Bull. Aust. Math. Soc. **90** (2014), 1–8
doi:10.1017/S0004972713001147

COUNTING SYMMETRIC COLOURINGS OF THE VERTICES OF A REGULAR POLYGON

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(Received 2 November 2013; accepted 5 November 2013; first published online 10 January 2014)

Abstract

A colouring of the vertices of a regular polygon is symmetric if it is invariant under some reflection of the polygon. We count the number of symmetric r-colourings of the vertices of a regular n-gon.

2010 *Mathematics subject classification*: primary 05A15, 05C15; secondary 05C25, 11A25.

Keywords and phrases: symmetric colouring, regular polygon, finite Abelian group, Möbius function, multiplicative function.

1. Introduction

Let G be a finite Abelian group and let $r \in \mathbb{N}$. An *r-colouring* of G is any mapping $\chi : G \to \{0, 1, \ldots, r-1\}$. Let r^G denote the set of all r-colourings of G. The group G naturally acts on r^G by

$$(a + \chi)(x) = \chi(x - a).$$

Colourings χ and ψ are *equivalent* if there is $a \in G$ such that $\chi(x - a) = \psi(x)$ for all $x \in G$ (that is, if they belong to the same orbit).

A *symmetry (proper symmetry)* of G is a mapping

$$G \ni x \mapsto a - x \in G \quad (G \ni x \mapsto 2a - x \in G),$$

where $a \in G$. A colouring $\chi \in r^G$ is *symmetric (properly symmetric)* if there is $a \in G$ such that

$$\chi(a - x) = \chi(x) \quad (\chi(2a - x) = \chi(x))$$

for all $x \in G$ (that is, if it is invariant under some symmetry (proper symmetry)).

Of special interest is the case $G = \mathbb{Z}_n$. Identifying $\mathbb{Z}_n$ with the vertices of a regular n-gon, we obtain that the symmetries (proper symmetries) of $\mathbb{Z}_n$ are the reflections of the polygon (reflections in an axis through one of the vertices). If n is odd,

Supported by NRF grants IFR2011033100072 and IFR1202220164, the John Knopfmacher Centre for Applicable Analysis and Number Theory, and the Friedel Sellschop Award.

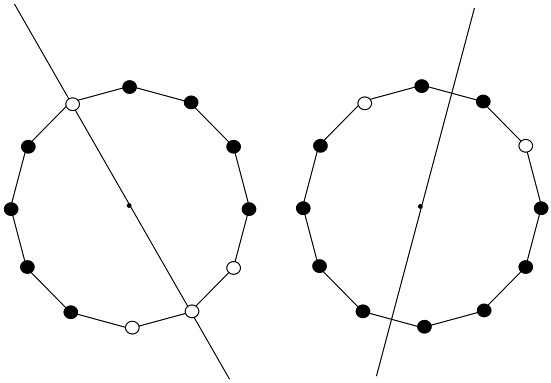

FIGURE 1. Two colourings of $\mathbb{Z}_{12}$.

the proper symmetries are the same as the symmetries, but if n is even, the proper symmetries form only half of the symmetries. A colouring of $\mathbb{Z}_n$ is symmetric (properly symmetric) if it is invariant under some reflection of the polygon (reflection in an axis through one of the vertices). For example, in Figure 1 the first colouring is properly symmetric, and the second is symmetric but not properly symmetric. Two colourings of $\mathbb{Z}_n$ are equivalent if one of them can be obtained from another by a rotation of the polygon.

In the case of $\mathbb{Z}_n$ proper symmetries look incomplete in comparison with symmetries. However, proper symmetries can be defined on any group (by taking them to be the mappings $x \mapsto ax^{-1}a$), while symmetries cannot.

It is well known that there are

$$N_r(n) = \frac{1}{n}\sum_{d|n} \varphi(d) r^{n/d}$$

classes of equivalent r-colourings of $\mathbb{Z}_n$, where φ is the Euler function (see [2]). In [3] it was shown that there are

$$s_r(n) = \begin{cases} r^{(n+1)/2} & \text{if } n \text{ is odd,} \\ \frac{1}{2}(r^{n/2+1} + r^{(m+1)/2}) & \text{if } n \text{ is even} \end{cases}$$

classes of equivalent properly symmetric r-colourings of $\mathbb{Z}_n$, where m is the greatest odd divisor of n, and

$$S_r(n) = \begin{cases} \sum_{d|n} d \prod_{p|n/d} (1-p) r^{(d+1)/2} & \text{if } n \text{ is odd,} \\ \sum_{d|n/2} d \prod_{p|n/2d} (1-p) r^{d+1} & \text{if } n \text{ is even} \end{cases}$$

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properly symmetric r-colourings of $\mathbb{Z}_n$, where p is a prime. Recently in [5], it was shown that there are

$$N_r^*(n) = \begin{cases} r^{(n+1)/2} & \text{if } n \text{ is odd,} \\ \frac{1}{2}(r^{n/2+1} + r^{n/2}) & \text{if } n \text{ is even} \end{cases}$$

classes of equivalent symmetric r-colourings of $\mathbb{Z}_n$.

In this note we count the number $C_r^*(n)$ of symmetric r-colourings of $\mathbb{Z}_n$. We prove the following result.

Theorem 1.1. *We have*

$$C_r^*(n) = \begin{cases} \displaystyle\sum_{d|n} d \prod_{p|n/d} (1-p) r^{(d+1)/2} & \text{if } n \text{ is odd,} \\ \displaystyle\sum_{d|n/2} d \prod_{p|n/2d} (1-p)(r^{d+1} + r^d) & \\ \qquad \displaystyle - \sum_{d|m} d \prod_{p|m/d} (1-p) r^{(d+1)/2} & \text{if } n \text{ is even,} \end{cases}$$

where m is the greatest odd divisor of n.

As in [3], we first establish a general formula for counting the number $C_r^*(G)$ of symmetric r-colourings of G (Section 2), and then deduce from it Theorem 1.1 (Section 3).

2. General formula

For every $\chi \in r^G$, let $[\chi]$ and $\mathrm{St}(\chi)$ denote the orbit and the stabiliser of χ, that is,

$$[\chi] = \{a + \chi : a \in G\} \quad \text{and} \quad \mathrm{St}(\chi) = \{a \in G : a + \chi = \chi\}.$$

Then $|[\chi]| = |G : \mathrm{St}(\chi)|$, and for every $\psi \in [\chi]$, $\mathrm{St}(\psi) = \mathrm{St}(\chi)$. Also let

$$Z(\chi) = \{a \in G : \chi(a - x) = \chi(x) \text{ for all } x \in G\}.$$

Thus, a colouring $\chi \in r^G$ is symmetric if and only if $Z(\chi) \neq \emptyset$.

Lemma 2.1. *If $a \in Z(\chi)$, then for every $b \in G$, $a + 2b \in Z(b + \chi)$.*

Proof. Indeed,

$$\begin{aligned}(b+\chi)(a+2b-x) &= \chi(a+2b-x-b) = \chi(a+b-x) \\ &= \chi(a-(x-b)) = \chi(x-b) = (b+\chi)(x).\end{aligned}$$

This completes the proof. □

Corollary 2.2. *If χ is symmetric, so is every $\psi \in [\chi]$.*

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Notice that the 'proper' version of Lemma 2.1 was better. If

$$Z'(\chi) = \{a \in G : \chi(2a - x) = \chi(x) \text{ for all } x \in G\}$$

and $a \in Z'(\chi)$, then for every $b \in G$, $a + b \in Z'(b + \chi)$, and consequently, $\bigcup_{\psi \in [\chi]} Z'(\psi) = G$. This made counting properly symmetric colourings easier. Now we can conclude only that if $a \in Z(\chi)$, then $a + 2G \subseteq \bigcup_{\psi \in [\chi]} Z(\psi)$, where

$$2G = \{2x : x \in G\}.$$

Lemma 2.3. *If $a \in Z(\chi)$ and $Y = \mathrm{St}\,(\chi)$, then $Z(\chi) = a + Y$.*

Proof. To see that $a + Y \subseteq Z(\chi)$, let $b \in Y$. Then

$$\chi(a + b - x) = \chi(a - (x - b)) = \chi(x - b) = (b + \chi)(x) = \chi(x),$$

so $a + b \in Z(\chi)$.

To see that $Z(\chi) \subseteq a + Y$, let $c \in Z(\chi)$. Then

$$(c - a)\chi(x) = \chi(x - (c - a)) = \chi(a - (c - x)) = \chi(c - x) = \chi(x).$$

Consequently, $c - a \in Y$, and so $c \in a + Y$.

Thus, $Z(\chi) = a + Y$. □

From Lemmas 2.1 and 2.3 we obtain that the following corollary.

Corollary 2.4. *If $a \in Z(\chi)$ and $Y = \mathrm{St}\,(\chi)$, then for every $b \in G$, $Z(b + \chi) = a + 2b + Y$, and $\bigcup_{\psi \in [\chi]} Z(\psi) = a + 2G + Y$.*

Define the subgroup $B(G)$ of G by

$$B(G) = \{x \in G : 2x = 0\}.$$

Lemma 2.5. *If $a \in Z(\chi)$ and $Y = \mathrm{St}\,(\chi)$, then $[\chi]$ decomposes into a disjoint union of subsets $\{\psi \in [\chi] : Z(\psi) = a + S\}$, where $S \in (2G + Y)/Y$, and each of the subsets consists of $|B(G/Y)|$ colourings.*

Proof. The first statement is obvious. For the second, it suffices to check that

$$|\{\psi \in [\chi] : Z(\psi) = a + Y\}| = |B(G/Y)|.$$

Let $b \in G$. Then by Corollary 2.4, $Z(b + \chi) = a + 2b + Y$. Consequently,

$$Z(b + \chi) = a + Y \Leftrightarrow 2b \in Y \Leftrightarrow b + Y \in B(G/Y).$$

This completes the proof. □

Lemma 2.6. *For every $a \in G$,*

$$|\{\chi \in r^G : a \in Z(\chi)\}| = \begin{cases} r^{(|G| + |B(G)|)/2} & \textit{if } a \in 2G, \\ r^{|G|/2} & \textit{otherwise}. \end{cases}$$

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PROOF. The number on the left is equal to the number of r-colourings of the family $\{\{x, a-x\} : x \in G\}$. Since $x = a - x$ if and only if $2x = a$, that number is

$$r^{|K_a|+(|G|-|K_a|)/2} = r^{(|G|+|K_a|)/2},$$

where $K_a = \{x \in G : 2x = a\}$. If $a \notin 2G$, then $K_a = \emptyset$. Let $a \in 2G$ and pick $x_0 \in K_a$. We claim that $K_a = x_0 + B(G)$.

To see that $x_0 + B(G) \subseteq K_a$, let $y \in B(G)$. Then $2(x_0 + y) = 2x_0 + 2y = a$, so $x_0 + y \in K_a$.

To see the converse inclusion, let $x \in K_a$. From $2x_0 = a$ and $2x = a$, we obtain that $2(x - x_0) = 0$, whence $x - x_0 \in B(G)$, and so $x \in x_0 + B(G)$. □

Let $\mu(Y, X)$ denote the *Möbius function* of the lattice of subgroups of A, that is,

$$\mu(Y, X) = \begin{cases} 1 & \text{if } Y = X, \\ -\sum_{Y \le Z < X} \mu(Y, Z) & \text{if } Y < X, \\ 0 & \text{otherwise.} \end{cases}$$

See [1, Ch. IV] for more information about the Möbius function and Möbius inversion.

For every subgroup $Y \le G$, let $R(Y)$ be a set of representatives of cosets of G by $2G + Y$. Also for every $a \in G$ and $Y \le G$, let

$$\delta_a(Y) = \begin{cases} 1 & \text{if } a \in 2G + Y, \\ 0 & \text{otherwise.} \end{cases}$$

The next theorem gives us a general formula for counting the number $C_r^*(G)$.

THEOREM 2.7. *We have*

$$C_r^*(G) = \sum_{X \le G} \sum_{Y \le X} \frac{|G/Y| \cdot \mu(Y, X)}{|B(G/Y)|} \sum_{a \in R(Y)} r^{(|G/X| + \delta_a(X) \cdot |B(G/X)|)/2}.$$

PROOF. For every $a \in G$ and for every $Y \le G$, let $C_a(Y)$ ($\overline{C}_a(G, Y)$) denote the number of all $\chi \in r^G$ such that $a \in Z(\chi)$ and $Y = \mathrm{St}\,(\chi)$ ($Y \subseteq \mathrm{St}\,(\chi)$). Notice that $\overline{C}_a(G, Y) = \sum_{Y \le X \le A} C_a(X)$ and $\overline{C}_a(G, Y) = \overline{C}_{a+Y}(G/Y, 0)$. Consequently, by Lemma 2.6,

$$\sum_{Y \le X \le G} C_a(X) = \begin{cases} r^{(|G/Y| + |B(G/Y)|)/2} & \text{if } a \in 2G + Y, \\ r^{|G/Y|/2} & \text{otherwise} \end{cases}$$

(since $a + Y \in 2(G/Y)$ if and only if $a \in 2G + Y$). Using the function $\delta_a(Y)$, we can rewrite this as

$$\sum_{Y \le X \le G} C_a(X) = r^{(|G/Y| + \delta_a(Y) \cdot |B(G/Y)|)/2}.$$

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Then applying Möbius inversion gives us

$$C_a(Y) = \sum_{Y \le X \le G} \mu(Y, X) r^{(|G/X| + \delta_a(X) \cdot |B(G/X)|)/2}.$$

Now for every $Y \le G$, let $C(Y)$ denote the number of all symmetric colourings χ with $\mathrm{St}\,(\chi) = Y$. From Lemma 2.5,

$$C(Y) = \sum_{a \in R(Y)} \frac{|G/Y| \cdot C_a(Y)}{|B(G/Y)|}.$$

Consequently,

$$\begin{aligned} C(Y) &= \sum_{a \in R(Y)} \sum_{Y \le X \le G} \frac{|G/Y| \cdot \mu(Y, X)}{|B(G/Y)|} r^{(|G/X| + \delta_a(X) \cdot |B(G/X)|)/2} \\ &= \sum_{Y \le X \le G} \frac{|G/Y| \cdot \mu(Y, X)}{|B(G/Y)|} \sum_{a \in R(Y)} r^{(|G/X| + \delta_a(X) \cdot |B(G/X)|)/2}. \end{aligned}$$

Finally, since $C_r^*(G) = \sum_{Y \le G} C(Y)$,

$$\begin{aligned} C_r^*(G) &= \sum_{Y \le G} \sum_{Y \le X \le G} \frac{|G/Y| \cdot \mu(Y, X)}{|B(G/Y)|} \sum_{a \in R(Y)} r^{(|G/X| + \delta_a(X) \cdot |B(G/X)|)/2} \\ &= \sum_{X \le G} \sum_{Y \le X} \frac{|G/Y| \cdot \mu(Y, X)}{|B(G/Y)|} \sum_{a \in R(Y)} r^{(|G/X| + \delta_a(X) \cdot |B(G/X)|)/2}, \end{aligned}$$

completing the proof. □

3. Proof of Theorem 1.1

Recall that the *classical Möbius function* is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes}, \\ 0 & \text{otherwise}, \end{cases}$$

and that it is in fact the Möbius function of the lattice of natural numbers with respect to the divisibility: if $d \mid n$, then $\mu(d, n) = \mu(n/d)$. Also recall that a function $f : \mathbb{N} \to \mathbb{C}$ is *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever m, n are relatively prime. For example, the functions $\mu(n)$ and $f(n) = n$ are multiplicative. The product of multiplicative functions is also a multiplicative function. If f is a multiplicative function, then for every $n \in \mathbb{N}$, one has

$$\sum_{d \mid n} \mu(d) f(d) = \prod_{p \mid n} (1 - f(p))$$

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(see [4, Theorem II.3.b]). Here, p is a prime, and for $n = 1$, the right-hand side of the equality is defined to be 1.

Define the function $\delta(n)$ by

$$\delta(n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Both $\delta(n)$ and $1/(2 - \delta(n))$ are multiplicative functions [3, Lemma].

Proof of Theorem 1.1. For every subgroup Y of $\mathbb{Z}_n$, define $R(Y)$ by

$$R(Y) = \begin{cases} \{0, 1\} & \text{if } 2\mathbb{Z}_n + Y \neq \mathbb{Z}_n, \\ \{0\} & \text{otherwise.} \end{cases}$$

Let d, k denote the orders of subgroups X, Y of $\mathbb{Z}_n$. Then $\mu(Y, X) = \mu(d/k)$, $|B(G/Y)| = 2 - \delta(n/k)$,

$$R(Y) = \begin{cases} \{0, 1\} & \text{if } n/k \text{ is even,} \\ \{0\} & \text{otherwise,} \end{cases}$$

$\delta_0(X) = 1$, $\delta_1(X) = \delta(n/d)$, and $\delta_1(X) \cdot |B(G/X)| = \delta(n/d)(2 - \delta(n/d)) = \delta(n/d)$. It follows from Theorem 2.7 that

$$\begin{aligned} C_r^*(n) &= \sum_{d|n} \sum_{k|d} \frac{\frac{n}{k}\mu(\frac{d}{k})}{2 - \delta(\frac{n}{k})}\Big(r^{((n/d)+2-\delta(n/d))/2} + \Big(1 - \delta\Big(\frac{n}{k}\Big)\Big) r^{((n/d)+\delta(n/d))/2}\Big) \\ &= \sum_{d|n} \sum_{k|n/d} \frac{\frac{n}{k}\mu(\frac{n}{kd})}{2 - \delta(\frac{n}{k})}\Big(r^{(d+2-\delta(d))/2} + \Big(1 - \delta\Big(\frac{n}{k}\Big)\Big) r^{(d+\delta(d))/2}\Big) \\ &= \sum_{d|n} d \sum_{k|n/d} \frac{k\mu(k)}{2 - \delta(dk)}(r^{(d+2-\delta(d))/2} + (1 - \delta(dk)) r^{(d+\delta(d))/2}). \end{aligned}$$

If n is odd, then $\delta(d) = \delta(dk) = 1$, and so

$$C_r^*(n) = \sum_{d|n} d \sum_{k|n/d} k\mu(k) r^{(d+1)/2} = \sum_{d|n} d \prod_{p|n/d} (1 - p) r^{(d+1)/2}.$$

Now suppose that n is even. Write $C_r^*(n) = S_1 - S_2$, where

$$\begin{aligned} S_1 &= \sum_{d|n} d \sum_{k|n/d} \frac{k\mu(k)}{2 - \delta(dk)}(r^{(d+2-\delta(d))/2} + r^{(d+\delta(d))/2}), \\ S_2 &= \sum_{d|n} d \sum_{k|n/d} \frac{k\delta(dk)\mu(k)}{2 - \delta(dk)} r^{(d+\delta(d))/2}. \end{aligned}$$

Consider S_1. If d is odd, then

$$\sum_{k|n/d} \frac{k\mu(k)}{2 - \delta(dk)} = \sum_{k|n/d} \frac{k\mu(k)}{2 - \delta(k)} = \prod_{p|n/d}\Big(1 - \frac{p}{2 - \delta(p)}\Big) = 0,$$

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since $f(k) = k/(2-\delta(k))$ is a multiplicative function, n/d is even and $f(2) = 1$. Thus,

$$S_1 = \sum d \sum_{k|n/d} \frac{k\mu(k)}{2}(r^{d/2+1} + r^{d/2}),$$

where the first sum is taken over all even $d \mid n$. Hence,

$$S_1 = \sum_{d|n/2} d \sum_{k|n/2d} k\mu(k)(r^{d+1} + r^d) = \sum_{d|n/2} d \prod_{p|n/2d} (1-p)(r^{d+1} + r^d).$$

Consider S_2. If d is odd, then

$$\sum_{k|n/d} \frac{k\delta(dk)\mu(k)}{2-\delta(dk)} = \sum_{k|n/d} \frac{k\delta(k)\mu(k)}{2-\delta(k)} = \prod_{p|n/d}\Big(1 - \frac{p\delta(p)}{2-\delta(p)}\Big) = \prod_{p|m/d}(1-p),$$

since $f(k) = k\delta(k)/(2-\delta(k))$ is a multiplicative function, n/d is even and $f(2) = 0$. If d is even, then

$$\sum_{k|n/d} \frac{k\delta(dk)\mu(k)}{2-\delta(dk)} = 0.$$

Hence,

$$S_2 = \sum_{d|m} d \prod_{p|m/d} (1-p) r^{(d+1)/2}. \qquad \square$$

This completes the proof.

In this way one can also determine the number $N_r^*(n)$. However, in [5] it is done more simply.

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