$\begin{cases} (-X + V - P(V + J^u))\eta = 0, \quad \mathrm{WF}(\eta) \subset E_u^*, \\ (X + V + \mathrm{div}_{\mathrm{vol}}(X) - P(V + J^u))\nu = 0, \quad \mathrm{WF}(\nu) \subset E_s^*, \end{cases} \tag{5}$$

as well as

$$\begin{cases} (-\mathcal{L}_X + V - P(V))m_V^s = 0, \quad \mathrm{WF}(m_V^s) \subset E_u^*, \\ (\mathcal{L}_X + V - P(V))m_V^u = 0, \quad \mathrm{WF}(m_V^u) \subset E_s^*. \end{cases} \tag{6}$$

The wavefront set bounds allow us to take the distributional pairing of the resonant and co-resonant states η and ν (respectively m_V^s and m_V^u). The resulting distribution is easily seen to be measure invariant by the flow. The next theorem asserts that this measure is actually the equilibrium state for the potential $V + J^u$ (respectively V). As a consequence, the presence of other resonances on C_0 (respectively C_{d_s}) is linked to mixing properties of the flow. In the rest of the paper, we denote by μ_W the equilibrium measure associated to the Hölder continuous potential W.

THEOREM 1.2. *Under Assumption 1, one has*

$$\begin{cases} \text{there exists } c > 0, \ \mu_{V+J^u} = c\eta \times \nu, \\ \text{there exists } c' > 0, \ \mu_V = c' m_V^s \wedge \alpha \wedge m_V^u, \ \alpha(X) = 1, \ \alpha(E_u \oplus E_s) = 0. \end{cases} \tag{7}$$

Let χ be a cutoff $\chi \in C_c^\infty([0, T + \epsilon[, [0, 1])$ such that $\chi \equiv 1$ on $[0, T]$ for any $\epsilon > 0$ and for $T > 0$ large enough. Then, one has the following. (The notation f^{*k} denotes the kth convolution product of a function f with itself.)

- *For any $f \in C^\infty(\mathcal{M})$,*

$$-\lim_{k\to+\infty} \int_0^{+\infty} (\chi')^{*k}(t)((\varphi_t)_*\nu)(f)\,dt = c\mu_{V+J^u}(f), \quad c > 0. \tag{8}$$

- *Fix an orientation of E_u^* and let $\omega \in C^0(\mathcal{M}; \Lambda^{d_s} E_u^*)$ be a non-negative section which does not vanish identically. Then, one has, for any $f \in C^\infty(\mathcal{M})$,*

$$-\lim_{k\to+\infty} \int_0^{+\infty} (\chi')^{*k}(t)((\varphi_t)_*(\omega \wedge \alpha \wedge m_V^u))(f)\,dt = c\mu_V(f), \quad c > 0. \tag{9}$$

Finally, one has the following.

- *If the flow is weakly mixing with respect to the equilibrium state μ_{V+J^u}, then*

$$\mathrm{Res}_0 \cap C_0 = \{P(V + J^u)\}.$$

- *If the flow is weakly mixing with respect to the equilibrium state μ_V, then*

$$\mathrm{Res}_{d_s} \cap C_{d_s} = \{P(V)\}.$$

In particular, for $V = 0$, the theorem gives a construction of the Sinai–Ruelle–Bowen (SRB) measure for the action on 0-forms (respectively the measure of maximal entropy for the action on d_s-forms) from a solution of (5) (respectively (6)) for $P(J^u) = 0$ (respectively for $P(0) = \mathrm{h_{top}}(\varphi_1)$).

1.3. *Ruelle zeta function.* The resolvents acting on $\mathscr{E}_0^k$ are linked to the (weighted) *Ruelle zeta function*:

$$\zeta_V(\lambda) := \prod_{\gamma\in\Gamma^\sharp} (1 - e^{-(\lambda+V_\gamma)T_\gamma}), \quad V_\gamma := \frac{1}{T_\gamma}\int_0^{T_\gamma} V(\gamma(t))\, dt, \tag{10}$$

where $\Gamma^\sharp$ is the set of primitive geodesics and T_γ denotes the period of the closed geodesic γ. This function can be shown to be convergent and holomorphic in a half-plane $\{\mathrm{Re}(\lambda) \gg 1\}$. In a celebrated paper [20], Giulietti, Liverani, and Pollicott proved that the function ζ_R^V admits a meromorphic extension to the whole complex plane. Another proof, using microlocal analysis, was given by Dyatlov and Zworski in [15].

More precisely, let us introduce the *Poincaré* map $\mathcal{P} : \gamma \in \Gamma \mapsto \mathcal{P}_\gamma := d\varphi_{-T_\gamma}|_{E_s\oplus E_u}$. The link between the resolvents on forms and the Ruelle zeta function is given by the *Guillemin trace formula*. In particular, [15, equation (2.5)] gives

$$\zeta_V'(\lambda)/\zeta_V(\lambda) = \sum_{k=0}^{n-1}(-1)^{k+d_s} e^{-\lambda t_0}\, \mathrm{tr}^\flat(\varphi_{-t_0}^*((\mathbf{P})_{|\mathscr{E}_0^k} - \lambda)^{-1})), \tag{11}$$

where the shift by a small time t_0 is a technicality to ensure that the pullbacked resolvent $\varphi_{-t_0}^*(\mathbf{P} - \lambda)^{-1}$ satisfies the wavefront set condition which makes its flat trace well defined (see [15, §4]). This shows that the meromorphic extension of ζ_V follows from the extension of the resolvent acting on the space $\mathscr{E}_0^k$ for any k (plus some additional arguments). Moreover, we get the poles of ζ_V by studying poles of each resolvent.

The study of the first pole of the Ruelle zeta function can be found in [31, Theorem 9.2].

THEOREM. (Parry and Pollicott) *Let V be a Hölder continuous potential, and suppose that the flow is Anosov and weakly topologically mixing. Then, the Ruelle zeta function ζ_V is non-zero and analytic in the half-plane $\{Re(\lambda) \geq P(V)\}$ except for a simple pole at $\lambda = P(V)$, where $P(V)$ is the topological pressure of the potential V.*

As a consequence of our first two theorems, we recover the theorem of Parry and Pollicott on the first pole of the Ruelle zeta function for topologically mixing Anosov flows with smooth potentials.

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We note that a consequence of this corollary and a standard Tauberian argument is the following asymptotic growth:

$$\sum_{\gamma\in\Gamma, T_\gamma\le t} e^{V_\gamma T_\gamma} = \sum_{\gamma\in\Gamma, T_\gamma\le t} e^{\int_0^{T_\gamma} V(\gamma(t))\,dt} \sim \frac{e^{tP(V)}}{tP(V)}.$$

For $V = 0$, this result is known as the prime orbit theorem and can be found in [31, Theorem 9.3].

1.4. *Regularity of the pressure.* Another consequence of our first theorems is the following regularity statement of the topological pressure. It was first established by Katok *et al* in [27] and Contreras in [13].

COROLLARY 1.1. (Smoothness of the topological pressure) *Let $\mathcal{V}_{\mathrm{t}}^\infty$ denote the set of smooth transitive Anosov flows. Then, it is an open set (this point follows from* [19, Proposition 1.6.30] *which proves that topological transitivity is preserved by orbit conjugacy and the structural stability of Anosov flows, see* [19, Corollary 5.4.7]*) and the maps*

$$P_1 : (X, V) \in \mathcal{V}_{\mathrm{t}}^\infty \times C^\infty(\mathcal{M}) \mapsto P_X(V),$$
$$P_2 : (X, V) \in \mathcal{V}_{\mathrm{t}}^\infty \times C^\infty(\mathcal{M}) \mapsto P_X(V + J^u),$$

where P_X denotes the topological pressure for the flow induced by the vector field X, are smooth.

1.5. *Outline of article.*

- In §2, we recall some important features of Anosov flows and of the thermodynamical formalism. Then, we will review microlocal methods for the study of Anosov flows. Finally, we will recall the construction of leaf measures.
- In §3, we recall the definition of Ruelle resonances using a parametrix construction. Then, we define in Theorem 3.1 a co-resonant state for the action on 0-forms. This allows us to precisely locate the critical axis at $\{\mathrm{Re}(\lambda) = P(V + J^u)\}$. The rest of the section is devoted to the study of resonant states on the critical axis and more precisely to the proofs of the results announced in §1.
- In §4, we prove the equivalent results for the action on d_s-forms. The strategies of the proofs remain the same, but some additional care is needed when adapting certain arguments to this case.
- In §5, we give a proof of the first part of Theorem 1.1 and of Corollary 1.1.

2. *Preliminaries*

2.1. *Anosov flow.* Our main assumption on the flow is that it is *Anosov*.

Definition 2.1. (Anosov) The flow φ_t is Anosov (or uniformly hyperbolic) if:

- there is a continuous splitting of the tangent space

$$T_x\mathcal{M} = E_u(x) \oplus E_s(x) \oplus \mathbb{R}X(x); \tag{12}$$

- the decomposition is flow-invariant, meaning that

$$\text{for all } t \in \mathbb{R}, \quad E_u(\varphi_t(x)) = (d\varphi_t)_x(E_u(x)), \quad E_s(\varphi_t(x)) = (d\varphi_t)_x(E_s(x)); \tag{13}$$

- there are uniform constants $C > 0$ and $\theta > 0$ such that for every $x \in M$, we have

$$|(d\varphi_t)_x(v_s)|_g \leq Ce^{-\theta t}|v_s|_g \quad \text{for all } t \geq 0, \text{ for all } v_s \in E_s(x).$$

$$|(d\varphi_t)_x(v_u)|_g \leq Ce^{-\theta|t|}|v_u|_g \quad \text{for all } t \leq 0, \text{ for all } v_u \in E_u(x).$$

We will denote by d_u and d_s the dimensions of E_u and E_s, respectively.

For a comprehensive introduction to the theory of Anosov flows, we refer to [19, Ch. 8]. An important class of examples is given by geodesic flows on the unit tangent bundle $\mathcal{M} = SM$ of a negatively curved closed Riemannian manifold M. We recall the stable manifold theorem, see for instance [25, Theorem 6.4.9].

For all $x \in \mathcal{M}$, there exists immersed submanifolds

$$\mathcal{W}^{s,u}(x) := \{y \in \mathcal{M} \mid d(\varphi_t(x), \varphi_t(y)) \to_{t\to\pm\infty} 0\}, \tag{14}$$

where $+$ (respectively $-$) corresponds to s (respectively u), called the (strong) stable (respectively unstable) manifolds, such that $T_x\mathcal{W}^{s,u} = E_{s,u}$. Moreover, $x \mapsto \mathcal{W}^{s,u}(x)$ are (Hölder continuous) foliations of $\mathcal{M}$. We also define the weak stable and unstable manifolds

$$\begin{aligned} \mathcal{W}^{ws,wu} &:= \{y \in \mathcal{M} \mid \text{there exists } t_0 \in \mathbb{R}, d(\varphi_t(x), \varphi_{t+t_0}(y)) \to_{t\to\pm\infty} 0\} \\ &= \bigcup_{t\in\mathbb{R}} \varphi_t(\mathcal{W}^{s,u}(x)), \end{aligned} \tag{15}$$

their tangent spaces are given respectively by $\mathbb{R}X \oplus E_s$ and $\mathbb{R}X \oplus E_u$.

A consequence of the existence of these (un)stable manifolds is the *local product structure*, see [25, Proposition 6.4.13].

For any $x_0 \in \mathcal{M}$, there exists a neighborhood V of x_0 such that for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\text{for all } x, y \in V, \quad d(x, y) \leq \delta \Rightarrow \text{there exists} |t| \leq \epsilon, \quad \mathcal{W}^u_\epsilon(\varphi_t(x)) \cap \mathcal{W}^s_\epsilon(y) =: \{[x, y]\},$$

where we denoted by $\mathcal{N}_\epsilon$ the ϵ-ball of the manifold $\mathcal{N}$. The point $[x, y]$ is called the Bowen bracket of x and y, and $t(x, y)$ is the Bowen time. For $q \in \mathcal{M}$, we define a local rectangle to be

$$R_q := \{\Psi_q(x, y) := [x, y] \mid x \in \mathcal{W}^u_\delta(q), \ y \in \mathcal{W}^{ws}_\delta(q)\}. \tag{16}$$

2.2. *Thermodynamical formalism.* We recall here the main features from the thermodynamical formalism we will need. The two main objects are the *topological pressure* and the *equilibrium state*. For an introduction to the thermodynamical formalism, we refer to [19, Ch. 4]. Consider a Hölder continuous and real valued potential V.

We first recall the *variational principle* (see [19, Theorem 9.3.4]), which we will state in the case of smooth flows.

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Let $(\mathcal{M}, g)$ be a closed Riemannian manifold and φ_t be a smooth flow on $\mathcal{M}$, and let $V : \mathcal{M} \to \mathbb{R}$ be a Hölder continuous potential, then

$$P(\varphi_1, V) := \sup_{\mu \in \mathfrak{M}(\varphi)} \left(h_\mu(\varphi_1) + \int_{\mathcal{M}} V \, d\mu \right), \tag{17}$$

where h_μ is the metric entropy and $\mathfrak{M}(\varphi)$ is the set of invariant-probability Borel measures, and $P(\varphi_1, V)$ is the *topological pressure* associated to V.

Now, we can define an *equilibrium state* as a measure that achieves the supremum, where the existence and uniqueness of such a measure can be obtained under Assumption 1, and we will use the following result (see [**19**, Theorem 7.3.6] and [**6**, Theorem 3.3]).

PROPOSITION 2.1. (Existence and uniqueness of equilibrium states) *Under Assumption 1, the equilibrium state associated to V is unique, ergodic, and has full support. If the flow is topologically mixing, then the flow is weak mixing with respect to the equilibrium state μ_V.*

In the case of an Anosov flow, an equivalent characterization of equilibrium state is given by the *Gibbs property* (see [**19**, Theorem 4.3.26]). Indeed, μ is the equilibrium state for V if and only if

$$\text{for all } \delta > 0, \text{ there exists } C > 0 \text{ for all } t > 0, \text{ for all } q \in \mathcal{M},$$
$$C^{-1} \le \mu(B_t(q, \delta)) e^{tP(V) - S_t V(q)} \le C. \tag{18}$$

Here, $B_t(q, \delta)$ denotes the *Bowen ball* defined in (28) and $S_t V(q) := \int_0^t V(\varphi_s q)\, ds$.

We define a special potential called the *unstable Jacobian* by the following formula:

$$J^u(x) := -\frac{d}{dt} \det(d\varphi_t(x)_{|E_u(x)})|_{t=0} =: -\frac{d}{dt} j_t(x)|_{t=0} = -\frac{d}{dt} \ln j_t(x)|_{t=0}, \tag{19}$$

where the determinant is taken with respect to the Riemannian measure vol.

The *equilibrium state* associated to the unstable Jacobian is the SRB measure whose pressure vanishes: $P(J^u) = 0$, see [**19**, Corollary 7.4.5].

2.3. *Anisotropic spaces.* To a given decomposition (12), we can associate a corresponding splitting of the cotangent space. This will be more natural as we will use microlocal analysis. For an introduction to microlocal analysis, we refer the reader to [**39**].

There is a continuous splitting $T_x^* M = E_u^*(x) \oplus E_s^*(x) \oplus E_0^*(x)$, defined by

$$E_s^*(x)(E_s(x) \oplus \mathbb{R}X(x)) = 0, \quad E_u^*(x)(E_u(x) \oplus \mathbb{R}X(x)) = 0, \quad E_0^*(x)(E_s(x) \oplus E_u(x)) = 0. \tag{20}$$

Moreover, this decomposition is flow-invariant and there exists constants $C, \theta > 0$ such that, uniformly in $x \in M$, we have

$$|(d\varphi_{-t})_x^T(\xi_s)| \le Ce^{-\theta t}|\xi_s| \quad \text{for all } t \ge 0, \text{ for all } \xi_s \in E_s^*(x),$$
$$|(d\varphi_{-t})_x^T(\xi_u)| \le Ce^{-\theta |t|}|\xi_u| \quad \text{for all } t \le 0, \text{ for all } \xi_u \in E_u^*(x).$$

The following result, which is due to Faure, Roy, and Sjöstrand in [**17**] constructs an anisotropic order function.

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There exists an order function $m(x, \xi)$ taking its values in $[-1, 1]$ and an escape function $G_m(x, \xi) := m(x, \xi) \log |\xi|$, such that:

- the order function $m(x, \xi)$ only depends on the direction $\xi/|\xi| \in S^*M$ for $|\xi| \geq 1$ and is equal to 1 (respectively -1) in a conical neighborhood of E_s^* (respectively E_u^*);
- the escape function decreases along trajectories, that is,

$$\text{there exists } R > 0, |\xi| \geq R, \quad \mathbf{X}(G_m)(x, \xi) \leq 0.$$

(For this inequality, one should work with an adapted metric and $\mathbf{X}$ denotes the symplectic lift of the vector field X.)

We fix now an order function m and consider the corresponding symbol class. We refer to [17, Appendix] for the detailed construction. What is important to understand here is that the order function $m(x, \xi)$ constructed below gives rise to a symbol class $S^{m(x,\xi)}$ on which we can perform quantization. These quantized symbols are called pseudo-differential operators of order $m(x, \xi)$ and can be viewed as bounded operators from the anisotropic Sobolev space $H^{m(x,\xi)}$ to $L^2(M)$. The anisotropic Sobolev space $H^{m(x,\xi)}$ is defined by means of an *elliptic* operator in the anisotropic symbol class $S_\rho^{m(x,\xi)}$ (see [17, Appendix, Definition 7]).

The symbol $\exp(G_m)$ belongs to the anisotropic class $S_\rho^{m(x,\xi)}$ for every $\rho < 1$, and if we fix a quantization Op (see for instance [39, Ch. 4]),

$$\hat{A}_m := \mathrm{Op}(\exp(G_m))$$

is a pseudo-differential operator which is elliptic and, up to changing the symbol by a $O(S^{m(x,\xi)-(2\rho-1)})$ term, it can be made formally self-adjoint and invertible on $C^\infty(M)$. For $s \in \mathbb{R}$, we define the Sobolev space of order $sm(x, \xi)$ to be $\mathcal{H}^s := \hat{A}_{sm}^{-1}(L^2(M))$. In the following, the L^2 spaces will be associated to the Riemannian volume form vol. The following properties hold:

- the space $\mathcal{H}^s$ is a Hilbert space with inner product

$$(\varphi_1, \varphi_2)_{\mathcal{H}^s} := (\hat{A}_{sm}\varphi_1, \hat{A}_{sm}\varphi_2)_{L^2},$$

  which makes $\hat{A}_{sm}$ a unitary operator from $\mathcal{H}^s$ to L^2;
- for a pseudo-differential operator $A \in \Psi^{sm(x,\xi)}$, A is an unbounded operator on $L^2(M)$ with domain given by $\mathcal{H}^s$;
- the space $\mathcal{H}^{-s}$ can be identified to the dual of $\mathcal{H}^s$ by

$$(\varphi, \psi)_{\mathcal{H}^s \times \mathcal{H}^{-s}} := (\hat{A}_{sm}\varphi, \hat{A}_{sm}^{-1}\psi)_{L^2}, \tag{21}$$

  and the duality extends the usual L^2-pairing;
- if $f \in C^\infty(\mathcal{M})$, then for any $\varphi \in \mathcal{H}^s$, $\psi \in \mathcal{H}^{-s}$,

$$(f\varphi, \psi)_{\mathcal{H}^s \times \mathcal{H}^{-s}} = (\varphi, \bar{f}\psi)_{\mathcal{H}^s \times \mathcal{H}^{-s}}. \tag{22}$$

We now use the microlocal techniques introduced in [17, 18] to define the *Ruelle resonances*. We first define the *transfer operator* associated to the Anosov flow.

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The transfer operator $e^{t\mathbf{P}} : L^2(\mathcal{M}, \mathrm{vol}) \to L^2(\mathcal{M}, \mathrm{vol})$ (where $\mathbf{P} := -X + V$) is given, for any $f \in C^\infty(M)$, by

$$e^{t\mathbf{P}} f(x) := \exp\left(\int_0^t V(\varphi_{-s}(x))\, ds\right) f(\varphi_{-t}(x)) =: \exp(S_t V(\varphi_{-t}x)) f(\varphi_{-t}(x)).$$

We define the exponential growth in the L^2-norm of $e^{t\mathbf{P}}$ by

$$C_0(\mathbf{P}) = C_0 := \limsup_{t\to+\infty} \frac{1}{t} \ln \|e^{t\mathbf{P}}\|_{L^2\to L^2}.$$

The fact that C_0 is finite is a consequence of $e^{t\mathbf{P}}$ being a semi-group.

We obtain the existence of the meromorphic extension of the resolvent to the whole complex plane and the fact that Ruelle resonances are contained in $\{\mathrm{Re}(\lambda) \le C_0\}$. We refer to [28, Theorem 9.11] for a proof of the following theorem.

THEOREM 2.1. (Faure and Sjöstrand) *There exists $c > 0$ such that for any $s > 0$, we have, for any λ such that $\mathrm{Re}(\lambda) > -cs + C_0$, that $\mathbf{P} - \lambda$ is Fredholm of index 0 as an operator*

$$\mathbf{P} - \lambda : \mathrm{Dom}(\mathbf{P}) \cap \mathcal{H}^s = \{u \in \mathcal{H}^s \mid \mathbf{P}u \in \mathcal{H}^s\} \to \mathcal{H}^s. \tag{23}$$

Moreover, the resolvent

$$R_+(\lambda) = (\mathbf{P} - \lambda)^{-1} = (-X + V - \lambda)^{-1} : \mathcal{H}^s \to \mathcal{H}^s \tag{24}$$

is well defined, bounded, and holomorphic for $\{\mathrm{Re}(\lambda) > C_0\}$ and has a meromorphic extension to $\{\mathrm{Re}(\lambda) > -cs + C_0\}$, which is independent of any choice. Thus, the resolvent, viewed as an operator $C^\infty(M) \to \mathcal{D}'(M)$, has a meromorphic extension to the whole complex plane. The poles of this extension are called the Ruelle resonances, and they are located in $\{\mathrm{Re}(\lambda) \le C_0\}$. On $\{\mathrm{Re}(\lambda) > C_0\}$, one has

$$(\mathbf{P} - \lambda)^{-1} f = \int_0^{+\infty} e^{t(\mathbf{P}-\lambda)} f\, dt = \int_0^{+\infty} \exp\left(\int_0^t V(\varphi_{-s}(x))\, ds\right) e^{-t(X+\lambda)} f\, dt. \tag{25}$$

Remark 2.1. The previous construction can be made for a general smooth Hermitian bundle $\mathscr{E}$ and in particular for the bundle of forms in the kernel of the contraction, see [15, Appendix C] for more details. (We insist on the fact that smoothness is important and this leads to some technical difficulties when working on forms.) We will not specify the dependence on $\mathscr{E}$ in the rest of the paper.

We sum up at the end of the subsection the characterization of generalized eigenfunctions: the *resonant states*.

A complex number λ_0 is a Ruelle resonance if and only if there exists a distribution $u \in \mathcal{D}'(\mathcal{M})$ with wavefront set contained in E_u^* (see for instance [16, Lemma 5.1]) such that $(\mathbf{P} - \lambda_0)u = 0$, and we will then say that u is a *resonant state*. We will sometimes write Res as the set of Ruelle resonances and Res_{λ_0} as the set of resonant states associated to λ_0:

$$\mathrm{Res}_{\lambda_0} := \{u \in \mathcal{D}'(\mathcal{M}) \mid (\mathbf{P} - \lambda_0)u = 0,\ \mathrm{WF}(u) \subset E_u^*\}. \tag{26}$$

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We have a corresponding version for the co-resonant states (defined in the next subsection). The wavefront set condition then becomes $\mathrm{WF}(u) \subset E_s^*$ and this will be the version of the proposition that we will use in the proof of Theorem 3.1.

We finish the subsection by discussing *generalized resonant states* and the presence of Jordan blocks. More precisely, if we consider the meromorphic extension $R_+(\lambda)$ constructed in Theorem 2.1, then $\lambda_0 \in \mathrm{Res}$ if and only if λ_0 is a pole of the meromorphic extension. In this case, the spectral projector at λ_0 is

$$\Pi_{\lambda_0}^+ = \frac{1}{2i\pi} \int_\gamma R_+(z)\, dz,$$

where γ is a small loop around λ_0. Moreover, we can use the analytic Fredholm theorem to deduce that the resolvent has the following expansion:

$$R_+(\lambda) = R_+^H(\lambda) + \sum_{j=1}^{N(\lambda_0)} \frac{(\mathbf{P} - \lambda_0)^{j-1}\Pi_{\lambda_0}^+}{(\lambda - \lambda_0)^j},$$

where $R_+^H(\lambda)$ is the holomorphic part near λ_0. The *generalized resonant states* are

$$\mathrm{Res}_{\lambda_0,\infty} := \Pi_{\lambda_0}^+(\mathcal{H}^s) = \Pi_{\lambda_0}^+(C^\infty(\mathcal{M})) = \{u \in \mathcal{H}^s \mid (\mathbf{P} - \lambda_0)^{N(\lambda_0)} u = 0\}. \tag{27}$$

Remark 2.2. We will say that the Ruelle resonance λ_0 has no Jordan block if $\mathrm{Res}_{\lambda_0,\infty} = \mathrm{Res}_{\lambda_0}$. Note that if $N(\lambda_0) = 1$, i.e., the resolvent has a pole of order at most 1, then there is no Jordan block. This will be used in Lemmas 3.5 and 4.2 to show that resonances on the critical axes have no Jordan blocks.

2.4. *Equilibrium states from dimension theory.* In this section, we recall Climenhaga's construction from [**10**] of leaf measures m_V^u and m_V^s using dimension theory.

There are two main ways equilibrium states are defined. The first one is through the use of *Markov partitions* and the second one is via the use of the *specification property*. A third approach is given by dimension theory: the goal is to generalize the idea of *Haussdorff dimension* and *Haussdorff measure* to a more dynamical setting. We recall the definition of the Haussdorff dimension for a metric space (X, δ). For $d \geq 0$ and $\epsilon > 0$, define the d-dimensional Haussdorff measure by

$$H_\epsilon^d(S) := \inf \left\{ \sum_{i=0}^{\infty} \mathrm{diam}(U_i)^d \mid S \subset \bigcup_{i \in \mathbb{N}} U_i,\ \mathrm{diam}(U_i) < \epsilon \right\}$$

for any subset S and where the infimum is taken over all countable covering of sets U_i with diameter less than ϵ. We define an outer measure by taking

$$H^d(S) = \lim_{\epsilon \to +\infty} H_\epsilon^d(S) \in [0, +\infty].$$

We then define the Haussdorff dimension of X to be

$$\dim_{\mathrm{Hauss}}(X) := \inf\{d \geq 0 \mid H^d(X) = +\infty\} = \sup\{d \geq 0 \mid H^d(X) = 0\}.$$

The idea of Climenhaga, Pesin, and Zelerowicz (already present in essence in [**22**, **24**], where the case of the measure of maximal entropy was treated) was to replace the sets with

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small diameters by more dynamical objects, namely, coverings should be made of *Bowen balls* (defined below) and we should let time $t \to +\infty$.

$$B_t(x, r) := \{y \in \mathcal{M} \mid \max_{s\in[0,t]} d(\varphi_s x, \varphi_s y) < r\}. \tag{28}$$

Let $\delta_0 > 0$ be the size of the local (un)stable manifolds, then fix thereafter $\delta \in]0, \delta_0[$ and define $\mathcal{W}^\bullet(x, \delta) := B(x, \delta) \cap \mathcal{W}^\bullet(x)$, where $\bullet = u, s, ws, wu$. We will not always specify in the rest of the paper the dependence in δ if it is not relevant to the argument.

Let $x \in \mathcal{M}$, consider $Z \subset \mathcal{W}^u(x, \delta)$. Define, for $T > 0$,

$$\mathbb{E}(Z, T) := \left\{\mathscr{E} \subset M \times [T, +\infty[,\ Z \subset \bigcup_{(x,t)\in\mathscr{E}} B_t(x, r, \mathcal{W}^u(x, \delta))\right\},$$

where $B_t(x, r, \mathcal{W}^u(x, \delta)) = \{y \in W^u(x, \delta) \mid \max_{[0,t]} d_{W^u(x,\delta)}(\varphi_s x, \varphi_s y) < r\}$. Let $\alpha \in \mathbb{R}$, then we define a measure $m^\alpha_{\mathcal{W}^u(x,\delta)} = m^\alpha$ by putting

$$m^\alpha(Z) = \lim_{T\to+\infty} \inf_{\mathscr{E}\in\mathbb{E}(Z,T)} \sum_{(x,t)\in\mathscr{E}} e^{S_t V(x)-t\alpha}.$$

We then retrieve the Carathéodory dimension as a threshold just like in the case of the Haussdorff measure, this is a result due to Pesin, see [**32**, Propositions 1.1 and 1.2]. We have, moreover, in this case that the measure for $\alpha = \alpha_C$ defines a Borel measure, see [**10**, Lemma 2.14].

The measure m^α defined above is an outer measure for any $\alpha \in \mathbb{R}$ and

$$P(V) := \inf\{\alpha \mid m^\alpha(W(x, \delta)) = +\infty\} = \sup\{\alpha \mid m^\alpha(\mathcal{W}^u(x, \delta)) = 0\}.$$

Moreover, $m^{P(V)}$ is a Borel measure, denoted by m^u_x and called the leaf measure. It satisfies

$$\text{for all } Z \subset \mathcal{W}^u(x_1, \delta), \quad m^u_x(Z) = \lim_{T\to+\infty} \inf_{\mathscr{E}\in\mathbb{E}(Z,T)} \sum_{(x,t)\in\mathscr{E}} e^{S_t V(x)-tP(V)}. \tag{29}$$

Up until now, we have defined a system of (unstable) leaf measures $\{m^u_x \mid x \in \mathcal{M}\}$ satisfying the two following conditions.

- Support: each measure m^u_x is supported in $\mathcal{W}^u(x, \delta)$.
- Compatibility: if $Z \subset \mathcal{W}^u(x_1, \delta) \cap \mathcal{W}^u(x_2, \delta)$ is a Borel set, then the two measures agree, i.e., $m^u_{x_1}(Z) = m^u_{x_2}(Z)$.

The set of measures defined in equation (29) has actually two more important properties: it is φ_t-conformal and behaves naturally with holonomies (see [**10**, Theorem 3.4]).

The system of (unstable) leaf measures $\{m^u_x \mid x \in \mathcal{M}\}$ defined in equation (29) is φ_t-conformal, namely, for any $x \in \mathcal{M}$ and $t \in \mathbb{R}$, the measures $(\varphi_t)_* m^u_x$ and $m^u_{\varphi_t x}$ are equivalent and more precisely,

$$m^u_{\varphi_t x}(\varphi_t Z) = \int_Z e^{tP(V)-S_t V(z)}\, dm^u_x(z). \tag{30}$$

In terms of Radon–Nikodym derivatives, we have

$$\frac{d((\varphi_{-t})_* m^u_{\varphi_t x})}{dm^u_x}(z) = e^{tP(V) - S_t V(z)}, \quad \frac{d((\varphi_t)_* m^u_x)}{dm^u_{\varphi_t x}}(\varphi_t z) = e^{S_t V(z) - tP(V)}. \tag{31}$$

We define the notion of *holonomy* between (weak-un)stable leaves.

Given $\mathcal{W}^u(x_1, \delta), \mathcal{W}^u(x_2, \delta)$ for $x_1, x_2 \in \mathcal{M}$, a *weak-stable δ-holonomy* between $\mathcal{W}^u(x_1, \delta)$ and $\mathcal{W}^u(x_2, \delta)$ is a homeomorphism $\pi : \mathcal{W}^u(x_1, \delta) \to \mathcal{W}^u(x_2, \delta)$ such that $\pi(z) \in \mathcal{W}^{ws}(z, \delta)$ for all $z \in \mathcal{W}^u(x_1, \delta)$.

To conclude, we give the change of variable formula for holonomies. We first introduce a useful function. Let $\delta_0 > 0$ be small enough, given $x \in \mathcal{M}$ and $y \in \mathcal{W}^{ws}(x, \delta_0)$, define

$$w^+_V(x, y) = S_{t(x,y)} V(x) + t(x, y) P(V) + \int_0^{+\infty} (V(\varphi_{t(x,y)+\tau}(x)) - V(\varphi_\tau(y)))\, d\tau. \tag{32}$$

Here, $t(x, y)$ is the Bowen time and the integral converges because $d(\varphi_{t+\tau} x, \varphi_\tau y) \to 0$ exponentially fast and V is Hölder continuous.

We define for $x \in \mathcal{M}$ and $y \in \mathcal{W}^{wu}(x, \delta_0)$ the quantity

$$w^-_V(x, y) = -S_{t(x,y)} V(x) + t(x, y) P(V) + \int_0^{+\infty} (V(\varphi_{t-\tau} x) - V(\varphi_{-\tau} y))\, d\tau.$$

We note that we have the following special values:

$$\text{for all } y \in \mathcal{W}^s(x), \quad w^+_V(x, y) = \int_0^{+\infty} (V(\varphi_\tau x) - V(\varphi_\tau y))\, d\tau,$$

as well as

$$w^+_V(x, \varphi_t x) = S_t V(x) - tP(V). \tag{33}$$

Finally, we have the cocycle relation

$$\text{for all } y, z \in \mathcal{W}^{ws}(x, \delta_0/2), \quad w^+_V(x, y) = w^+_V(x, z) + w^+_V(z, y). \tag{34}$$

The holonomy we will mostly be interested in will be given by $\pi : \mathcal{W}^u(q, \delta) \to \mathcal{W}^u(p, \delta)$, $\pi(x) := [x, p]$ and we can now state the second change of variable theorem, see [**10**, Theorem 3.4] for more details.

Consider the system of (unstable) leaf measures $\{m^u_x \mid x \in \mathcal{M}\}$ defined in equation (29). Let $\mathcal{W}^u(x_1, \delta), \mathcal{W}^u(x_2, \delta)$ for $x_1, x_2 \in \mathcal{M}$ and let $\pi : \mathcal{W}^u(x_1, \delta) \to \mathcal{W}^u(x_2, \delta)$ be a weak-stable δ_0 holonomy. Then, the measures $\pi_* m^u_{x_1}$ and $m^u_{x_2}$ are equivalent, and we have

$$\frac{d(\pi_* m^u_{x_1})}{dm^u_{x_2}}(\pi(z)) = e^{w^+_V(z, \pi z)}. \tag{35}$$

3. *Resolvent acting on functions*

In this section, we study the action of $\mathbf{P}$ on 0-forms. The following proofs could all be written in the formalism of currents as it is done in §4 for the d_s-forms, but we remark that by fixing a smooth volume form, one can reduce to studying the action of $\mathbf{P}$ on function,

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which we will do here. The goal is to locate the critical axis, show that it is given by $\{\mathrm{Re}(\lambda) = P(V + J^u)\}$, and to study the co-resonant states associated to resonances on this axis.

3.1. *Construction of the co-resonant state.* We first prove that $P(V + J^u)$ is indeed a resonance, called the *first resonance*. In the case of a null potential ($V = 0$), this is trivial as constant functions lie trivially in the kernel of the flow. In this case, the equilibrium state (which is usually referred to as the SRB measure) can be identified as the co-resonant state for the Ruelle resonance 0.

Following §2.3, we can define Ruelle resonances for the L^2-adjoint $\mathbf{P}^* = X + V - \mathrm{div}_{\mathrm{vol}}(X)$. The resonant states for $\mathbf{P}^*$ are referred to as *co-resonant* states for $\mathbf{P}$ and their span has the same dimension as the span of the resonant states. Equivalently, we can define the adjoint by duality, using relation (21), as

$$\text{for all } f, g \in C^\infty(\mathcal{M}), \quad (\mathbf{P}f, g)_{\mathcal{H}^s \times \mathcal{H}^{-s}} = (f, \mathbf{P}^* g)_{\mathcal{H}^s \times \mathcal{H}^{-s}}, \tag{36}$$

from which we can define an unbounded operator

$$\mathbf{P}^* : \mathrm{Dom}(\mathbf{P}^*) \cap \mathcal{H}^{-s} = \{u \in \mathcal{H}^{-s} \mid \mathbf{P}^* u \in \mathcal{H}^{-s}\} \to \mathcal{H}^{-s}. \tag{37}$$

Now, a direct adaptation of [5, Lemma 5.3] yields the following lemma.

LEMMA 3.1. (Co-resonant states) *Let $\lambda \in \mathbb{C}$, then λ is a Ruelle resonance for $\mathbf{P}$ if and only if $\overline{\lambda}$ is a Ruelle resonance for $\mathbf{P}^*$. In this case, the space of resonant and co-resonant states have the same dimension m. If we consider $u_1, \ldots, u_m \in \mathcal{D}'_{E_u^*}(\mathcal{M}) := \{u \in \mathcal{D}' \mid \mathrm{WF}(u) \subset E_u^*\}$ and $v_1, \ldots, v_m \in \mathcal{D}'_{E_s^*}(\mathcal{M}) := \{u \in \mathcal{D}' \mid \mathrm{WF}(u) \subset E_s^*\}$ as two bases of the resonant states for $\mathbf{P}$ and $\mathbf{P}^*$, respectively, satisfying $(u_i, v_j) = \delta_{i,j}$, then the projector $\Pi_0(\lambda)$ on the space of resonant states for the resonance λ is given by*

$$\Pi_0(\lambda) = \sum_{i=1}^m u_i \otimes v_i. \tag{38}$$

In [10, Theorem 3.10], Climenhaga gives a local product construction of the equilibrium state. For this, let us first notice that Climenhaga's construction is still valid when swapping the unstable and stable foliations.

We can thus define a system of stable leaf measures $\{m_x^s \mid x \in \mathcal{M}\}$ which is compatible and supported in $\mathcal{W}^s(x)$. Moreover, m_x^s is defined using equation (29), but using backward Bowen-balls, see [10, §3.3] for details. The set $\{m_x^s \mid x \in \mathcal{M}\}$ is φ_t-conformal in the sense of equation (30) and the change of variable by holonomy is the one explicated in equation (35) but with w^+ replaced by w^-.

We can then extend the leaf measures to the weak (un)stable foliations

$$m_x^{ws} := \int_{-\delta}^{\delta} m_{\varphi_t x}^s \, dt, \quad m_x^{wu} := \int_{-\delta}^{\delta} m_{\varphi_t x}^u \, dt. \tag{39}$$

To state the main result of this subsection, we introduce further notation,

$$z = \Psi_q(x, y), \quad R_q^u(z) = \mathcal{W}^u(z, \delta) \cap R_q, \;\; R_q^{ws}(z) = \mathcal{W}^{ws}(z, \delta) \cap R_q,$$

where R_q and Ψ_q are defined in equation (16). Observe that

$$z = [x, y], \quad x = [z, q], \quad y = [q, z]. \tag{40}$$

We adapt the product construction of Climenhaga [10, Theorem 3.10] to obtain the following local construction of the co-resonant state. In the following, we add an additional index W to denote the leaf measures constructed using equation (29) from the Hölder continuous potential W. For any $q \in \mathcal{M}$, we define three measures on R_q by putting, for a Borel set $Z \subset R_q$,

$$m_1^q(Z) := \int_Z e^{w^+_{J^u+V}(z,[z,q]) + w^-_{J^u}(z,[q,z])} \, d((\Psi_q)_*(m^u_{q,J^u+V} \times m^{ws}_{q,J^u}))(z), \tag{41}$$

$$m_2^q(Z) := \int_{R_q^{ws}} \int_{Z\cap R_q^u(y)} e^{w^-_{J^u}(z,y)} \, dm^u_{y,J^u+V}(z) \, dm^{ws}_{q,J^u}(y), \tag{42}$$

$$m_3^q(Z) := \int_{R_q^u} \int_{Z\cap R_q^{ws}(x)} e^{w^+_{J^u+V}(z,x)} \, dm^{ws}_{x,J^u}(z) \, dm^u_{q,V+J^u}(x). \tag{43}$$

THEOREM 3.1. (Construction of the co-resonant state) *For any $q \in \mathcal{M}$, the three measures m_1^q, m_2^q, and m_3^q coincide, and there is a unique non-zero and finite Borel measure ν on $\mathcal{M}$ such that for all $Z \subset R_q$, one has $\nu(Z) = m_1(Z)$. This Borel measure satisfies*

$$\mathbf{P}^*\nu = P(V + J^u)\nu, \quad \mathrm{WF}(\nu) \subset E_s^* \tag{44}$$

and is therefore a co-resonant state. In other words, $P(V + J^u)$ is a Ruelle resonance.

Remark 3.1. Climenhaga's original construction is obtained by taking the two potentials to be equal, in other words, all leaf measures are constructed using V and the resulting Borel measure is a non-zero multiple of the equilibrium state μ_V. We observe that when $V = 0$, both constructions coincide and we retrieve the known fact that the SRB measure is the co-resonant state in this case. Of course, by taking the adjoint, we obtain a construction of a resonant state η.

Proof. We mostly follow the strategy of the proof of [10, Theorem 10], while only changing the necessary details. We first prove that the three formulas indeed agree and define the same local measure. Starting from a measurable function $\zeta : \mathcal{W}^u(q, \delta) \times \mathcal{W}^{ws}(q, \delta) \to]0, +\infty[$, one sees that

$$\int_Z \zeta(\Psi_q^{-1}(z)) d((\Psi_q)_*(m^u_{q,V+J^u} \times m^{ws}_{q,J^u}))(z) = \int_{\Psi_q^{-1}(Z)} \zeta(x, y) \, d((m^u_{q,V+J^u} \times m^{ws}_{q,J^u}))(x, y).$$

Now, this last expression can be seen to be equal to equation (42), using the change of variable by holonomy in equation (35) with the holonomy $\pi : \mathcal{W}^u(q, \delta) \to R_q^u(y)$, $\pi(x) = [x, y]$ and with $\zeta(z) := e^{w^+_{V+J^u}(z,[z,q]) + w^-_{J^u}(z,[q,z])}$.

$$\int_{R_q^u} \zeta([x,y])\mathbf{1}([x,y])\,dm^u_{q,V+J^u}(x) = \int_{R_q^u(y)\cap Z} \zeta(z)\,d(\pi_* m^u_{q,V+J^u})(z)$$
$$= \int_{R_q^u(y)\cap Z} \zeta(z) e^{w^+_{V+J^u}([z,q],z)}\,dm^u_{y,V+J^u}(z) = \int_{R_q^u(y)\cap Z} e^{w^-_{J^u}(z,[q,z])}\,dm^u_{y,V+J^u}(z),$$

where we used the cocycle relation (34). Similarly, we obtain the equivalent equation (43). Next, we show that the set of local measures $\{\nu_q \mid q \in \mathcal{M}\}$ are compatible in the sense that for any Borel set $Z \subset R_q \cap R_p$, one has $\nu_q(Z) = \nu_p(Z)$. This will then define a global Borel measure. If $p \in \mathcal{W}^{ws}(q,\delta)$, then we deduce that the above relation follows from equation (42) and the fact that the system measures $\{m^s_x \mid x \in \mathcal{M}\}$ is compatible on the intersection. Similarly, the relation for $p \in \mathcal{W}^u(q,\delta)$ follows from the compatibility of $\{m^{ws}_x \mid x \in \mathcal{M}\}$ and equation (43). The general case then follows from the local product structure. Call ν the global Borel measure we obtain. The fact that ν is non-zero and finite follows from the analog fact on the local measures ν_q and the compactness of the space $\mathcal{M}$. What is left to prove is equation (44). It suffices to show that one has

$$\text{for all } f \in C^\infty(\mathcal{M}),\ \langle e^{t\mathbf{P}} f, \nu\rangle = e^{tP(V+J^u)}\langle f, \nu\rangle, \tag{45}$$

which will clearly imply the first part of equation (44). Because the measure ν is constructed from its restrictions, it will be enough to prove that

$$\text{for all } f \in C^\infty(\mathcal{M}),\quad \mathrm{Supp}(f) \cup \mathrm{Supp}(e^{t\mathbf{P}} f) \subset R_q \Rightarrow \nu(e^{t\mathbf{P}} f) = e^{tP(V+J^u)}\nu(f). \tag{46}$$

We now use equation (43) to compute explicitly, using the φ_t-conformality in equation (30):

$$\nu^{V+J^u}(e^{t\mathbf{P}} f) = \int_{R_q^{ws}} \int_{Z\cap R_q^u(y)} e^{S_t V(\varphi_{-t} z)} f(\varphi_{-t} z) e^{w^-_{J^u}(z,y)}\,dm^u_{x,V+J^u}(z)\,dm^{ws}_{q,J^u}(y).$$

The cocycle relation (34) gives $e^{w^-_{J^u}(z,y)} = e^{w^-_{J^u}(z,\varphi_{-t}z)} e^{w^-_{J^u}(\varphi_{-t}z,y)}$. Using equation (33) to get $w^+_{J^u}(z,\varphi_{-t}z) = S_t J^u(\varphi_{-t}z)$, this means that

$$\int_{Z\cap R_q^u(y)} e^{S_t V(\varphi_{-t} z)} f(\varphi_{-t} z) e^{w^-_{J^u}(z,y)}\,dm^u_{x,V+J^u}(z)$$
$$= \int_{Z\cap R_q^u(y)} e^{S_t(V+J^u)(\varphi_{-t} z)} f(\varphi_{-t} z) e^{w^-_{J^u}(\varphi_{-t}z,y)}\,dm^u_{x,V+J^u}(z)$$
$$= \int_{Z\cap R_q^u(y)} e^{S_t(V+J^u)(z)} f(z) e^{w^-_{J^u}(z,y)}\,d((\varphi_{-t})_* m^u_{x,V+J^u})(z)$$
$$= \int_{Z\cap R_q^u(y)} e^{tP(V+J^u)-S_t(V+J^u)(z)} e^{S_t(V+J^u)(z)} f(z) e^{w^-_{J^u}(z,y)}\,dm^u_{\varphi_{-t}x,V+J^u}(z)$$
$$= e^{tP(V+J^u)} \int_{Z\cap R_q^u(y)} f(z) e^{w^-_{J^u}(z,y)}\,dm^u_{\varphi_{-t}x,V+J^u}(z),$$

where we used the φ_t-conformality, see equation (31). This implies equation (45) and thus

$$\mathbf{P}^*\nu = P(V+J^u)\nu.$$

The last thing we need to do is to bound the wavefront set of the measure ν. For this, we will need some regularity result on the conditional measures of the SRB measures and the following adaptation of [**5**, Lemma 2.9].

LEMMA 3.2. *The wavefront set of* ν *is included in* E_s^*.

Proof. Consider $q \in \mathcal{M}$ and $\xi \notin E_s^*(q)$, and we shall prove that (q, ξ) is not in the wavefront set. Choose a phase function S such that $d_q S = \xi$ and a cutoff χ near q, we then use equation (43) to obtain

$$\nu_q^{V+J^u}(\chi e^{i\frac{S}{h}}) \\ := \int_{R_q^u} \int_{Z \cap R_q^{ws}(x)} \chi(z) e^{i\frac{S}{h}} e^{w_V^+(z,x)+t(P(V+J^u)-P(V))} e^{w_{J^u}^+(z,x)}\, dm_{x,J^u}^{ws}(z)\, dm_{q,V+J^u}^u(x).$$

Now, we can use [**10**, Corollary 3.11] to obtain that

$$\frac{d\mu_x^{ws}}{dm_{x,J^u}^{ws}}(z) = \frac{e^{w_{J^u}^+(z,x)}}{m_x^{ws}(R_q^{ws}(x))}, \quad h(x) := m_x^{ws}(R_q^{ws}(x)),$$

where μ_x^{ws} is the conditional measure of the SRB measure on the weak unstable manifold, in other words,

$$\nu_q^{V+J^u}(\chi e^{i\frac{S}{h}}) \\ := \int_{R_q^u} h(x) \bigg(\int_{Z \cap R_q^{ws}(x)} \chi(z) e^{i(S/h)} e^{w_V^+(z,x)+t(P(V+J^u)-P(V))} \mu_x^{ws}(z) \bigg)\, dm_{q,V+J^u}^u(x).$$

Now, we use [**14**, Theorem 3.9] to obtain that the density μ_x^{ws} is smooth along the leaves $R_q^{ws}(x)$. By this, we mean that

$$\|\mu_x^{ws}\|_{C^k(R_q^{ws}(x))} := \sup_{z \in R_q^{ws}(x)} \sup_{X_1,\ldots,X_k \in S_z R_q^{ws}(x)} |X_1 \cdots X_k(\mu_x^{ws}(z)_{|R_q^{ws}(x)})|$$

is finite for any k and that the norm depends continuously in x.

From the proof of [**14**, Corollary 4.4], one sees that the smoothness of the potential V implies that $w_V^+(z, x)$ is smooth along the leaves $R_q^{ws}(x)$. More precisely, the proof shows that given a function which is smooth along the leaves, the function defined by the last integral in equation (32) is smooth along the leaves, therefore proving that the holonomy factor is smooth along the leaves. (Note that the fact that the unstable Jacobian is smooth along the leaves in the sense above is non-trivial because it is only Hölder continuous in x.)

Using the fact that each leaf is a smooth submanifold, we can perform integration by parts (in z) to show that the inner integral is $O(h^\infty)$ as long as $dS_{|R_w^{ws}}(x)$ does not vanish. However, $\xi \notin E_s^*$, so this can be ensured locally near p. This proves that $\xi \notin E_s^*$ and we have thus shown that $\mathrm{WF}(\nu) \subset E_s^*$. □

We can now finish the proof of Theorem 3.1. For this, we use the characterization in equation (26) to deduce that ν is indeed a co-resonant state, i.e., $P(V + J^u)$ is a Ruelle resonance. □

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3.2. *Critical axis.* In this subsection, we prove that the set of Ruelle resonances is contained in the half-plane $\{\lambda \mid \mathrm{Re}(\lambda) \leq P(V + J^u)\}$. The case $V = 0$ is enlightening; in this case, the critical axis is the imaginary axis. No resonance has positive real part, indeed, in the case of a volume preserving Anosov flow, this follows directly from Theorem 2.1, while in the general case, it follows from the general bound $\|e^{-Xt} f\|_{L^\infty} \leq \|f\|_{L^\infty}$, see [**5**, Corollary 4.16]. We insist on the fact that the functional space L^∞ is not really part of the L^2-anisotropic Sobolev spaces scale and, thus, this part of the proof needs the introduction of some 'finer' measure theory. The goal of this subsection is thus to construct, using the co-resonant state ν defined in Theorem 3.1, a norm $\|.\|_V$ such that

$$\text{for all } f \in C^\infty(\mathcal{M}), \quad \|e^{t\mathbf{P}} f\|_V \leq e^{tP(V+J^u)} \|f\|_V. \tag{47}$$

LEMMA 3.3. *Let ν be as in Theorem 3.1, for $f \in C^\infty(\mathcal{M})$, define*

$$\|f\|_V := \|f\|_{L^1(\mathcal{M},\nu)} = \int_{\mathcal{M}} |f|(z)\, d\nu(z).$$

Then, this defines a norm on $C^\infty(\mathcal{M})$ such that

$$\text{for all } f \in C^\infty(\mathcal{M}), \quad \|e^{t\mathbf{P}} f\|_V \leq e^{tP(V+J^u)} \|f\|_V.$$

We will denote by $L^1(\mathcal{M}, \nu)$ the completion for $\|.\|_V$ of $C^\infty(\mathcal{M})$.

Proof. The fact that $\nu(|f|) \geq 0$ is a consequence of ν being a measure (i.e., of ν being a distribution which is non-negative on non-negative functions). The homogeneity and the triangle inequality are clear. Suppose now $\|f\|_V = 0$, then use equation (41) as well as equation (29) to see that the co-resonant state ν gives positive measure to any non-empty open set of $\mathcal{M}$. Thus, if f is non-zero, then $|f|$ is positive on a small open set, which is a contradiction. The change of variable formula (47) is a direct consequence of equation (45), which was shown in the proof of Theorem 3.1. □

As a direct consequence, we adapt [**5**, Corollary 4.16] to show that there is no Ruelle resonance in the half-plane $\{\mathrm{Re}(z) > P(V + J^u)\}$.

LEMMA 3.4. *The set of Ruelle resonances for the potential V is contained in the half-plane $\{\mathrm{Re}(\lambda) \leq P(V + J^u)\}$.*

Proof. Recall from the proof of Theorem 2.1 (see [**28**, Theorem 9.11]) that we can construct an operator $\tilde{Q}(\lambda)$ such that, as an equality on operators acting on $C^\infty(M)$,

$$(\mathbf{P} + \lambda)\tilde{Q}(\lambda) = \mathrm{Id} - \tilde{R}(\lambda), \tag{48}$$

where the remainder is given by

$$\tilde{R}(\lambda) = -\int_T^{T+\epsilon} \chi'(t) e^{-t\lambda} e^{t\mathbf{P}}\, dt,$$

where χ is a cutoff supported in $[0, T + \epsilon[$ and constant equal to 1 on $[0, T]$, for some suitable choices of ϵ and T.

Let $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > P(V + J^u)$. Let us show that it is not in the Ruelle spectrum. The projector $\Pi_0(\lambda)$ on the eigenvalue $z = 0$ of the Fredholm operator $F(\lambda) := \mathrm{Id} - \tilde{R}(\lambda)$ is given by the integral

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$$\Pi_0(\lambda) = \frac{1}{2\pi i} \int_{|z|=\epsilon} (z\mathrm{Id} - \tilde{R}(\lambda))^{-1}\, dz \tag{49}$$

for a radius $\epsilon > 0$ small enough. Note that by the proof of Theorem 2.1, this is the spectral projector for any anisotropic Sobolev space $\mathcal{H}^s$ for which $\mathrm{Re}(\lambda) \geq -cs + C_0$ and thus equation (49) also holds as a map $C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$. However, if $f \in L^1(\mathcal{M}, \nu)$, then, using Lemma 3.3, one has

$$\begin{aligned}\|\tilde{R}(\lambda) f\|_V &\leq \int_T^{T+\epsilon} |\chi'(t)| e^{-t\mathrm{Re}(\lambda)} \times \|e^{t\mathbf{P}} f\|_V\, dt \\ &\leq \int_T^{T+\epsilon} |\chi'(t)| e^{-t(\mathrm{Re}(\lambda) - P(V+J^u))} \times \|f\|_V\, dt \leq \frac{1}{2}\|f\|_V,\end{aligned}$$

if T is chosen large enough. In particular, this shows that $F(\lambda)$ is invertible on $L^1(\mathcal{M}, \nu)$ with inverse in $\mathcal{L}(L^1(\mathcal{M}, \nu))$ and thus $\Pi_0(\lambda) = 0$, meaning that λ is not a Ruelle resonance. □

3.3. *Resonances on the critical axis.* In this subsection, we investigate some properties of Ruelle resonances on the critical axis and prove the first part of Theorem 1.1. The main results are that Ruelle resonances on the critical axis have no Jordan block and that the first resonance is simple.

The following two lemmas justify that resonances on the critical axis have no Jordan block. The strategy is the same as in [**5**, Lemma 5.1], where the space L^∞ is replaced by the new functional space $L^1(\mathcal{M}, \nu)$.

LEMMA 3.5. *Let* $\lambda + P(V + J^u) \in \{\mathrm{Re}(\lambda) = P(V + J^u)\}$ *be on the critical axis. Then,* $\tilde{R}(\lambda)$ *has spectrum included in the closed unit disk.*

Proof. Consider $B \in \Psi^0(\mathcal{M})$ such that its principal symbol is equal to 1 except in a conic neighborhood V of E_0^*. We can then write $\tilde{R} = \tilde{R}B + (\mathrm{Id} - B)\tilde{R}(\mathrm{Id} - B) + B\tilde{R}(\mathrm{Id} - B)$, where $(\mathrm{Id} - B)\tilde{R}(\mathrm{Id} - B)$ is smoothing thus compact (because $\mathrm{WF}(\tilde{R})$ does not intersect E_0^*, see [**28**, Theorem 9.11] for a detailed computation of the wavefront set). Moreover, we have $\|\tilde{R}B + B\tilde{R}(\mathrm{Id} - B)\|_{L^2\to L^2} \leq \frac{1}{2}$ if T is chosen large enough. This shows that the essential spectrum of $\tilde{R}$ is contained in $B(0, 1/2)$ and that the spectrum outside this ball consists of isolated eigenvalues. Using the $L^2 \to L^2$ boundedness of $\tilde{R}(\lambda)$, we see that for large values of $|z|$, we get the converging Neumann series (in $\mathcal{H}^s$)

$$(z - \tilde{R}(\lambda))^{-1} = z^{-1} \sum_{k\geq 0} z^{-k} \tilde{R}(\lambda)^k. \tag{50}$$

Now, we use the bound in equation (47) for $f \in C^\infty(\mathcal{M})$:

$$\begin{aligned}\|\tilde{R}(\lambda) f\|_{L^1(\mathcal{M},\nu)} &\leq \int_T^{T+\epsilon} e^{-tP(V+J^u)} |\chi'(t)| \times \|e^{t\mathbf{P}} f\|_{L^1(\mathcal{M},\nu)}\, dt \\ &\leq \int_T^{T+\epsilon} e^{-tP(V+J^u)} |\chi'(t)| e^{tP(V+J^u)} \|f\|_{L^1(\mathcal{M},\nu)}\, dt \leq \|f\|_{L^1(\mathcal{M},\nu)},\end{aligned}$$

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to get that the above Neumann series actually converges for $|z| > 1$ in $\mathcal{L}(L^1(\mathcal{M}, \nu))$ and is analytic in $|z| > 1$. By density of smooth functions in $\mathcal{H}^s$, this proves that the eigenvalues are contained in the closed unit disk. □

The next lemma, together with Remark 2.2, justifies that resonances on the critical axis have no Jordan blocks.

LEMMA 3.6. *A complex number $\lambda_0 \in P(V + J^u) + i\mathbb{R}$ is a Ruelle resonance if and only if $\tilde{R}(\lambda_0)$ has an eigenvalue at 1. Moreover, if $\tilde{R}(\lambda_0)$ has a eigenvalue on the unit circle, then it is equal to 1 and the eigenfunctions correspond to resonant states at the resonance λ_0. Finally, write $\Pi_0(\lambda_0)$ to be the spectral projector of $\mathrm{Id} - \tilde{R}(\lambda_0)$ on the kernel (see equation (49)). Then, we have the convergence, in $\mathcal{L}(\mathcal{H}^s)$, for any $s > 0$,*

$$\Pi_0(\lambda_0) = \lim_{k \to +\infty} \tilde{R}(\lambda_0)^k. \tag{51}$$

Proof. Suppose that $\tau \in \mathbb{S}^1$ is an eigenvalue of $\tilde{R}(\lambda_0)$. We first check that for any smooth function u, one has $\mathbf{P}\tilde{R}(\lambda_0)u = \tilde{R}(\lambda_0)\mathbf{P}u$. In other words, the operator $\tilde{R}(\lambda_0)$ commutes with $\mathbf{P}$ so that the space $\mathrm{Ran}(\Pi_\tau(\lambda_0))$ associated to the eigenvalue $\tau \in \mathbb{S}^1$ can be decomposed into generalized eigenspaces of $\mathbf{P}$. Suppose now that $u \in \mathrm{Ran}(\Pi_\tau(\lambda_0))$ is an eigenvector for $\mathbf{P}$ for the eigenvalue μ, then we get

$$\tau u = \tilde{R}(\lambda_0)u = -\int_0^\infty \chi'(t)e^{-t\lambda_0 + t\mu}u(x)\,dt = \widehat{\chi'}(-i\lambda_0 + i\mu)u.$$

This implies that $|\widehat{\chi'}(-i\lambda_0 + i\mu)| = 1$. However, using Lemma 3.4, we see that the real part of μ is smaller than $P(V + J^u)$, so that

$$1 = |\tau| = |\widehat{\chi'}(-i\lambda_0 - i\mu)| \leq \int_0^\infty |\chi'(t)|e^{t\mathrm{Re}(\mu) - tP(V+J^u)}\,dt \leq 1.$$

This implies that $\mathrm{Re}(\mu) = P(V + J^u)$ and that $\cos(t(-i\lambda_0 + i\mu))$ is constant on the support of χ'. This is possible only if $\lambda_0 = \mu$, which gives $\tau = 1$ and shows that u is a resonant state for the Ruelle resonance λ_0.

The previous discussion justifies that there exists an integer N such that

$$\Pi_0(\lambda_0) := \{u \in \mathcal{D}'_{E_u^*}(\mathcal{M}) \mid (\mathbf{P} - \lambda_0)^N u = 0\}.$$

We will now prove that $\tilde{R}(\lambda_0)$ has no Jordan block at 1 and deduce that $N = 1$. Note that the right-hand side of the previous equality does not depend on the choice of cutoff function and so that needs to be the case for the right-hand side. Let f be an eigenfunction of $\tilde{R}(\lambda_0)$, then

$$R(\lambda_0)f = \mathrm{Id} = -\int_{\mathbb{R}} \chi'(t)e^{-t\lambda_0 + t\mathbf{P}}f\,dt = f.$$

The above relation holds for any cutoff function χ supported in $[0, T + \epsilon]$ and equal to 1 on $[0, T]$. The only condition being that T must be large enough to ensure the validity of Lemma 3.5. Approximating the Dirac mass at T (for T large enough) yields $e^{-T\lambda_0 + T\mathbf{P}}f = f$ and f is thus a resonant state at λ_0. The converse is clear and we showed that the eigenfunctions of $\tilde{R}(\lambda_0)$ at 1 coincide with the resonant states at λ_0. Let $u \in C^\infty$, then near $z = 1$,

$$(\tilde{R}(\lambda_0) - z)^{-1} u = \tilde{R}_H(z) + \sum_{j=1}^{N} \frac{(\tilde{R}(\lambda_0) - 1)^{j-1} \Pi_0(\lambda_0) u}{(z-1)^j}, \tag{52}$$

where $R_+^H(z)$ is the holomorphic part near $z = 1$. We note that $(\tilde{R}(\lambda_0) - 1)^{N-1} \Pi_0(\lambda_0) u$ is an eigenevector of $\tilde{R}(\lambda)$ for 1. By the previous discussion, it is thus a resonant state θ associated to λ_0. By Lemma 3.1, we can consider a co-resonant state μ such that the pairing $(\theta, \mu) \neq 0$. On one hand, using equation (52) gives, near $z = 1$,

$$\langle (\tilde{R}(\lambda_0) - z)^{-1} u, \mu \rangle_{\mathcal{H}^s \times \mathcal{H}^{-s}} = \frac{(\theta, \mu)}{(z-1)^N} + O\bigg(\frac{1}{(1-z)^{N-1}} \bigg). \tag{53}$$

However, on the other hand, for any $|z| > 1$, one has, for any $k \geq 0$,

$$z^{-k} \langle \tilde{R}(\lambda_0)^k u, \mu \rangle = z^{-k} \langle u, \tilde{(}R(\lambda_0)^*)^k \mu \rangle = z^{-k} \langle u, \mu \rangle.$$

Summing the Neumann series then gives

$$|\langle (\tilde{R}(\lambda_0) - z)^{-1} u, \mu \rangle_{\mathcal{H}^s \times \mathcal{H}^{-s}}| \leq \frac{|\langle u, \mu \rangle|}{|z|^{-1} - 1}. \tag{54}$$

Going back to equation (41), this is possible only if $N = 1$. In particular, resonances on the critical axis have no Jordan block. Finally, we use that

$$\tilde{R}(\lambda_0) = \Pi_0(\lambda_0) + K(\lambda_0), \quad K(\lambda_0) \Pi_0(\lambda_0) = \Pi_0(\lambda_0) K(\lambda_0) = 0,$$

where the spectral radius of $K(\lambda_0)$ on $\mathcal{H}^s$ is strictly smaller than 1. Remember that if $r < 1$ is the spectral radius of $K(\lambda_0)$, then for any $\epsilon > 0$ small enough, one has

$$\| K(\lambda_0)^n \|_{\mathcal{H}^s \to \mathcal{H}^s} \leq (r + \epsilon)^n \to 0.$$

Thus, as bounded operators on $\mathcal{H}^s$, we get, as $n \to +\infty$, $\tilde{R}(\lambda_0)^n = \Pi_0(\lambda_0) + K(\lambda_0)^n \to \Pi_0(\lambda_0)$. □

We now identify the co-resonant states associated to resonances on the critical axis to equivariant measures with wavefront set contained in E_s^*, and show that the space of (co)-resonant states at the first resonance is one-dimensional and spanned by the measure constructed in Theorem 3.1. First, define for $v \in C^\infty(\mathcal{M}, [0, +\infty[)$ a (complex) Radon measure μ_v^λ such that

$$\mu_v^\lambda : u \in C^\infty(\mathcal{M}) \mapsto \langle \Pi_0(\lambda) u, v \rangle, \quad \mathrm{WF}(\mu_v^\lambda) \subset E_s^*, \quad \mu_v^\lambda(\mathbf{P} u) = \bar{\lambda} \mu_v^\lambda(u). \tag{55}$$

PROPOSITION 3.1. (Resonant states on the critical axis) *Let $\lambda \in P(V + J^u) + i\mathbb{R}$ be a Ruelle resonance on the critical axis. The space of co-resonant states at λ is equal to $\Pi_0^*(\lambda)(C^\infty(\mathcal{M})) = \Pi_0^*(\lambda)(\mathcal{H}^{-s})$. More precisely, we have the following isomorphism:*

$$\Pi_0^*(\lambda)(C^\infty(\mathcal{M})) = \mathrm{Span}\{\zeta_v^\lambda, v \in C^\infty(\mathcal{M}; [0, +\infty[)\}.$$

Finally, all these measures are absolutely continuous with respect to the measure $\mu_1^{P(V+J^u)}$ with bounded densities.

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Proof. Let $v \in C^\infty(\mathcal{M}, [0, +\infty[)$ and first consider $\lambda = P(V + J^u)$. Then, using Lemma 3.6, we get $\tilde{R}(\lambda)^n \to \Pi_0(\lambda)$ in $\mathcal{L}(\mathcal{H}^s)$. Now, for any $u \in C^\infty(\mathcal{M}, [0, +\infty[)$,

$$\langle \tilde{R}(P(V + J^u))^k u, v\rangle = -\int_0^{+\infty} \chi'^{*k}(t) e^{-tP(V+J^u)} \bigg(\int_{\mathcal{M}} e^{-t\mathbf{P}^*} vu \, d\mathrm{vol} \bigg) \, dt \geq 0.$$

This proves that $\langle \Pi_0(P(V + J^u))u, v\rangle \geq 0$, which gives that μ_v is a non-negative Radon measure for $v \geq 0$. Consider a general resonance on the critical axis $\lambda = ib + P(V + J^u)$ with $b \in \mathbb{R}$ and $v \leq v' \in C^\infty(\mathcal{M}, [0, +\infty[)$, we get

$$\text{for all } u \in C^\infty(\mathcal{M}, \mathbb{R}),\ |\langle \tilde{R}^k(\lambda)u, v\rangle| \leq \langle \tilde{R}^k(P(V + J^u))|u|, v\rangle \leq \langle \tilde{R}^k(P(V + J^u))|u|, v'\rangle.$$

Passing to the limit, we get first that μ_v^λ defines an order 0 distribution and thus defines a (complex) measure. Moreover, the last inequality actually proves that $\mu_v^\lambda \ll \mu_{v'}$ with a density bounded by 1. In particular, one gets that every measure μ_v^λ is absolutely continuous with respect to the measure μ_1. □

We can now prove the first part of Theorem 1.1 for the action on 0-forms.

PROPOSITION 3.2. *Under Assumption 1, the first Ruelle resonance $P(V + J^u)$ for the action on* 0*-forms is simple with a space of co-resonant state spanned by* ν.

Proof. Consider a co-resonant state μ for the first Ruelle resonance $P(V + J^u)$. We know by the previous proposition that μ is a measure with wavefront set contained in E_s^* and we will suppose that it is a non-negative measure. The wavefront set condition, together with [**26**, Corollary 8.2.7], justifies that one can restrict the distribution μ to any unstable manifold $\mathcal{W}^u(x)$. We will denote by $\mu_x := \mu_{|\mathcal{W}^u(x)}$. More explicitly, for a smooth function $f \in C^\infty(\mathcal{M})$, we can define, for some $s \in \mathbb{R}$,

$$(f_{|\mathcal{W}^u(x)}, \mu_x) := (f\delta_{\mathcal{W}^u(x)}, \mu) = (f\delta_{\mathcal{W}^u(x)}, \mu)_{\mathcal{H}^s, \mathcal{H}^{-s}},$$

where the bracket denotes the distributional pairing and $\delta_{\mathcal{W}^u(x)}$ denotes the integration over the unstable manifold $\mathcal{W}^u(x)$ defined by

$$(f, \delta_{\mathcal{W}^u(x)}) := \int_{\mathcal{W}^u(x)} f(y) \, d\mathrm{vol}_{\mathcal{W}^u(x)}(y). \tag{56}$$

The fact that the distributional pairing coincides with the $\mathcal{H}^s \times \mathcal{H}^{-s}$ pairing is a consequence of the wavefront set condition on the distributions. In particular, μ_x is a non-negative distribution as the product of two non-negative distributions. The restrictions $\{\mu_x \mid x \in \mathcal{M}\}$ define a system of $\mathcal{W}^{ws}$-transversal measures in the sense of [**10**]. The strategy to show that the first resonance is simple is to use [**10**, Corollary 3.12], which asserts that the only system (up to constant rescaling) of $\mathcal{W}^{ws}$-transversal measures which satisfies φ_t-conformality (see equation (30)) and the change of variable by holonomy (see equation (35)) is $m^u_{V+J^u}$.

The system of measures $\{\mu_x \mid x \in \mathcal{M}\}$ satisfies φ_t-conformality. This is a consequence of the fact that μ is a co-resonant state. More precisely, for any smooth function $f \in C^\infty(\mathcal{M})$, we can write

$$(f\delta_{\mathcal{W}^u(x)}, \mu)_{\mathcal{H}^s\times\mathcal{H}^{-s}} = e^{-tP(V+J^u)}(e^{t\mathbf{P}}(f\delta_{\mathcal{W}^u(x)}), \mu)_{\mathcal{H}^s\times\mathcal{H}^{-s}}.$$

Using $e^{t\mathbf{P}}(f\delta_{\mathcal{W}^u(x)}) = e^{t\mathbf{P}}fe^{-tX}\delta_{\mathcal{W}^u(x)}$, then yields

$$\int_{\mathcal{W}^u(x)} f(y)\,d\mu_x(y) = \int_{\mathcal{W}^u(\varphi_t x)} e^{-tP(V+J^u)}e^{S_tV(\varphi_{-t}y)}f(\varphi_{-t}y)\det(d\varphi_{-t})_{|E_u(y)}\,d\mu_{\varphi_t x}(y).$$

This rewrites

$$\int_{\mathcal{W}^u(x)} f(y)\,d\mu_x(y) = \int_{\mathcal{W}^u(\varphi_t x)} e^{-tP(V+J_u)}e^{S_t(V+J^u)(\varphi_{-t}y)}f(\varphi_{-t}y)\,d\mu_{\varphi_t x}(y),$$

which is exactly φ_t-conformality for the potential $V + J^u$.

The system of measures $\{\mu_x \mid x \in \mathcal{M}\}$ satisfies the change of variable by holonomy. Investigating the proof of [10, Corollary 3.12], we see that we only need to show the change of variable formula for the standard δ_0-holonomy given by $\mathcal{W}^u(x) \ni z \mapsto \pi(z) := [z, x'] \in \mathcal{W}^u(x')$. We have the following facts.

- For all $x, x' \in \mathcal{M}$, $x \neq x'$ close, one has $L_{x,x'} : \phi \in C^0(\mathcal{M}) \mapsto (\mu_x - \mu_{x'}, \phi)/d(x, x')^\alpha$ is linear.
- For any $\phi \in C^0(\mathcal{M})$, one has $\sup_{x\neq x'} |L_{x,x'}(\phi)| < +\infty$ by equation (A.1).
- The space $C^0(\mathcal{M})$ is a Banach space.

We can apply the Banach–Steinhauss theorem to get $\sup_{x\neq x'} \|L_{x,x'}\|_{\mathrm{op}} = C < +\infty$, i.e.,

$$\text{for all } \phi \in C^0(\mathcal{M}),\ \text{for all } x, x' \in \mathcal{M},\quad |(\mu_x - \mu_{x'}, \varphi)| \leq Cd(x, x')^\alpha.$$

Now consider $\mathcal{W}^u(x) \ni z \mapsto \pi(z) := [z, x'] \in \mathcal{W}^u(x')$. Our goal is to relate μ_x and $\pi^*_{x'}\mu_{x'}$. Consider $f \in C^0(\mathcal{W}^u(x, \delta_0))$. Since the foliation is Hölder, we can extend f locally on a small rectangle by making it constant on weak-stable manifolds, and this defines a continuous function $\bar{f}$. We are now left with relating $\mu_x(\bar{f})$ and $\mu_{x'}(\bar{f})$.

First, consider the Bowen time $t = t(x, x')$ such that $\varphi_t x \in \mathcal{W}^s(x')$. Then, φ_t-conformality shows that $d[(\varphi_t)_*\mu_x]/d\mu_{\varphi_t x}(z) = e^{S_t(V+J^u)(z)-tP(V+J^u)}$. Now, for any $\tau > 0$, we can use φ_t-conformality again to get $d[(\varphi_\tau)_*\mu_{\varphi_t x}]/d\mu_{\varphi_{t+\tau}x}(z) = e^{S_\tau(V+J^u)(z)-\tau P(V+J^u)}$ and $d[(\varphi_\tau)_*\mu_{x'}]/d\mu_{\varphi_\tau x'}(y) = e^{S_\tau(V+J^u)(y)-\tau P(V+J^u)}$. Now, we have that $d(\varphi_{t+\tau}x, \varphi_\tau x') \leq Ce^{-\eta\tau}$ and the above continuity then shows that $|\mu_{\varphi_{t+\tau}x}(\bar{f}) - \mu_{\varphi_\tau x'}(\bar{f})| \to 0$ when $\tau \to +\infty$. This means that the overall holonomy factor is given by the following limit:

$$\exp(\lim_{\tau\to+\infty} S_t(V + J^u)(z) - tP(V + J^u) + S_\tau(V + J^u)(\varphi_{\tau+t}z) - S_\tau(V + J^u)(\varphi_\tau y)),$$

which is exactly equal to $e^{w^+_{V+J^u}(z,y)}$, see the proof of [10, Theorem 3.9] for a more detailed argument.

The first resonance $P(V + J^u)$ is simple. We can use [10, Corollary 3.12] to claim that there is a constant $c > 0$ such that for any $x \in \mathcal{M}$, one has $\mu_x = cm^u_{x,V+J^u}$. We now would like to deduce that this implies $\mu = c\nu$ by proving a Fubini-like formula for distributions with wavefront set in E^*_s. For this, we first need to disintegrate the volume measure with

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respect to the unstable foliation and an arbitrary (smooth) transversal complementary foliation G. More precisely, we will use [4, Proposition 1.6], which states that in a product neighborhood U around a point $q \in \mathcal{M}$, there is a continuous function $\delta_u : U \to \mathbb{R}_+$ such that

$$\text{for all } f \in C^\infty(\mathcal{M}), \int_U f\, d\mathrm{vol} = \int_{G_{\mathrm{loc}}(p)} \left(\int_{\mathcal{W}^u(y)} f(z)\delta_u(z)\, d\mathrm{vol}_{\mathcal{W}^u(y)}(z) \right) d\mathrm{vol}_p^G(y),$$

where $G_{\mathrm{loc}}(p)$ is the connected component of the element of the partition of G containing p. Actually, the *conditional density* δ_u is smooth along the leaves in the following sense:

$$\|\delta_u\|_{C^k(R_q^{ws}(x))} := \sup_{z \in \mathcal{W}^u(x)} \sup_{X_1,\ldots,X_k \in S_z\mathcal{W}^u(x)} |X_1 \cdots X_k(\delta_u(z))_{|\mathcal{W}^u(x)}|$$

is finite for any k and that the norm depends continuously in x. This regularity condition allows to integrate by parts in the inner integral and, in particular, an immediate adaptation of [4, Lemma 1.9] yields $\mathrm{WF}(\delta_u(z)\delta_{\mathcal{W}^u(y)}) \subset E_u^* \oplus E_0^*$. Here, we have denoted by $\delta_u(z)\delta_{\mathcal{W}^u(y)}$ the distribution:

$$f \mapsto \int_{\mathcal{W}^u(y)} f(z)\delta_u(z)\, d\mathrm{vol}_{\mathcal{W}^u(y)}(z).$$

In particular, we consider $f_n \in C^\infty(\mathcal{M})$ such that $f_n \to \mu$ is $\mathcal{D}'_\Gamma(\mathcal{M})$, where Γ is a conic neighborhood of E_s^* which does not intersect $E_u^* \oplus E_0^*$, and we can use the continuity of the distributional product (see [28, Lemma 4.2.7]) to write for any $f \in C^\infty(\mathcal{M})$,

$$\begin{aligned}
(f, \mu) &= \lim_{n\to+\infty} (f, f_n) = \lim_{n\to+\infty} \int_{G_{\mathrm{loc}}(p)} (\delta_u(z)\delta_{\mathcal{W}^u(y)}, f \times f_n)\, d\mathrm{vol}_p^G(y) \\
&= \int_{G_{\mathrm{loc}}(p)} (\delta_u(z)\delta_{\mathcal{W}^u(y)}, f \times \mu)\, d\mathrm{vol}_p^G(y) \\
&= \int_{G_{\mathrm{loc}}(p)} \mu_y(f \times \delta_u)\, d\mathrm{vol}_p^G(y) = c \int_{G_{\mathrm{loc}}(p)} m^u_{y,V+J^u}(f \times \delta_u)\, d\mathrm{vol}_p^G(y) \\
&= c(f, \eta),
\end{aligned}$$

where we used the continuity of the density and the fact that μ_x is of order 0. Together with the previous lemma, this proves that all co-resonant states are proportional, and thus that $P(V + J^u)$ is simple and concludes the proof. □

We can now prove Theorem 1.2 for the action on 0-forms.

PROPOSITION 3.3. *The equilibrium state (see equation (17)) μ_{V+J^u} is equal to $= c\eta \times \nu$ or is given by the averaging formula (8). Finally, if the flow is weak mixing with respect to the equilibrium state μ_{V+J^u}, then the only resonance on the critical axis is $P(V + J^u)$.*

Proof. The co-resonant state ν is absolutely continuous with respect to μ_{V+J^u}. We follow the proof of [10, Theorem 3.10] and prove that there exists a constant $C > 0$ independent of $q \in \mathcal{M}$ and $t \geq 0$ such that

$$\text{for all } t \geq 0, \text{ for all } q \in \mathcal{M}, \quad \nu(B_t(q, r)) \leq C e^{S_t(V+J^u)(q) - tP(V+J^u)}, \tag{57}$$

which suffices as μ_{V+J^u} is the the Gibbs state (see [19, Theorem 4.3.26]). From the proof of [10, Theorem 3.10], we get that

$$\text{there exists } C>0 \text{ for all } q\in\mathcal{M}, \text{ for all } t\geq 0, \quad m^u_{V+J^u}(B_t(q,\delta)) \leq Ce^{S_t(V+J^u)(q)-tP(V+J^u)}.$$

Now, we can use equation (43), the fact that the integrand $e^{w^-_{J^u}(z,x)}$ is uniformly bounded on $\mathcal{M}$, and $m^{ws}_{q,J^u}(R^{ws}_q)$ is uniformly bounded in q (see [10, Theorem 3.1]) to get equation (57).

Product formula and averaging formula. Let η be the resonant state, which we can obtain by applying Theorem 3.1 to the dual $\mathbf{P}^*$. We have $\mathrm{WF}(\eta)\subset E_u^*$, $\mathrm{WF}(\nu)\subset E_s^*$ and we have $E_s^*\cap E_u^*\cap(T^*\mathcal{M}\setminus\{0\})=\emptyset$, which shows that the distributional product of η and ν is well defined. Actually, we can use the duality in equation (21) to define the product:

$$\text{for all } f\in C^\infty(\mathcal{M}), \quad (\eta\times\nu)(f):=(\eta, f\nu)_{\mathcal{H}^s\times\mathcal{H}^{-s}}=(f\eta,\nu)_{\mathcal{H}^s\times\mathcal{H}^{-s}}, \tag{58}$$

where the equality is justified by equation (22). From the fact that ν and η are non-negative measures (see Proposition 3.1), and equation (58), we see that this is also the case of their product $\eta\times\nu$. It is easily seen to be invariant by the flow:

$$\begin{aligned}(\eta\times\nu)(Xf) &= (X^*\eta, f\nu)-(f\eta, X\nu)\\ &= P(V+J^u)(\eta, f\nu)-(V\eta, f\nu)+(\mathrm{div}_{\mathrm{vol}}(X)\eta, f\nu)\\ &\quad -P(V+J^u)(f\eta,\nu)+(f\eta, V\nu)-(\mathrm{div}_{\mathrm{vol}}(X)f\eta,\nu)=0.\end{aligned}$$

We prove that the product is absolutely continuous with respect to μ_{V+J^u}. For this, we use Proposition 3.1 and the previous proposition to get $\Pi_0(P(V+J^u))1=c\eta$ for some $c>0$. Now, this means that we can compute $(\eta\times\nu)(f)$ for $f\in C^\infty(\mathcal{M})$:

$$\begin{aligned}c(\eta\times\nu)(f) &= (\Pi_0(P(V+J^u))1, f\nu)_{\mathcal{H}^s\times\mathcal{H}^{-s}}\\ &= -\lim_{k\to+\infty}\int_0^{+\infty}(\chi')^{*k}(t)e^{-tP(V+J^u)}(1, e^{t\mathbf{P}^*}(f\nu))\,dt\\ &= -\lim_{k\to+\infty}\int_0^{+\infty}(\chi')^{*k}(t)(e^{tX}f,\nu)\,dt.\end{aligned}$$

We use the fact that ν is absolutely continuous with respect to μ_{V+J^u} with bounded density to get that $|\nu(e^{tX}f)|\leq C\mu_{V+J^u}(|e^{tX}f|)\leq C\mu_{V+J^u}(|f|)$ because μ_{V+J^u} is invariant. In particular, we get, by integrating,

$$\text{for all } f\in C^\infty(\mathcal{M}), \quad (\eta\times\nu)(|f|)\leq C'\mu_{V+J^u}(|f|)\Rightarrow(\eta\times\nu)\ll\mu_{V+J^u}.$$

This proves that the density of $(\mu\times\nu)$ is invariant by the flow and measurable for the equilibrium state μ_{V+J^u}. However, this last measure is known to be ergodic so the density is constant, which proves the two formulas.

Weak mixing implies no other resonances on the critical axis. From the averaging formula (8), we see that if ν gives zero measure to an invariant Borel set, then it is also the case for μ_{V+J^u} and the two measures are actually equivalent.

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Next, we use [33, Theorem VII.14] to see that the flow is weakly mixing if and only if the only eigenvalue is 1 and it is a simple eigenvalue. In other words, if

$$\begin{cases} Xf = i\lambda f, \\ f \in L^2(\mathcal{M}, \mu_{V+J^u}) \end{cases} \tag{59}$$

has no solution except for $\lambda = 0$ and f constant. Using Proposition 3.1, the presence of another co-resonant state on the critical axis is equivalent to the existence of an absolutely continuous measure with respect to ν. Its density with respect to ν is in $L^\infty(\mathcal{M}, \nu) = L^\infty(\mathcal{M}, \mu_{V+J^u}) \subset L^2(\mathcal{M}, \mu_{V+J^u})$ and is thus a solution of the system above. This shows that weak mixing of the flow with respect to μ_{V+J^u} implies that there is no other resonance on the critical axis. □

4. *Acting on the bundle of d_s-forms*

We have studied the case of 0-forms before and for this, we have fixed a Riemmanian volume vol to embed smooth functions into distributions. As a consequence, the co-resonant state ν from Theorem 3.1 has a rather convoluted form. Note however that changing the metric does not change the critical axis as the pressure of two cohomologic functions are the same. For d_s-forms, such an identification is not possible as the bundles involved are not line bundles anymore, and we are led to adopt the more general and intrinsic viewpoint of *currents*. Loosely speaking, in this new formalism, a distribution is an element of the topological dual of 0-forms. More generally, we will think of 'distributions' on d_s-forms as continuous linear forms on $C^\infty(\mathcal{M}, \Lambda^{d_s} T^*\mathcal{M})$ and will call this a *d_s-current*. The construction of anisotropic Sobolev space is also valid on this space, as noted before in Remark 2.1. (Smoothness is needed to apply the microlocal approach. Here, the bundles E_s^* and E_u^* are only Hölder continuous but $\mathscr{E}_0^k$ is smooth.) We will focus our attention on forms in the kernel of the contraction by the flow, which are given by

$$\mathscr{E}_0^k := \{u \in C^\infty(\mathcal{M}; \Lambda^k T^*\mathcal{M}) \mid \iota_X u = 0\} = C^\infty(\mathcal{M}; \Lambda^k(E_u^* \oplus E_s^*)).$$

4.1. *Critical axis.*

More generally, the action of $\mathbf{P}$ will be extended, by duality, to *currents*. This idea can be traced back at least to Ruelle and Sullivan although no spectral theory was involved in [34]. We define a k-current T as a continuous linear form on the space of smooth k-forms. On manifolds, especially when no canonical choice of smooth volume form is available, currents allow us to obtain an intrinsic definition of distributions. As such, the space of smooth k-forms is embedded canonically (up to choosing an orientation) in the space of $(n-k)$-currents by the formula

$$\text{for all } \varphi \in C^\infty(\mathcal{M}; \Lambda^k T^*\mathcal{M}), \quad \varphi : \alpha \in C^\infty(\mathcal{M}; \Lambda^{n-k} T^*\mathcal{M}) \mapsto \int_{\mathcal{M}} \alpha \wedge \varphi.$$

Note that the set of smooth k-forms is dense in the space of homogeneous currents of degree $n-k$, see [9, Theorem 12, §15] for a precise statement and more generally [9, Ch. 3] for an introduction to currents. In the rest of the paper, we will write $\mathcal{D}'(\mathcal{M}; \Lambda^{d_u}(E_u^* \oplus E_s^*))$ for the space of sections of currents of degree d_s which are canceled by the contraction ι_X. They can be thought of as linear combinations of elements

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of $\Lambda^{d_u}(E_u^* \oplus E_s^*)$ with distributional coefficients. We first introduce a useful (Hölder continuous) splitting;

$$\Lambda^q(E_u^* \oplus E_s^*) = \bigoplus_{k=0}^{q}(\Lambda^k E_s^* \wedge \Lambda^{q-k} E_u^*) =: \bigoplus_{k=0}^{q} \Lambda_k^q. \tag{60}$$

We prove that the system of (unstable) leaf measures $\{m_x^u \mid x \in \mathcal{M}\}$ defines a section of currents of Sobolev regularity $\mathcal{H}^s(\mathcal{M}; \Lambda^{d_s}(E_u^* \oplus E_s^*))$ and is actually a co-resonant state for $\lambda = P(V)$. This point reduces to a computation of the wavefront set of the system of leaf measures m_V^u by equation (26). This is done by adapting the argument of Lemma 3.2 and we note that the proof is actually easier in this case as we do not have to deal with the smoothness of the unstable Jacobian J^u along the leaves. We will then use the 'L^1-norm' associated to m_V^u to prove that no Ruelle resonances exist in the half-plane $\{\mathrm{Re}(\lambda) > P(V)\}$ by mimicking the proof of Lemma 3.4.

However, because we work on forms, the co-resonant state m_V^u will be non-zero only when tested on sections with values in $\Lambda_0^{d_s}$ and will fail to define a norm on the rest of the decomposition. This will explain the need of a more complicated norm, which we will construct in Proposition 4.2.

More importantly, the decomposition in equation (67) is only Hölder continuous which prevents us from applying the microlocal strategy to obtain a meromorphic extension of the resolvent acting on sections with values in $\Lambda_k^{d_s}$.

Indeed, in general, one can only expect to have a Hölder continuous section $C^\alpha(\mathcal{M} : \Lambda_k^{d_s})$ for some $\alpha > 0$. The pairing against m_V^u will however still be justified because the co-resonant state is of order 0.

PROPOSITION 4.1. *The system of leaf measures $\{m_x^u \mid x \in \mathcal{M}\}$ from equation (29) defines a section of $\mathcal{D}'(\mathcal{M}; \Lambda^{d_u}(E_u^* \oplus E_s^*))$, which we will denote by m_V^u. Moreover, one has*

$$\mathcal{L}_{\mathbf{P}^*} m_V^u = P(V) m_V^u, \quad \mathrm{WF}(m_V^u) \subset E_s^*, \tag{61}$$

and thus $m_V^u \in \mathcal{H}^s(\mathcal{M}; \Lambda^{d_u}(E_u^* \oplus E_s^*))$ and it is a co-resonant state associated to the Ruelle resonance $P(V)$. We will call $P(V)$ the first resonance.

Moreover, for any $1 \le k \le d_s$ and $\omega_k \in C^\alpha(\mathcal{M}; \Lambda_k^{d_s})$, one has $m_V^u(\omega_k) = 0$.

Proof. Our first goal is to give meaning to $m_V^u(\varphi)$ for $\varphi \in C^\infty(\mathcal{M}; \Lambda^{d_s} T^*\mathcal{M})$. First, the compatibility statement allows us to only define the duality locally, so let $\varphi \in C^\infty(\mathcal{M}; \Lambda^{d_s} T^*\mathcal{M})$ be supported in R_q. We can define the duality as follows:

$$m_V^u(\varphi) := \int_{\mathcal{M}} m_V^u \wedge \alpha \wedge \varphi = \int_{\mathcal{W}^u(q,\delta)} \left(\int_{R_q^{ws}(x)} e^{w_V^+(y,x)} (\varphi \wedge \alpha)(y) \right) dm_V^u(x), \tag{62}$$

where $\alpha \in E_0^*$, $\alpha(X) = 1$. We see that the previous definition makes sense as $\varphi \wedge \alpha$ is a $d_s + 1$ form and as $w_V^+(x, y)$ is smooth in y, in the sense of Lemma 3.2, and continuous in x. The formula clearly defines a current of degree 0 so $m_V^u \in D'(\mathcal{M}; \Lambda^{d_u}(E_u^* \oplus E_s^*))$.

Let $\omega_k \in C^\alpha(\mathcal{M} : \Lambda_k^{d_s})$ for some k. Then, the fact that m_V^u is a measure on $\mathcal{W}^u(q, \delta)$ shows that the pairing $m_V^u(\omega_k)$ is well defined. If $k \ge 1$, we can use equation (62) and the definition of E_s^* to get that $m_V^u(\omega_k) = 0$.

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To prove the first part of equation (61), we can consider a d_s form φ supported in R_q such that $e^{-t\mathbf{P}}\varphi$ is also supported in R_q. We have to prove that $m_V^u(e^{t\mathbf{P}}\varphi) = e^{tP(V)}m_V^u(\varphi)$. We first use the Leibniz rule to get $\mathcal{L}_X(\varphi \wedge \alpha) = \mathcal{L}_X\varphi \wedge \alpha + \varphi \wedge \mathcal{L}_X\alpha = \mathcal{L}_X\varphi \wedge \alpha$. This allows us to compute explicitly the action of the propagator on the current m_V^u. More precisely, we use the cocycle relation in equation (34), equation (33), as well as φ_t conformality in equation (30):

$$\begin{aligned} m_V^u(e^{t\mathbf{P}}\varphi) &= \int_{\mathcal{W}^u(q,\delta)} \bigg(\int_{R_q^{ws}(x)} e^{w_V^+(y,x)} e^{S_t V(\varphi_{-t} y)} e^{t\mathcal{L}_X}(\varphi \wedge \alpha)(y) \bigg)\, dm_V^u(x) \\ &= \int_{\mathcal{W}^u(q,\delta)} \bigg(\int_{R_q^{ws}(\varphi_t x)} e^{w_V^+(\varphi_t w,x)} e^{S_t V(w)} (\varphi \wedge \alpha)(w) \bigg)\, dm_V^u(x) \\ &= \int_{\mathcal{W}^u(q,\delta)} e^{w_V^+(\varphi_t x,x)} \bigg(\int_{R_q^{ws}(\varphi_t x)} e^{w_V^+(\varphi_t w,w)} e^{w_V^+(w,\varphi_t x)} e^{S_t V(w)} (\varphi \wedge \alpha)(w) \bigg)\, dm_V^u(x) \\ &= \int_{\mathcal{W}^u(q,\delta)} e^{2tP(V)} e^{-S_t V(x)} \bigg(\int_{R_q^{ws}(\varphi_t x)} e^{w_V^+(w,\varphi_t x)} (\varphi \wedge \alpha)(w) \bigg)\, dm_V^u(x) \\ &= e^{tP(V)} \int_{\mathcal{W}^u(q,\delta)} \bigg(\int_{R_q^{ws}(z)} e^{w_V^+(z,w)} (\varphi \wedge \alpha)(z) \bigg)\, dm_V^u(z) = e^{tP(V)} m_V^u(\varphi). \end{aligned}$$

For the wavefront set condition, we mimic the argument of Lemma 3.2, and consider a smooth d_s-form χ supported in R_q and S a phase function such that $dS(q) = \xi \notin E_s^*$, and compute

$$m_V^u(e^{i(S/h)}\chi) = \int_{\mathcal{W}^u(q,\delta)} \bigg(\int_{R_q^{ws}(x)} e^{w_V^+(y,x)} e^{i(S(y)/h)} (\chi \wedge \alpha)(y) \bigg)\, dm_V^u(x).$$

Now, the proof is easier than for Lemma 3.2 as the integrand is easily seen to be smooth along the weak-stable leaves (because the potential V is smooth) uniformly in x. We can perform integration by parts and show that the integrand is $O(h^\infty)$ as long as dS does not vanish on $R_q^{ws}(x)$, which can be ensured near q by the definition of E_s^*. This shows that $\xi \notin \mathrm{WF}(m_V^u)$ and thus $\mathrm{WF}(m_V^u) \subset E_s^*$. This shows that $P(V)$ is a Ruelle resonance with the associated Ruelle co-resonant state given by m_V^u. □

If we consider the adjoint $\mathbf{P}^*$, the above theorem would give that m_V^s is a resonant state for $P(V)$. We also get a product construction of the equilibrium measure. In this case, it is actually easier than in the case of functions, as the product is given by the usual wedge product (extended to distributions with convenient wavefront sets). The following lemma is a re-writing of [**10**, Theorem 3.10] and corresponds to the second part of Theorem 1.2.

LEMMA 4.1. (Equilibrium state as a product) *There exists $c > 0$ such that*

$$m_V^u \wedge \alpha \wedge m_V^s = c\mu_V.$$

Similarly to the case of functions, we now prove that $\{\mathrm{Re}(\lambda) = P(V)\}$ is the critical axis by using the 'L^1-norm' associated to the co-resonant state m_V^u. Nevertheless, because we work on forms, the following norm will only be non-zero for a section with values in $\Lambda_0^{d_s}$.

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LEMMA 4.2. *We define a norm on $C^0(\mathcal{M}; \Lambda_0^{d_s})$ by posing*

$$\text{for all } \varphi \in C^0(\mathcal{M}; \Lambda_0^{d_s}), \quad \|\varphi\|_{V,0} := m_V^u(|\varphi|). \tag{63}$$

This norm satisfies the bound

$$\text{for all } \varphi \in C^0(\mathcal{M}; \Lambda_0^{d_s}), \quad \|e^{t\mathbf{P}}\varphi\|_{V,0} \leq e^{tP(V)}\|\varphi\|_{V,0}. \tag{64}$$

Proof. Suppose $\varphi \in C^0(\mathcal{M}; \Lambda_0^{d_s})$, the bundle is one-dimensional and it thus makes sense to talk of $|\varphi| \wedge \alpha$ as a $d_s + 1$ density. If $\varphi(q) \neq 0$, then by continuity, $\varphi \neq 0$ on a small open set. Then, $\|\varphi\|_{V,0} > 0$ because m_V^u gives a positive measure to any open set. The bound in equation (64) follows from the fact that m_V^u is a co-resonant state. The last point follows from the first two points by a direct adaptation of the proof of Lemma 3.4. □

Remark 4.1. The one-dimensional nature of $\Lambda_0^{d_s}$ will be crucial to obtain a fine description of the resonant states on the critical axis. Recall that in the proof of Proposition 3.1, it was shown that the space of resonant states was given by $\Pi_0(C^\infty)$ and that $u \leq Cv$ implied that $\Pi_0(u)$ was absolutely continuous with respect to $\Pi_0(v)$. One could then deduce the simplicity of the first resonance from the ergodicity of the flow and the non-presence of other resonances on the critical axis by the mixing properties of the flow. Here, we will use a similar approach by showing that the set of resonant states is given by $\Pi_0(\Lambda_0^{d_s})$.

We now use this norm and a 'shift' to define inductively a norm on $C^0(\mathcal{M}; \Lambda_k^{d_s})$ for any k. We use Assumption 1 to define the set of (normalized) trivialization of E_u:

$$\mathscr{V}^u := \{(X_{u,h}^j)_{1\leq j\leq d_u} \in C^0(\mathcal{M}; E_u) \mid (E_u) = \operatorname{Span}\{X_{u,h}^j,\ 1 \leq j \leq d_u\},\ \|X_{u,h}^j\|_{C^0} = 1\},$$

and its dual counterpart

$$\mathscr{F}^u := \{(Y_{u,h}^j)_{1\leq j\leq d_s} \in C^0(\mathcal{M}; E_u^*) \mid (E_u^*) = \operatorname{Span}\{Y_{u,h}^j,\ 1 \leq j \leq d_s\},\ \|Y_{u,h}^j\|_{C^0} = 1\}.$$

PROPOSITION 4.2. *We define a norm inductively on $C^0(\mathcal{M}; \Lambda_k^{d_s})$, $k \geq 1$, by posing, for $f \in C^0(\mathcal{M}; \Lambda_k^{d_s})$,*

$$\|f\|_{V,k} := \sup_{(X_{u,h}^j)\in\mathscr{V}^u} \sup_{(Y_{u,h}^i)\in\mathscr{F}^u} \sum_{i=1}^{d_s}\sum_{j=1}^{d_u} \|\iota_{X_u^j} f \wedge Y_u^i\|_{V,k-1}. \tag{65}$$

This norm satisfies the bound

$$\text{for all } \varphi \in C^0(\mathcal{M}; \Lambda_k^{d_s}), \quad \|e^{t\mathbf{P}}\varphi\|_{V,k} \leq Ce^{t(P(V)-k\eta)}\|\varphi\|_{V,k} \tag{66}$$

for some $C, \eta > 0$. As a consequence, there are no Ruelle resonances in $\{\operatorname{Re}(\lambda) > P(V)\}$ and if λ is a Ruelle resonance on the critical axis, then it has no Jordan block.

Proof. Contracting with a vector in E_u and then wedging with a vector in E_u^* sends a section of $C^0(\mathcal{M}; \Lambda_k^{d_s})$ to $C^0(\mathcal{M}; \Lambda_{k-1}^{d_s})$, so the formula makes sense by induction. Let us prove that the norm takes finite values. For this, we use the fact that m_V^u is of order zero and Lemma 4.2 to get first that for $f \in C^0(\mathcal{M}; \Lambda_0^{d_s})$, one has $\|f\|_{V,0} \leq C\|f\|_0$. This implies, for any $f \in C^0(\mathcal{M}; \Lambda_1^{d_s})$,

$$\sum_{i=1}^{d_u}\sum_{j=1}^{d_s} \|\iota_{X_u^j} f \wedge Y_u^i\|_{V,0} \le C' \sum_{i,j} \|X_u^j\|_{C^0}\|Y_u^i\|_{C^0}\|f\|_0 \le C''\|f\|_0$$

with a constant C'' independent of the choice of cover or partition of unity. This shows that $\|.\|_{V,1}$ takes finite values and a quick induction shows that it is also the case of $\|.\|_{V,k}$ for any k. The triangle inequality, homogeneity, and non-negativity of $\|.\|_{V,k}$ follow from Lemma 4.2 and an induction. Suppose now that $f \in C^0(\mathcal{M}; \Lambda_k^{d_s})$ and $f \ne 0$. Then, f is non-zero on a small open set, and since $(X_{s,h}^j)$ and (Y_u^i) are local bases, then there is (i_0, j_0) such that $\iota_{X_{s,h}^{j_0}} f \wedge Y_u^{i_0}$ is continuous and non-zero on that open set. By induction and the fact that m_V^u gives a positive measure to any non-empty open set, we get that $\|f\|_{V,k} > 0$ and $\|.\|_{V,k}$ thus defines a norm. We prove equation (66):

$$\begin{aligned}\iota_{X_u^j} e^{t\mathbf{P}} f \wedge Y_u^i &= e^{t\mathbf{P}}(\iota_{(d\varphi_{-t})_{\varphi_t y}(X_u^j)} f \wedge e^{t\mathcal{L}_X} Y_u^i)(y)\\ &= e^{t\mathbf{P}} \iota_{(d\varphi_{-t})_{\varphi_t y}(X_u^j)(\varphi_t y)} f(y) \wedge e^{t\mathcal{L}_X} Y_u^i(y).\end{aligned}$$

Using the invariance of the Anosov decomposition in equation (13), we get

$$\begin{aligned}&\sum_{i=1}^{d_s}\sum_{j=1}^{d_u} e^{t\mathbf{P}} \iota_{(d\varphi_{-t})_{\varphi_t y}(X_u^j)(\varphi_t y)} f(y) \wedge e^{t\mathcal{L}_X} Y_u^i(y)\\ &\quad = \sum_{i,j} \|e^{t\mathcal{L}_X} Y_u^i\| \|(d\varphi_{-t})_{\varphi_t y}(X_u^j)\| e^{t\mathbf{P}} \iota_{(d\varphi_{-t})_{\varphi_t y}(X_u^j)/\|(d\varphi_{-t})_{\varphi_t y}(X_u^j)\|} f(y) \wedge \frac{e^{t\mathcal{L}_X} Y_u^i}{\|e^{t\mathcal{L}_X} Y_u^i\|}.\end{aligned}$$

We then have $\|e^{t\mathcal{L}_X} Y_u^i\|, \|(d\varphi_{-t})_{\varphi_t y}(X_u^j)\| \le Ce^{-t\eta}$ for some (uniform) $C, \eta > 0$ by the Anosov property. Finally, we obtain

$$\sum_{i=1}^{d_s}\sum_{j=1}^{d_u} \|\iota_{X_u^j} e^{t\mathbf{P}} f \wedge Y_u^i\|_{V,k-1} \le Ce^{-2\eta t} \sum_{i=1}^{d_s}\sum_{j=1}^{d_u} \|e^{t\mathbf{P}}(\iota_{\tilde{X}_u^j} f \wedge \tilde{Y}_u^i)\|_{V,k-1}$$

for $\tilde{X}_u^j := (d\varphi_{-t})_{\varphi_t y}(X_u^j)/\|(d\varphi_{-t})_{\varphi_t y}(X_u^j)\|$ and $\tilde{Y}_u^i = e^{t\mathcal{L}_X} Y_u^i/\|e^{t\mathcal{L}_X} Y_u^i\|$. Passing to the supremum thus gives

$$\|e^{t\mathbf{P}} f\|_{V,k-1} \le Ce^{(P(V)-2(k-1)\eta)t}\|f\|_{V,k-1} \Rightarrow \|e^{t\mathbf{P}} f\|_{V,k} \le Ce^{(P(V)-2k\eta)t}\|f\|_{V,k}.$$

To conclude the proof of equation (66), we only need to initialize the induction, and this is exactly the statement of Lemma 4.2.

Consider $f \in C^\infty(\mathcal{M}; \mathscr{E}_{d_s})$ and consider its decomposition

$$f = \sum_{k=0}^{d_s} \omega_k, \quad \omega_k \in C^\alpha(\mathcal{M}; \Lambda_k^{d_s}).$$

We define a norm on $C^\infty(\mathcal{M}; \mathscr{E}_{d_s})$ with the following bound when using the propagator:

$$\|f\|_V := \sum_{k=0}^{d_s} \|\omega_k\|_{V,k}, \quad \|e^{t\mathcal{L}_X} f\|_V \le Ce^{tP(V)}\|f\|_V.$$

This is the only thing we need to mimic the argument of Lemmas 3.4 and 3.5, and prove that no resonance exists in $\{\mathrm{Re}(\lambda) > P(V)\}$ and that resonances on the critical axis have no Jordan block. $\square$

We will now prove Theorem 1.1 for the action on d_s-forms.

PROPOSITION 4.3. (Critical axis for d_s-forms) *Under Assumption 1, the first resonance $P(V)$ is simple and the space of co-resonant states is spanned by m_V^u.*

Proof. The argument of Lemma 3.6 goes through and the projector $\Pi_0(\lambda)$ on the resonant states is obtained as the limit of $\tilde{R}(\lambda)^n$ as an operator $\mathcal{H}^s \to \mathcal{H}^s$. The formula defining $\tilde{R}(\lambda)$ is the same. In particular, as in the proof of Proposition 3.1, we obtain every co-resonant state by the following construction:

$$\text{for all } \lambda \in P(V) + i\mathbb{R}, \text{ for all } v \in C^\infty(\mathcal{M}; \Lambda^{d_s}(E_u^* \oplus E_s^*)), \quad \Pi_0(\lambda)^*(v) = \lim_{n\to+\infty} (\tilde{R}(\lambda)^*)^n(v),$$

where the convergence holds in $\mathcal{H}^s$. Consider the pairing between $C^\infty(\mathcal{M}; \Lambda^{d_s}(E_u^* \oplus E_s^*))$ and $C^\infty(\mathcal{M}; \Lambda^{d_u}(E_u^* \oplus E_s^*))$ defined by $\langle v, u\rangle := \int_{\mathcal{M}} u \wedge \alpha \wedge v$, extended to $\mathcal{H}^s \times \mathcal{H}^{-s}$.

Co-resonant states define currents of order 0. Consider two smooth forms $\omega \in C^\infty(\mathcal{M}; \Lambda^{d_s}(E_u^* \oplus E_s^*))$ and $\theta \in C^\infty(\mathcal{M}; \Lambda^{d_u}(E_u^* \oplus E_s^*))$. We first expand these forms:

$$\omega = \sum_{k=0}^{d_s} \omega_k, \quad \omega_k \in C^\alpha(\mathcal{M}; \Lambda_k^{d_s}), \quad \theta = \sum_{k=0}^{d_s} \theta_k, \quad \theta_k \in C^\alpha(\mathcal{M}; \Lambda_k^{d_u}). \tag{67}$$

Note that the regularity of the component is only Hölder continuous here because of the regularity of the stable and unstable foliation. We see that $\omega_k \wedge \alpha \wedge \theta_l$ is a continuous n-form which is non-zero if and only if $k + l = d_s$. The previous discussion justifies the following convergence:

$$\begin{aligned}\langle \omega, \Pi_0(\lambda)^*\theta\rangle &= -\lim_{k\to+\infty} \int_{\mathbb{R}} (\chi'(t))^{*k} e^{-t\lambda} \langle e^{t\mathbf{P}}\omega, \theta\rangle \, dt \\ &= -\lim_{k\to+\infty} \int_{\mathbb{R}} (\chi'(t))^{*k} e^{-t\lambda} \sum_{k=0}^{d_s} \langle e^{t\mathbf{P}}\omega_k, \theta_{d_s-k}\rangle \, dt.\end{aligned}$$

Here, we have noticed that the integration over all the manifold is an n-current of order zero and thus extends to continuous n-forms.

By Assumption 1, we can find a trivialization basis $(Y_u^j)_{1\le j\le d_s} \in C^0(\mathcal{M}; E_u^*)$ such that $E_u^* = \mathrm{Span}\{Y_u^j, \ 1 \le j \le d_s\}$ and $(Y_s^j)_{1\le j\le d_u} \in C^0(\mathcal{M}; E_s^*)$ such that $E_s^* = \mathrm{Span}\{Y_s^j, \ 1 \le j \le d_u\}$. Let $Y_s^1 \wedge \cdots \wedge Y_s^{d_s} \wedge \alpha \wedge Y_s^1 \wedge \cdots \wedge Y_s^{d_u} =: \tilde{\omega} \wedge \alpha \wedge \tilde{\theta}$ be a non-vanishing continuous n-form which we suppose to be positive. We notice to start that, using the Anosov property, there is an $\eta > 0$ such that

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$$\begin{aligned}\int_{\mathcal{M}} &|e^{t\mathbf{P}}(Y_u^1 \wedge \cdots Y_u^{d_s-1} \wedge Y_s^1) \wedge \alpha \wedge Y_u^{d_s} \wedge Y_s^2 \wedge \cdots \wedge Y_s^{d_s}| \\ &\leq Ce^{-t\eta} \int_{\mathcal{M}} |e^{t\mathbf{P}}(Y_u^1 \wedge \cdots Y_u^{d_s-1}) \wedge \alpha \wedge Y_u^{d_s} \wedge Y_s^1 \wedge Y_s^2 \wedge \cdots \wedge Y_s^{d_s}| \\ &\leq Ce^{-2t\eta} \int_{\mathcal{M}} |e^{t\mathbf{P}}(Y_u^1 \wedge \cdots Y_u^{d_s-1} \wedge Y_u^{d_s}) \wedge \alpha \wedge Y_s^1 \wedge Y_s^2 \wedge \cdots \wedge Y_s^{d_s}| \\ &\leq Ce^{-2t\eta} \int_{\mathcal{M}} e^{t\mathbf{P}}(\tilde{\omega}) \wedge \alpha \wedge \tilde{\theta}.\end{aligned}$$

Thus, a quick induction yields the following bound:

$$\text{for all } 0 \leq k \leq d_s, \quad |\langle e^{t\mathbf{P}}\omega_k, \theta_{d_s-k}\rangle| \leq C\|\omega\|_{C^0} e^{-2tk\eta} \langle e^{t\mathbf{P}}\tilde{\omega}, \tilde{\theta}\rangle.$$

Now, choose a smooth section ω and θ such that ω_0 and θ_{d_s} are non-vanishing on $\mathcal{M}$. Note that such an ω can be obtained by approximating in C^0 norm a non-vanishing continuous section by smooth sections. Then, we get the convergence of

$$\ell := \langle \omega, \Pi_0(\lambda)^*\theta\rangle = -\lim_{k\to+\infty} \int_{\mathbb{R}} (\chi'(t))^{*k} e^{-t\lambda} \langle e^{t\mathbf{P}}\omega_0, \theta_0\rangle (1 + O(e^{-\eta t}))\, dt.$$

Consider thus an arbitrary $\epsilon > 0$ and an integer $k_0 \geq 0$ such that for any $k \geq k_0$, one has

$$-\epsilon \leq \left| \int_{\mathbb{R}} (\chi'(t))^{*k} e^{-t\lambda} \langle e^{t\mathbf{P}}\omega_0, \theta_0\rangle (1 + O(e^{-\eta t}))\, dt + \ell \right| \leq \epsilon.$$

The support of the $(\chi'(t))^{*k}$ goes to infinity with k, so we can suppose that for $k \geq k_0$, one has $1 - \epsilon \leq 1 + O(e^{-\eta t}) \leq 1 + \epsilon$ on the support of the integrand. In other words, we have obtained

$$\text{for all } k \geq k_0, \quad \frac{-\epsilon}{1+\epsilon} \leq \left| \int_{\mathbb{R}} (\chi'(t))^{*k} e^{-t\lambda} \langle e^{t\mathbf{P}}\omega_0, \theta_0\rangle\, dt + \ell \right| \leq \frac{\epsilon}{1-\epsilon}.$$

This also proves the boundedness of the above integral with ω_0 and θ_0 replaced by $\tilde{\omega}$ and $\tilde{\theta}$, and this shows that

$$\begin{aligned}\langle \omega, \Pi_0(\lambda)^*\theta\rangle &= \lim_{k\to+\infty} \int_{\mathbb{R}} (\chi'(t))^{*k} e^{-t\lambda} \langle e^{t\mathbf{P}}\omega_0, \theta_0\rangle\, dt \\ &\leq C\|\omega\|_0 \limsup \int_{\mathbb{R}} |(\chi'(t))^{*k}| e^{-tP(V)} \langle e^{t\mathbf{P}}\tilde{\omega}, \tilde{\theta}\rangle\, dt \leq C'\|w\|_{C^0}.\end{aligned}$$

The above bound actually holds for any θ and ω, and we get that every co-resonant state is an order zero current.

Restrictions of co-resonant state on unstable manifolds are well defined. Following the strategy of the proof of Proposition 3.2, we consider a co-resonant state θ which satisfies $\mathrm{WF}(\theta) \subset E_s^*$. For any $x \in \mathcal{M}$, we can thus define the restriction of θ to $\mathcal{W}^u(x)$. We will denote $\theta_x := \theta_{|\mathcal{W}^u(x)}$. More explicitly, for a smooth function $f \in C^\infty(\mathcal{M})$, we can define

$$(f_{|\mathcal{W}^u(x)}, \theta_x) := (f[\mathcal{W}^u(x)], \theta) = (f[\mathcal{W}^u(x)], \theta)_{\mathcal{H}^s, \mathcal{H}^{-s}},$$

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where the bracket denotes the distributional pairing and $[\mathcal{W}^u(x)]$ denotes the d_u-current which consists in integrating over the unstable manifold $\mathcal{W}^u(x)$. Then, θ_x is seen to be of order zero, in other words, $\{\theta_x \mid x \in \mathcal{M}\}$ defines a system of measures on the unstable leaves.

The system of measures $\{\theta_x \mid x \in \mathcal{M}\}$ satisfies φ_t-conformality. This is again a consequence of the fact that θ is a co-resonant state. More precisely, for any smooth function $f \in C^\infty(\mathcal{M})$, we can write

$$(f[\mathcal{W}^u(x)], \theta)_{\mathcal{H}^s \times \mathcal{H}^{-s}} = e^{-tP(V)}(e^{t\mathbf{P}}(f[\mathcal{W}^u(x)], \theta)_{\mathcal{H}^s \times \mathcal{H}^{-s}}.$$

Using $e^{t\mathbf{P}}(f[\mathcal{W}^u(x)]) = e^{t\mathbf{P}} f[\mathcal{W}^u(\varphi_t x)]$ then yields

$$\theta_x(f) = e^{-tP(V)}\theta_{\varphi_t x}(e^{S_t V(\varphi_{-t} y)} f(\varphi_{-t} y)),$$

which is exactly the φ_t-conformality for the potential V.

The system of measures $\{\theta_x \mid x \in \mathcal{M}\}$ satisfies the change of variable by holonomy. This a consequence of the continuity of $x \mapsto \theta_x$, which is shown as in the proof of Proposition 3.2. Indeed, another way of obtaining θ_x is to write θ as a linear combination of smooth d_u-form with distributional coefficients in $\mathcal{H}^{-s}$. Then, the restriction θ_x is obtained by pulling back the smooth forms and the coefficients. The pullback on the coefficients are well defined by the wavefront set condition and we deduce the continuity property by the continuity of the coefficients.

Following the proof of Proposition 3.2 and using [**10**, Corollary 3.12], we obtain the existence of a constant $c > 0$ such that for any $x \in \mathcal{M}$, one has $\theta_x = cm^u_{x,V}$.

The first resonance $P(V)$ is simple. The first part of the proof shows that it suffices to prove that for any $\omega \in C^\alpha(\mathcal{M}; \Lambda_0^{d_s})$, one has $(\omega, \theta) = c(\omega, m^u_V)$. Note that such a ω writes $\omega = f \mathrm{vol}_{\mathcal{W}^u(x)}$ for some $f \in C^0(\mathcal{M})$, we can then write

$$\theta_x(f) := (f[\mathcal{W}^u(x)], \theta)_{\mathcal{H}^s \times \mathcal{H}^{-s}} = (f\delta_{\mathcal{W}^u(x)}, \mathrm{vol}_{\mathcal{W}^u(x)} \wedge \alpha \wedge \theta)_{\mathcal{H}^s \times \mathcal{H}^{-s}},$$

where $\delta_{\mathcal{W}^u(x)}$ is the integration of function on the unstable manifold of x (see equation (56)). Remark that here, the last bracket is the usual distributional pairing. In particular, θ_x is the restriction to the unstable manifold $\mathcal{W}^u(x)$ of the measure $\mathrm{vol}_{\mathcal{W}^u} \wedge \alpha \wedge \theta$. Using the last part of the proof of Proposition 3.2 then yields that

$$\text{for all } f \in C^0(\mathcal{M}), \quad (\omega, \theta) = (\mathrm{vol}_{\mathcal{W}^u} \wedge \alpha \wedge \theta)(f) = c(\mathrm{vol}_{\mathcal{W}^u} \wedge \alpha \wedge m^u_V) = c(\omega, m^u_V).$$

This concludes the proof. □

We can now prove the second part of Theorem 1.2 for d_s-forms.

PROPOSITION 4.4. *The equilibrium state μ_V is given by the averaging formula (8).*

Finally, if the flow is weak mixing with respect to the equilibrium state μ_V, then the only resonance on the critical axis is $P(V)$.

Proof. The averaging formula is derived from the product formula of Lemma 4.1 and the expression of the projector $\Pi_0(P(V))$ using the exact same argument as in the proof of Proposition 3.3. Suppose now that there is another co-resonant state θ associated to a resonance $P(V) + i\lambda$ on the critical axis. We use the fact that θ is of order 0 as well as

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[35, Ch. 6, equation (2.6)] to get that there exists a Borel measure μ_θ and a μ_θ^λ-measurable function T_θ with values in $\Lambda^{d_s}(E_u^* \oplus E_s^*)$ normalized μ_θ^λ-almost everywhere such that

$$\text{for all } \omega \in C^\infty(\mathcal{M}; \Lambda^{d_s}(E_u^* \oplus E_s^*)), \quad \langle \omega, \theta \rangle = \int_{\mathcal{M}} \langle \omega(x), T_\theta(x) \rangle \, d\mu_\theta^\lambda(x),$$

where the bracket here denotes the scalar product on $\Lambda^{d_s}(E_u^* \oplus E_s^*)$. Moreover, the measure μ_θ^λ is defined by

$$\text{for all } U \subset \mathcal{M} \text{ open set}, \quad \mu_\theta^\lambda(U) := \sup_{\|w\|_{C^0} \le 1, \operatorname{supp}(\omega) \subset U} \langle \omega, \theta \rangle. \tag{68}$$

The proof of the previous proposition showed that there is a $C > 0$ such that

$$\text{for all } \omega \in C^\infty(\mathcal{M}; \Lambda^{d_s}(E_u^* \oplus E_s^*)), \quad \langle \omega, \theta \rangle \le C \langle \theta, m_V^u \rangle \Rightarrow \mu_\theta \ll \mu_{m_V^u} =: \mu.$$

Now, using equations (62), (68), as well as the bound

$$\text{there exists } C > 0 \quad \text{for all } q \in \mathcal{M}, \quad \text{for all } t \ge 0, \quad m_V^u(B_t(q, \delta)) \le C e^{S_t(V)(q) - tP(V)},$$

(which can be found in the proof of [10, Theorem 3.10]), we get that μ satisfies an upper Gibbs bound. In other words, we have $\mu \ll \mu_V$. In particular, the Radon–Nikodym density $h := d\mu_\theta^\lambda / d\mu$ is an element of $L^\infty(\mathcal{M}, \mu_V)$.

We use [33, Theorem VII.14] to see that the flow is weakly mixing if and only if the only eigenvalue is 1 and it is a simple eigenvalue. In other words, if

$$\begin{cases} Xf = i\lambda f, \\ f \in L^2(\mathcal{M}, \mu_V) \end{cases} \tag{69}$$

has no solution except for $\lambda = 0$ and f constant. However, then, we have shown that h is a non-trivial solution of the above system which contradicts weak mixing. □

5. *Other critical axes and regularity of the pressure*

In this section, we will prove the statement about critical axes of Theorem 1.1 as well as Corollary 1.1.

Proof. We start from the Guillemin trace formula,

$$e^{\lambda t_0} \operatorname{tr}^\flat(\varphi_{-t_0}^* ((\mathbf{P})_{\mathscr{E}_0^k} - \lambda)^{-1}) = \sum_{\gamma \in \Gamma} \frac{T_{\gamma^\sharp} e^{-(\lambda + V_\gamma) T_\gamma} \operatorname{tr}(\Lambda^k \mathcal{P}_\gamma)}{|\det(\mathrm{Id} - \mathcal{P}_\gamma)|}.$$

We know by Lemma 4.2 that the critical axis of $(\mathbf{P})_{\mathscr{E}_0^{d_s}}$ is $\{\mathrm{Re}(\lambda) = P(V)\}$. We will prove that the other functions are analytic in a half-plane $\{\mathrm{Re}(\lambda) > P(V) - \epsilon\}$ for some $\epsilon > 0$. We list the eigenvalues of the Poincaré map

$$e^{\lambda_1^-(\gamma)} \le \cdots \le e^{\lambda_{d_u}^-(\gamma)} \le -e^{\eta T_\gamma} \le e^{\eta T_\gamma} \le e^{\lambda_1^+(\gamma)} \le \cdots \le e^{\lambda_{d_s}^+(\gamma)} \tag{70}$$

for some uniform constant $\eta > 0$ given by the Anosov property. Now we can compute

$$\operatorname{tr}(\Lambda^k \mathcal{P}_\gamma) := \sigma_k(e^{\lambda_1^-(\gamma)}, \ldots, e^{\lambda_{d_u}^-(\gamma)}, e^{\lambda_1^+(\gamma)}, \ldots, e^{\lambda_{d_s}^+(\gamma)}),$$

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where σ_k is the kth symmetric polynomial. We see that the maximum value of $\mathrm{tr}(\Lambda^k \mathcal{P}_\gamma)$ is attained at $k = d_s$, where one can choose all eigenvalues larger than 1 without choosing any other eigenvalues. In particular, there is a constant $C > 0$, independent of γ and k such that if $k \neq d_s$, then

$$|\mathrm{tr}(\Lambda^k \mathcal{P}_\gamma)| \leq C \ \mathrm{tr}(\Lambda^{d_s} \mathcal{P}_\gamma) e^{-\eta T_\gamma}.$$

However now, we know that for $k = d_s$, the trace is analytic and converges in $\{\mathrm{Re}(\lambda) > P(V)\}$. In particular, because all terms are positive, if $\epsilon < \eta$, then for any $\lambda \in (P(V) - \epsilon, P(V))$, one has

$$\sum_{\gamma \in \Gamma} \frac{T_{\gamma^\sharp} e^{-(\lambda + V_\gamma) T_\gamma} \ \mathrm{tr}(\Lambda^k \mathcal{P}_\gamma)}{|\det(\mathrm{Id} - \mathcal{P}_\gamma)|} \leq C \sum_{\gamma \in \Gamma} \frac{T_{\gamma^\sharp} e^{-(\lambda - \eta + V_\gamma) T_\gamma} \ \mathrm{tr}(\Lambda^{d_s} \mathcal{P}_\gamma)}{|\det(\mathrm{Id} - \mathcal{P}_\gamma)|} < +\infty$$

and this shows that the left-hand side is analytic in the region $\{\mathrm{Re}(\lambda) > P(V) - \epsilon\}$ as claimed. □

Now we prove Corollary 1.1.

Proof. From [3, Theorem 1], we can find a C^1 neighborhood $\mathcal{U}$ of (X_0, V_0) and an anisotropic Sobolev space $\mathcal{H}^{s_0}$ such that for any $(X, V) \in \mathcal{U}$, the operator $-X + V$ has discrete spectrum in $\{\mathrm{Re}(\lambda) \geq -1\}$. Now, the pressure $P_X(V + J^u)$ is obtained as a simple eigenvalue (from Theorem 1.1) of a smooth family of operators acting on $\mathcal{H}^{s_0}$. One deduces the regularity statement by a perturbation argument of the resolvent, just like in the proof of [3, Corollary 2]. The regularity of $P_X(V)$ is obtained by the same argument but on d_s-forms. □

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A. *Appendix. Hölder continuity of a distributional product*

In this appendix, we fix a continuous function φ and an order zero distribution θ such that $\mathrm{WF}(\theta) \subset E_s^*$. This means that the distributional product $(\delta_{\mathcal{W}^u(x)}, \varphi\theta)$ is well defined. We will prove the following result.

PROPOSITION A.1. *There exists $C(\varphi) > 0, \alpha > 0$ and $\epsilon > 0$ such that if $d(x, x') < \epsilon$, then*

$$|(\delta_{\mathcal{W}^u(x)}, \varphi\theta) - (\delta_{\mathcal{W}^u(x')}, \varphi\theta)| \leq C(\varphi) d(x, x')^\alpha. \tag{A.1}$$

Here, the exponent α only depends on the unstable foliation.

Proof. The proof is mostly contained in the proof of [38, Proposition 6]. Indeed, to get equation (A.1), one needs to prove superpolynomial decay of the Fourier transform of $\delta_{\mathcal{W}^u(x)} - \delta_{\mathcal{W}^u(x')}$ for $\xi \notin \Omega$, where a small conic neighborhood of $E_0^* \oplus E_u^*$ with a constant that is Hölder continuous. In other words,

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$$\text{for all } \xi \notin \Omega, \quad |\widehat{\delta_{W^u(x)}}(\xi) - \widehat{\delta_{W^u(x')}}(\xi)| \leq C_N d(x, x')^\alpha \langle \xi \rangle^{-N},$$

where the Fourier transforms should be, strictly speaking, defined locally using charts. The super polynomial decay is proved in [38, Proposition 6] using integration by parts. Inspection of the proof shows that all C^k norms of the operator L_l, which is used to integrate by parts, depend Hölder continuously on the base point x. Weich only uses continuity in the proof to get uniform constant C_N, but using the Hölder continuity instead to bound the difference yields the refinement above.

On Ω, we use the fact that the Fourier transform of θ decays super polynomially and that $|\widehat{\delta_{W^u(x)}}(\xi) - \widehat{\delta_{W^u(x')}}(\xi)|$ has order zero. Actually, using the Hölder continuity of the foliation, the above difference is again controlled by $d(x, x')^\alpha$ and using the definition of the distributional product yields equation (A.1). □

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