CHARACTERS ON C^*-ALGEBRAS, THE JOINT NORMAL SPECTRUM, AND A PSEUDO-DIFFERENTIAL C^*-ALGEBRA

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Let $\{A_i; i \in \Omega\}$ be a family of C^*-algebras acting on a Hilbert space H and let A be the C^*-algebra that they generate. We shall assume throughout that C^*-algebras always contain the identity operator. Let $M(A)$ denote the space of characters, that is, multiplicative linear functionals, acting on A, with the weak star topology. We obtain here a natural characterisation of $M(A)$ as a subset of the product space determined by $\{M(A_i); i \in \Omega\}$. In the case of singly generated C^*-algebras this characterisation is related to the joint normal spectrum (**5**) of a family of operators.

It has already been shown in (**7**) how the special case where A is generated by two commutative C^*-algebras may be used as a C^*-basis for certain theorems concerning pseudo-differential operators on a Hilbert space. Complete details of such an application are given in Section 2. The proof of the special case given in (**7**) is much simplified below.

1. Characters on C^*-algebras

Definition 1.1. The collection of permanent points of the product space $\times_{i\in\Omega} M(A_i)$ is the set P of $x = (x_i)_{i\in\Omega}$ such that $\left\|\prod_{i\in\Gamma} a_i\right\| = 1$ whenever Γ is a finite subset of Ω and a_i is a positive contraction in A_i such that $x_i(a_i) = 1$, for i in Γ.

Theorem 1.2. *$M(A)$ is naturally homeomorphic to P with the relative product topology.*

Proof. Let α be the natural mapping from $M(A)$ to $\times_{i\in\Omega} M(A_i)$ where, for x in $M(A)$, $\alpha(x)_i$ is the restriction of x to A_i. Clearly α is injective since the A_i generate A. Moreover the range of α is contained in P. To see this let Γ be a finite subset of Ω, x a character of $M(A)$ and let a_i, i in Γ, be positive contractions with $\alpha(x)_i(a_i) = x(a_i) = 1$. Then

$$1 = \prod_{i\in\Gamma} x(a_i) = x\left(\prod_{i\in\Gamma} a_i\right) \leqq \left\|\prod_{i\in\Gamma} a_i\right\| \leqq 1.$$

It suffices, then, to show that P is contained in the range of α in order to complete the proof. Suppose $x=(x_i)_{i\in\Omega}$ belongs to P and let J_x be the C^*-algebra generated by the sets S_i, i in Ω, where $S_i=\{a;\, a\in A_i,\, a\geqq 0,\, x_i(a)=0\}$. Since the identity operator and S_i generate A_i as a Banach algebra, it follows that J_x is closed under left and right multiplication by elements of A. Consequently J_x is an ideal in A. But now $CI+J_x$ (where I is the identity operator) is closed and contains a set of generators for each A_i, and so $CI+J_x=A$.

We now use the permanence of x to conclude that $CI+J_x$ is a direct sum.

Let $s=\sum_{i\in\Gamma} b_i a_i$ where Γ is finite, a_i is a positive contraction in A_i with $x_i(a_i)=0$, and b_i belongs to A. Since x belongs to P we have

$$\left\|\prod_{i\in\Gamma}(I-a_i)\right\|=1 \tag{1.1}$$

and so there exist unit vectors f_n in H such that $\|df_n\|\to 1$, as $n\to\infty$, where d is the operator of (1.1). This is easily seen to imply that $\|(I-a_i)f_n\|\to 1$, as $n\to\infty$, for each i in Γ. Consequently, by a simple calculation, or from the spectral theorem, we see that $\|a_if_n\|\to 0$, as $n\to\infty$, for each i. Thus, for λ in C, we have

$$\begin{aligned}\|\lambda I+s\| &\geqq \|\lambda d+ds\| \geqq \|\lambda df_n+dsf_n\|\\ &\geqq |\lambda|\,\|df_n\|-\|dsf_n\|\end{aligned}$$

Letting $n\to\infty$ we see that $\|\lambda I+s\|\geqq|\lambda|$. But we shall show shortly that the collection S of such s is a dense linear subspace of J_x and so it follows that $CI+J_x$ is a direct sum. The mapping $\lambda I+b\to\lambda$, for λ in C and b in J_x, now defines a character, $\bar{z}$ say, which clearly has the property that $\alpha(\bar{z})=x$.

To see that S is dense in J_x it is sufficient to show that it is a linear space. But this follows from the fact that if a and a' are two positive elements of a C^*-algebra with $x(a)=x(a')=0$ for some character x then $a=dc$ and $a'=d'c$ with c positive such that $x(c)=0$. For example we may take $c=(a+a')^{\frac{1}{2}}$, $d=\lim_{n\to\infty} a(n^{-1}+a+a')^{-\frac{1}{2}}$ and $d'=\lim_{n\to\infty} a'(n^{-1}+a+a')^{-\frac{1}{2}}$ [see F. Combes, *Bull. Sc. Math.* **94** (1970), p. 167].

Remarks. A condition which obviously ensures the non-voidness of $M(A)$ is that the C^*-algebras be independent in the sense that $\left\|\prod_{i\in\Gamma} a_i\right\|=\prod_{i\in\Gamma}\|a_i\|$ for all a_i in A_i and all finite subsets Γ of Ω. In this case $M(A)$ is as big as it can be, namely the whole product space. Independence is often realised in concrete situations and in particular for certain algebras of singular integral operators (see (**6**, Example 1), (**7**, Section 3), (**9**, Chapter 1), (**3**, Theorem 8) and (**4**, Theorem 4)) which are generated by a projection P and a commutative algebra of operators whose norms are determined on PH and $(I-P)H$. The concept is also of interest from a purely abstract viewpoint. Sample: can two maximal abelian algebras or C^*-algebras be independent?

Suppose now that each algebra A_i is singly generated, as a unital C^*-algebra, by an operator T_i, so that $M(A_i)$ is homeomorphic to a (possibly empty) compact subset of the complex plane. In this case $M(A)$ can also be identified with the joint normal spectrum of S. G. Lee (**5**), together with the natural product topology.

Definition 1.3. The joint normal spectrum of $\{T_i; i\in\Omega\}$, a family of operators on a Hilbert space, is the collection of points $\underline{\lambda}=(\lambda_i)_{i\in\Omega}$ in $\underset{i\in\Omega}{\times} C$ such that for any finite subfamily Γ of Ω, the operator

$$T_\Gamma=\sum_{i\in\Gamma}\{(T_i-\lambda_i)^*(T_i-\lambda_i)+(T_i-\lambda_i)(T_i-\lambda_i)^*\} \tag{1.2}$$

is not invertible. The collection of such points is denoted $\sigma_n(\{T_i; i\in\Omega\})$.

It is quite simple to verify that if each T_i is hyponormal ($T_i^*T_i\geqq T_iT_i^*$) then the joint normal spectrum is the joint approximate spectrum of the family, which is defined in a similar fashion except that the terms $(T_i-\lambda_i)(T_i-\lambda_i)^*$ of (1.2) are omitted. (This is equivalent to the usual definition given in (**1**).) The following theorem may be regarded, therefore, as a generalisation of (**1**, Corollary 4) which concerned hyponormal operators. It should be noted that $\sigma_n(\ )$ may be void, even in the case of self-adjoint operators, but is not so if the family commutes (**1**, Proposition 2).

Theorem 1.4. (Lee (**5**)). *Let $\{T_i; i\in\Omega\}$ be a family of operators on a Hilbert space which generate a unital C^*-algebra A. Then $(\lambda_i)_{i\in\Omega}$ belongs to the joint normal spectrum of $\{T_i, i\in\Omega\}$ if and only if there exists x in $M(A)$ such that $\lambda_i=x(T_i)$, i in Ω.*

Proof. Suppose that $\underline{\lambda}$ is not in the joint normal spectrum and that for some finite set Γ, the operator T_Γ, as in (1.2), is invertible. Then, for some positive number δ, we have $T_\Gamma\geqq\delta I$. Applying a character x to this inequality shows that $x(T_i)=\lambda_i$, i in Ω, is impossible. It remains only to show therefore, that a point $\underline{\lambda}$ in the joint normal spectrum arises from a character in $M(A)$. We may assume, without loss of generality, that $\lambda_i=0$ for all i in Ω. Let T be any operator in the star algebra generated by the family $\{T_i; i\in\Omega\}$ so that

$$T=\sum_{i\in\Gamma}(S_iT_i+R_iT_i^*)$$

for some finite set Γ. Since the positive operator T_Γ is not invertible there exists a sequence of unit vectors f_n such that $T_if_n\to 0$ as $n\to\infty$. Thus $(T_\Gamma f_n, f_n)\to 0$, as $n\to\infty$ which ensures that $T_if_n\to 0$ and $T_i^*f_n\to 0$, as $n\to\infty$, for i in Γ. Thus $\|\lambda I+T\|\geqq|\lambda|$ and we conclude that the closed star algebra J generated by $\{T_i; i\in\Omega\}$, is a proper ideal in A of codimension one. Thus there is a character x on A such that $0=x(T_i)$, $i\in\Omega$, and this completes the proof.

2. A pseudo-differential C^*-algebra

Theorem 1.2 is particularly well suited to the development of the theory of pseudo-differential operators on a Hilbert space which lie in an algebra generated by multiplication operators and differential operators and we shall now demonstrate this for a simple but fundamental case. We shall be able to avoid discussions of symbol smoothness conditions and interpretations of Fourier integral expressions.

Let A_1 be the C^*-algebra of operators on $L^2(R)$ consisting of multiplication operators M_ϕ whose symbols ϕ are continuous and have equal limits at $+\infty$ and $-\infty$, and let A_2 be the C^*-algebra F^*A_1F of differential operators (or Fourier multipliers)

$D_\phi = F^* M_\phi F$ where F is the Fourier–Plancherel transform. Let A be the pseudo-differential C^*-algebra generated by A_1 and A_2. This algebra contains the operators $T = \sum_{i=1}^{n} M_{\phi_i} D_{\psi_i}$ which are usually called pseudo-differential operators in view of the fact that they also have a Fourier integral expression

$$Tf(x) = \frac{1}{\sqrt{(2\pi)}} \int a(x, y) e^{ixy} Ff(y)\, dy \tag{2.1}$$

where $a(x, y) = \sum_{i=1}^{n} \phi_i(x)\psi_i(y)$ is the symbol function of T. The integral can be understood as the inverse Fourier–Plancherel transform. Let us refer to any operator in the algebra generated by A_1 and A_2 as a pseudo-differential operator, so that such an operator is typically of the form

$$T = \sum_{i=1}^{n} \prod_{j=1}^{m} M_{\phi_{ij}} D_{\psi_{ij}}, \tag{2.2}$$

and has an associated symbol function $a(x, y) = \sum_{i=1}^{n} \prod_{j=1}^{m} \phi_{ij}(x)\psi_{ij}(y)$.

Clearly $M(A_1)$ and $M(A_2)$ are homeomorphic to $\tilde{R} = R \cup \{\infty\}$, the one point compactification of R.

Theorem 2.1. *$M(A)$ is homeomorphic to the compact subset $\tilde{R} \times \{\infty\} \bigcup \{\infty\} \times \tilde{R}$ of $\tilde{R} \times \tilde{R}$.*

Proof. It suffices to show, by Theorem 1.2, that the compact set is precisely P, the set of permanent points of $\tilde{R} \times \tilde{R} = M(A_1) \times M(A_2)$. Suppose first that x and y are real numbers and that ϕ and ψ are positive continuous functions of compact support with $\phi(z) < \phi(x) = 1$, for $z \neq x$, and $\psi(z) \leqq \psi(y) = 1$, for z in R. The representation (2.1) shows that $M_\phi D_\psi$ is compact (in fact Hilbert-Schmidt) and so

$$\|D_\psi M_\phi\|^2 = \|M_\phi D_{\psi^2} M_\phi\| = \lambda,$$

where λ is the largest eigenvalue of the positive compact operator $M_\phi D_{\psi^2} M_\phi$. Plainly $\lambda < 1$ because ϕ has a strict maximum at x, and it follows that (x, y) does not belong to P.

Suppose now that ϕ is a positive continuous function on $\tilde{R}$, with maximum value 1 at x in $\tilde{R}$, and that ψ is any positive continuous function with maximum 1 at ∞. We wish to show that $\|D_\psi M_\phi\| = 1$ and so conclude that (x, ∞) is a permanent point. To this end fix $\varepsilon > 0$ and choose f in $L^2(R)$ so that $\|f\| = 1$ and $\|\phi f\| > 1 - \varepsilon$. For t in R let $g_{(t)}$ denote the t-translate of a function g, so that $FM_{e^{itx}} g = (Fg)_{(t)}$. Then

$$\|D_\psi M_\phi\| = \|M_\psi F M_\phi M_{e^{itx}}\| \geqq \|\psi F \phi e^{itx} f\| = \|\psi (F\phi f)_{(t)}\|$$

and it is clear that t may be chosen large enough so that this dominates $\|F\phi f\| - \varepsilon$. Thus we have $\|D_\psi M_\phi\| \geqq \|\phi f\| - \varepsilon > 1 - 2\varepsilon$ and so $\|D_\psi M_\phi\| = 1$. In a similar fashion, or by applying F, it may be shown that the points (∞, y), for y in $\tilde{R}$, belong to P, and so the theorem is proved.

As we have already observed, the integral expression (2.1) shows that $M_\phi D_\psi$ is compact if ϕ and ψ are compactly supported. Thus, by approximating, $M_\phi D_\psi$ is compact if the symbols are continuous and vanish at infinity, and this leads to the fact that commutators of operators in A are compact. Consequently, the commutator ideal of A, denoted com A, is contained in K, the space of compact operators. In fact com $A = K$. This is because of a well known theorem [**3**, p. 141] concerning irreducible C^*-algebras (that is, having no reducing subspaces) which contain a compact operator, and the fact that com A is irreducible. Since this latter property is occasionally alluded to without proof (e.g. (**9**, p. 6), (**8**, p. 500 line 14) and (**7**, Theorem 6.5)), for completeness we give one in Lemma 2.2 below.

In view of the identity com $A = K$ we have the natural mappings

$$A \xrightarrow[\pi]{} A/K \xrightarrow[\eta]{} C(M(A)) \tag{2.3}$$

where the right hand side denotes continuous functions on $M(A)$, and in fact $\eta\pi(M_\phi D_\psi)$ is just the restriction of $\phi(x)\psi(y)$ to $M(A)$. If T is a pseudo-differential operator, as in (2.2), with associated symbol function $a(x, y)$, then we can conclude from (2.3) that T is a Fredholm operator if and only if $a(x, y)$ is non-vanishing on $M(A)$. Equivalently, if and only if $|a(x, y)|$ is bounded away from zero for large enough $|x|$ and $|y|$. In particular the essential spectrum of T is a closed path.

The abstract index group of $C(M(A))$, and therefore of A/K, is clearly $Z \oplus Z$. However, all Fredholm operators in A have Fredholm index equal to zero. In particular, although two Fredholm operators T_1 and T_2, which belong to A, can be connected by a path of Fredholm operators in $B(L^2(R))$, it is not necessarily true that this path may be chosen to lie in A. This, of course, is only the case if T_1 and T_2 have the same abstract index. This triviality of the Fredholm index is easy to see. In fact let T be a Fredholm operator in A with index $j(T)$. Then $\eta\pi(T)$ is a function of the form $\phi(x)\psi(y)$, for (x, y) in $M(A)$, where ϕ and ψ are invertible functions in $C(\tilde{R})$. Consequently $\pi(T) = \pi(M_\phi D_\psi)$. But then $j(T) = j(M_\phi D_\psi) = j(M_\phi) + j(D_\psi) = 0 + 0 = 0$, since M_ϕ and D_ψ are invertible.

Remarks.

1. A somewhat deeper reason why zero is the index of a Fredholm pseudo-differential operator in A is that, according to (**7**, Theorem 6.6), the index is given by the asymptotic winding number $w(a(x, y))$ of the symbol $a(x, y)$. This is the winding number of the image of $x^2 + y^2 = r$ (positively oriented) under $a(x, y)$, where r is suitably large. If one starts with continuous symbols ϕ_i and ψ_i which are allowed to have differing limits at $+\infty$ and $-\infty$, or, more generally still, to have vanishing asymptotic oscillation at $+\infty$ and $-\infty$, then a similar theory holds to the one above (see for example (**7**, Theorems 6.5 and 6.7), (**2**, Theorem C) and (**8**, Theorem 1)) and in this case the Fredholm index, which may be non-zero, is $w(a(x, y))$. For example the pseudo-differential operator $M_{\tan^{-1}x} + iD_{\tan^{-1}y}$ has index 1.

2. The algebra A above is $\Psi(C(\tilde{R}), C(\tilde{R}))$ in the notation of [**7**] and is therefore the most basic pseudo-differential C^*-algebra $\Psi(E, F)$ determined by the two symbol spaces $E = F = C(\tilde{R})$. The study of these algebras, their pseudo-differential operators

and their symbol spaces parallels, to some extent, the theory of Toeplitz algebras and their symbol spaces, and it seems that, as with Toeplitz operators, there exist interesting index theorems and generalisations for the more spectacular symbol functions.

3. Although it is not usual to consider C^*-algebras which include F itself as a generator, it is not too difficult to show that $C^*(\{F, M_\phi; \phi \text{ in } C(\tilde{R})\})$ is just $A+FA+F^2A+F^3A$. In fact the quotient of this algebra by the ideal of compact operators is just the crossed product of $A/K=C(M(A))$ by a homeomorphism of $M(A)$ of period 4. Of course, $M(A)$ is homeomorphic to an "infinity sign" and the action of the homeomorphism may be illustrated, not too cryptically we hope, by the following diagram

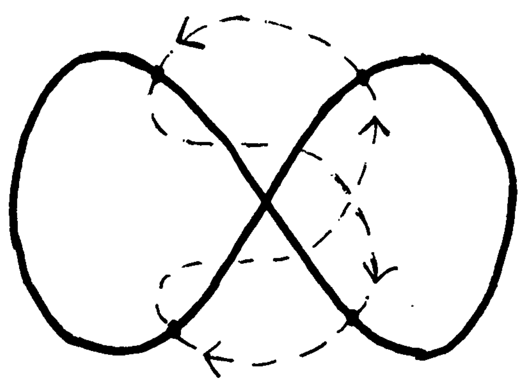

Diagram. The homeomorphism induced by the Plancherel transform.

Lemma 2.2. com A *is irreducible.*

Proof. First note that A is irreducible. This is because Lat A_1 is the collection of subspaces $M_E=\{f; f=0$ a.e. on $E\}$ for measurable subsets E of R, and because elementary properties of the Fourier transform soon show that

$$\text{Lat } A \subset \text{Lat } A_1 \cap \text{Lat } A_2 = \text{Lat } A_1 \cap F^* \text{ Lat } A_1 = \{\{0\}\}.$$

It suffices then to show that ker com $A=\{0\}$. Indeed, suppose that this is the case and M is a non-trivial subspace in Lat com A. Then for any T in A, $T(\text{com } A)M \subset M$. Since M does not belong to Lat A it must be that $(\text{com } A)M$ is not dense in M. Thus there exists $g \neq 0$ in M with $((\text{com } A)M, g)=\{0\}$ which means that $(\text{com } A)g=\{0\}$, a contradiction.

Suppose that f is any function in $L^2(R)$, not identically zero, and let ϕ be a continuous function with support contained in a finite interval, $[-N, N]$ say, such that $\phi f \neq 0$. Let θ be any continuous function such that D_θ has trivial kernel and θ vanishes at infinity. We have

$$D_{e^{itx}} D_\theta M_\phi f = (D_\theta \phi f)_{(t)}$$

and so t may be chosen so that this function does not have support contained in $[-N, N]$. Thus $(D_\psi M_\phi - M_\phi D_\psi)f \neq 0$ where $\psi = e^{itx}\theta$ and so ker com $A=\{0\}$, completing the proof.

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