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PINCHING THEOREMS FOR A COMPACT MINIMAL SUBMANIFOLD IN A COMPLEX PROJECTIVE SPACE

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Abstract

We give a formula for the Laplacian of the second fundamental form of an n-dimensional compact minimal submanifold M in a complex projective space CP^m. As an application of this formula, we prove that M is a geodesic minimal hypersphere in CP^m if the sectional curvature satisfies $K \geq 1/n$, if the normal connection is flat, and if M satisfies an additional condition which is automatically satisfied when M is a CR submanifold. We also prove that M is the complex projective space $CP^{n/2}$ if $K \geq 3/n$, and if the normal connection of M is semi-flat.

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1. Introduction

The theory of submanifolds in a complex projective space CP^m is one of the most interesting objects in differential geometry. We have three typical classes of submanifolds in CP^m, complex submanifolds, totally real submanifolds and CR submanifolds, according to the behavior of the tangent bundle of a submanifold with respect to the action of the almost complex structure of the ambient manifold CP^m. For these submanifolds, there are many interesting results (see [1, 6, 12]).

In the present paper, we first study general submanifolds in a complex projective space CP^m of constant holomorphic sectional curvature 4, and give the Laplacian of the second fundamental form of an n-dimensional minimal submanifold M in CP^m, which corresponds to a formula for the Laplacian of the second fundamental form of a minimal submanifold in a unit sphere given by Simons [9].

Moreover, we prepare some inequalities for the second fundamental form which are useful to prove pinching theorems. Based on these results we study an n-dimensional compact minimal submanifold M in CP^m whose sectional curvature K satisfies $K \geq 1/n$. In particular, we prove that if the sectional curvature K of an n-dimensional compact minimal CR submanifold M in CP^m with flat normal connection satisfies

$K \geq 1/n$, then M is the geodesic minimal hypersphere in CP^m. The geodesic minimal hypersphere is given by $\pi(S^1(\sqrt{1/2m}) \times S^{2m-1}(\sqrt{(2m-1)/2m}))$ in CP^m, where $\pi : S^{2m+1} \longrightarrow CP^m$ is the Hopf fibration and $S^k(r)$ is a k-dimensional sphere (see [10]).

This is a generalization of the result in Kon [4] for a compact real minimal hypersurface M in CP^m .

We also prove that if the sectional curvature K of an n-dimensional compact minimal submanifold M in CP^m satisfies $K \geq 3/n$, then M is the complex projective space $CP^{n/2}$ under the assumption that the normal connection of M is semi-flat.

The concept of a semi-flat normal connection of a submanifold in a complex projective space is closely related to that of a flat normal connection of a submanifold in a sphere.

2. Preliminaries

Let $\tilde{M}$ denote a Kähler manifold of complex dimension m (real dimension $2m$). We denote by J the almost complex structure of $\tilde{M}$. The Hermitian metric of $\tilde{M}$ is denoted by g.

Let M be a real n-dimensional Riemannian manifold immersed in $\tilde{M}$. We denote by the same g the Riemannian metric on M induced from that of $\tilde{M}$. We denote by $\tilde{\nabla}$ the Levi-Civita connection in $\tilde{M}$ and by ∇ the connection induced on M. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \tilde{\nabla}_X V = -A_V X + D_X V,$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the normal connection. A normal vector field V on M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M. We call both A and B the *second fundamental form* of M that are related by $g(B(X, Y), V) = g(A_V X, Y)$.

For the second fundamental form B and A, we define ∇B and ∇A, the covariant derivative of the second fundamental form, by

$$(\nabla_X B)(Y, Z) = D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z),$$
$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{D_X V} Y - A_V(\nabla_X Y).$$

Then we have $g((\nabla_X B)(Y, Z), V) = g((\nabla_X A)_V Y, Z)$. The *mean curvature vector field* μ of M is defined to be $\mu = (1/n)\mathrm{tr}\, B$, where tr B is the trace of B. If $\mu = 0$, then M is said to be *minimal*.

For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX is the normal part of JX. For any vector field V normal to M, we put

$$JV = tV + fV,$$

where tV is the tangential part of JV and fV is the normal part of JV. Then P and f are skew-symmetric with respect to g and $g(FX, V) = -g(X, tV)$. We also have $P^2 = -I - tF$, $FP + fF = 0$, $Pt + tf = 0$ and $f^2 = -I - Ft$.

Next we define the covariant derivatives of P, F, t and f by $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$, $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$, $(\nabla_X t)V = \nabla_X(tV) - tD_X V$ and $(\nabla_X f)V = D_X(fV) - fD_X V$, respectively. We then have $(\nabla_X P)Y = A_{FY}X + tB(X, Y)$, $(\nabla_X F)Y = -B(X, PY) + fB(X, Y)$, $(\nabla_X t)V = -PA_V X + A_{fV}X$ and $(\nabla_X f)V = -FA_V X - B(X, tV)$.

We denote by $T_x(M)$ and $T_x(M)^\perp$ the tangent space and the normal space of M at x, respectively.

DEFINITION 2.1. A submanifold M in a Kähler manifold $\tilde{M}$ with almost complex structure J is called a *CR submanifold* in $\tilde{M}$ if there exists a differentiable distribution $\mathcal{D} : x \longrightarrow \mathcal{D}_x \subset T_x(M)$ on M satisfying the following conditions:

(i) H is holomorphic, that is $J\mathcal{D}_x = \mathcal{D}_x$ for each $x \in M$; and
(ii) the complementary orthogonal distribution $\mathcal{D}^\perp : x \longrightarrow \mathcal{D}_x^\perp \subset T_x(M)$ is anti-invariant, that is $J\mathcal{D}_x^\perp \subset T_x(M)^\perp$ for each $x \in M$.

In the following, we put $h = \dim \mathcal{D}_x$, $q = \dim \mathcal{D}_x^\perp$ and $\operatorname{codim} M = 2m - n = p$. If $q = 0$, then a *CR* submanifold M is a complex submanifold in $\tilde{M}$, and if $h = 0$, then M is a totally real submanifold in $\tilde{M}$. If $p = q$, then a *CR* submanifold M is called a *generic submanifold*. Any real hypersurface is a generic submanifold.

We use the following theorem (see [12, p. 217]).

THEOREM 2.2. *In order for a submanifold M in a Kähler manifold $\tilde{M}$ to be a CR submanifold, it is necessary and sufficient that $FP = 0$.*

We suppose that the ambient manifold $\tilde{M}$ is a complex projective space CP^m of constant holomorphic sectional curvature 4. The Riemannian curvature tensor $\tilde{R}$ of CP^m is given by

$$\begin{aligned}\tilde{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ - g(JX, Z)JY + 2g(X, JY)JZ,\end{aligned} \tag{2.1}$$

for any vector fields X, Y and Z of CP^m. Thus the *equation of Gauss* and the *equation of Codazzi* are given respectively by

$$\begin{aligned}R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(PY, Z)PX \\ &\quad - g(PX, Z)PY + 2g(X, PY)PZ \\ &\quad + A_{B(Y,Z)}X - A_{B(X,Z)}Y, \\ (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) &= g(PY, Z)FX - g(PX, Z)FY + 2g(X, PY)FZ.\end{aligned}$$

We define the curvature tensor $R^\perp$ of the normal bundle of M by

$$R^\perp(X, Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V,$$

where X and Y are vector fields tangent to M and V is a vector field normal to M. Then we have the *equation of Ricci*:

$$\begin{aligned} &g(R^{\perp}(X, Y)U, V) + g([A_V, A_U]X, Y) \\ &\quad = g(FY, U)g(FX, V) - g(FX, U)g(FY, V) + 2g(X, PY)g(fU, V), \end{aligned}$$

where $[A_V, A_U] = A_V A_U - A_U A_V$. If the normal curvature tensor $R^{\perp}$ of M satisfies $R^{\perp}(X, Y)V = 0$ for any vector fields X and Y tangent to M and any vector field V normal to M, then the normal connection of M is said to be *flat*. If $R^{\perp}$ satisfies $R^{\perp}(X, Y)V = 2g(X, PY)fV$, then the normal connection of M is said to be *semi-flat*.

In the following, we denote by A_a the second fundamental form in the direction of v_a, where $\{v_1, \ldots, v_p\}$ is an orthonormal basis for $T_x(M)^{\perp}$, $p = 2m - n$. We denote by $|\cdot|$ the length of the tensor. From the equation of Ricci, we have the following.

LEMMA 2.3. *Let M be an n-dimensional submanifold in CP^m. If the normal connection of M is flat, then*

$$\begin{aligned} \sum_{a,b} |[A_a, A_b]|^2 &= 2\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2) \\ &\quad - 8\sum_a g(tfv_a, tfv_a) + 4\sum_{i,a} g(Pe_i, Pe_i)g(fv_a, fv_a), \\ \sum_{i,a} g([A_{fa}, A_a]e_i, Pe_i) &= 2\sum_a \operatorname{tr} A_a A_{fa} P \\ &= 2\Big(\sum_a g(tfv_a, tfv_a) - \sum_{i,a} g(Pe_i, Pe_i)g(fv_a, fv_a)\Big), \\ \sum_{a,b} g([A_a, A_b]tv_a, tv_b) &= \sum_{a,b}(g(A_a tv_b, A_b tv_a) - g(A_a tv_a, A_b tv_b)) \\ &= \sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2) \\ &\quad - 2\sum_a g(tfv_a, tfv_a), \end{aligned}$$

where we have put $A_{fa} = A_{fv_a}$.

LEMMA 2.4. *Let M be an n-dimensional submanifold in CP^m. If the normal connection of M is semi-flat, then*

$$\sum_{a,b} |[A_a, A_b]|^2 = 2\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2),$$

$$\sum_{i,a} g([A_{fa}, A_a]e_i, Pe_i) = 2\sum_a g(tfv_a, tfv_a),$$

$$\sum_{a,b} g([A_a, A_b]tv_a, tv_b) = \sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2).$$

In the following we give an example of a compact CR submanifold in CP^m with semi-flat normal connection.

EXAMPLE 1. Let S^{2m+1} be a $(2m+1)$-dimensional unit sphere and N be a $(n+1)$-dimensional submanifold immersed in S^{2m+1}. With respect to the Hopf fibration $\pi : S^{2m+1} \longrightarrow CP^m$, we consider the following commutative diagram (see [5, 8, 12]):

$$\begin{array}{ccc} N & \longrightarrow & S^{2m+1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & CP^m. \end{array}$$

We denote by (ϕ, ξ, η, G) the contact metric structure on S^{2m+1}. The horizontal lift with respect to the connection η will be denoted by $*$. Then $(JX)^* = \phi X^*$ and $G(X^*, Y^*) = g(X, Y)^*$ for any vectors X and Y tangent to CP^m. A submanifold N in S^{2m+1} is tangent to the totally geodesic fibre of π and the structure vector field ξ is tangent to N.

Let α be the second fundamental form of N in S^{2m+1}. Then we have the relations of the second fundamental form α of N and B of M:

$$\begin{aligned} (\nabla_{X^*}\alpha)\,(Y^*, Z^*) &= [(\nabla_X B)\,(Y, Z) + g(PX, Y)FZ + g(PX, Z)FY]^*, \\ (\nabla_{X^*}\alpha)\,(Y^*, \xi) &= [fB(X, Y) - B(X, PY) - B(Y, PX)]^*, \\ (\nabla_{X^*}\alpha)\,(\xi, \xi) &= -2(FPX)^*, \end{aligned}$$

for any vectors X, Y and Z tangent to M. From the third equation, we see that if the second fundamental form α of N is parallel, then $FP = 0$ and M is a CR submanifold of CP^m by Theorem 2.2.

Let $K^\perp$ be the curvature tensor of the normal bundle of N. Then

$$\begin{aligned} G(K^\perp(X^*, Y^*)V^*, U^*) &= [g(R^\perp(X, Y)V, U) - 2g(X, PY)g(fV, U)]^*, \\ G(K^\perp(X^*, \xi)V^*, U^*) &= g((\nabla_X f)V, U)^* \end{aligned}$$

for any vectors X and Y tangent to M and any vectors V and U normal to M. Therefore, the normal connection of N in S^{2m+1} is flat if and only if the normal connection of M is semi-flat and $\nabla f = 0$ (see [7, 8, 12]).

We put

$$N = S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k), \quad n + 1 = \sum_{i=1}^{k} m_i, \quad 1 = \sum_{i=1}^{k} r_i^2,$$

where $m_1, \ldots, m_k$ are odd numbers. Then $n + k$ is also odd. The second fundamental form α of N is parallel in S^{2m+1}. We can see that $M = \pi(N)$ is a generic submanifold in $CP^{(n+k-1)/2}$ with flat normal connection. $\pi(S^1(r_1) \times S^n(r_2))$ is called a geodesic

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hypersphere in $CP^{(n+1)/2}$ (see [10]). Moreover, M is a CR submanifold in CP^m $(m > (n+k-1)/2)$ with semi-flat normal connection and $\nabla f = 0$.

If $r_i = (m_i/(n+1))^{1/2}$ $(i = 1, \ldots, k)$, then M is a generic minimal submanifold in $CP^{(n+k-1)/2}$. Then we have $|A|^2 = \sum_a \operatorname{tr} A_a^2 = (n-1)q$, $q = k-1$.

If M is a complex submanifold in CP^m, the normal connection of M is semi-flat if and only if M is totally geodesic (see [3]).

3. Minimal submanifolds with flat normal connection

In this section, we give a pinching theorem for n-dimensional compact minimal submanifolds in a complex projective space CP^m with flat normal connection. For the proof of a theorem, we first give the Simons' type integral formula for a compact minimal submanifold in CP^m (see [9]).

We use the following lemma [2, p. 81].

LEMMA 3.1. *Let M be a minimal submanifold in a Riemannian manifold $\bar{M}$. Then*

$$(\nabla^2 B)(X, Y) = \sum_i (\nabla_{e_i}\nabla_{e_i} B)(X, Y)$$
$$= \sum_i \Big((R(e_i, X)B)(e_i, Y) + (\bar{\nabla}_X(\bar{R}(e_i, Y)e_i)^{\perp})^{\perp} + (\bar{\nabla}_{e_i}(\bar{R}(e_i, X)Y)^{\perp})^{\perp} \Big),$$

where $\{e_1, \ldots, e_n\}$ denotes an orthonormal basis of $T_x(M)$, and $\bar{\nabla}$ is the Levi-Civita connection in $\bar{M}$.

We compute the equation in Lemma 3.1 for an n-dimensional minimal submanifold M in a complex projective space CP^m of constant holomorphic sectional curvature 4. Since CP^m is locally symmetric, using (2.1),

$$\sum_i (\bar{\nabla}_X(\bar{R}(e_i, Y)e_i)^{\perp})^{\perp}$$
$$= \sum_i (\bar{R}(B(X, e_i), Y)e_i + \bar{R}(e_i, B(X, Y))e_i$$
$$+ \bar{R}(e_i, Y)B(X, e_i))^{\perp} - \sum_i B(X, (\bar{R}(e_i, Y)e_i)^T),$$
$$= 3(fB(X, PY) + FtB(X, Y) - B(X, P^2Y) + FA_{FY}X),$$
$$\sum_i (\bar{\nabla}_{e_i}(\bar{R}(e_i, X)Y)^{\perp})^{\perp}$$
$$= \sum_i (\bar{R}(B(e_i, e_i), X)Y$$
$$+ \bar{R}(e_i, B(e_i, X))Y + \bar{R}(e_i, X)B(e_i, Y))^{\perp} - \sum_i B(e_i, (\bar{R}(e_i, X)Y)^T)$$
$$= FA_{FX}Y - FA_{FY}X + fB(X, PY) + 2fB(PX, Y)$$
$$- 3B(PX, PY) - 2\sum_i g(A_{Fe_i}e_i, X)FY - \sum_i g(A_{Fe_i}e_i, Y)FX.$$

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Thus we obtain

$$
\begin{aligned}
g(\nabla^2 B, B) &= \sum_{i,j,k} g((\nabla_{e_i}\nabla_{e_i} B)(e_j, e_k), B(e_j, e_k)) \\
&= \sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) + 3\Big(\sum_a \mathrm{tr}\, A_{Ftv_a} A_a \\
&\quad - 2\sum_a \mathrm{tr}\, A_a A_{fa} P - \sum_a \mathrm{tr}\, P^2 A_a^2 + \sum_a \mathrm{tr}\,(A_a P)^2 \\
&\quad + \sum_{a,b} g(A_a t v_a, t v_b)\mathrm{tr}\, A_b \\
&\quad + \sum_{a,b}(g(A_a t v_b, A_b t v_a) - g(A_a t v_a, A_b t v_b))\Big).
\end{aligned}
$$

On the other hand,

$$
\sum_a \mathrm{tr}\, A_{Ftv_a} A_a = -\sum_a \mathrm{tr}\, A_a^2 + \sum_a \mathrm{tr}\, A_{fa}^2,
$$

$$
-\sum_a \mathrm{tr}\, P^2 A_a^2 + \sum_a \mathrm{tr}\,(A_a P)^2 = \frac{1}{2}\sum_a |[P, A_a]|^2.
$$

Hence we have the following lemma.

LEMMA 3.2. *Let M be an n-dimensional minimal submanifold in CP^m. Then*

$$
\begin{aligned}
g(\nabla^2 B, B) &= g(\nabla^2 A, A) \\
&= \sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) \\
&\quad + 3\Big(-\sum_a \mathrm{tr}\, A_a^2 + \sum_a \mathrm{tr}\, A_{fa}^2 - 2\sum_a \mathrm{tr}\, A_a A_{fa} P + \frac{1}{2}\sum_a |[P, A_a]|^2 \\
&\quad + \sum_{a,b}(g(A_a t v_b, A_b t v_a) - g(A_a t v_a, A_b t v_b))\Big).
\end{aligned}
$$

We prepare the following lemma.

LEMMA 3.3. *Let M be an n-dimensional minimal submanifold in CP^m. If U is a parallel section in the normal bundle of M, then*

$$
\begin{aligned}
\mathrm{div}(\nabla_{tU} tU) &= (n-1) g(tU, tU) + 3g(PtU, PtU) - \sum_a g(A_a tU, A_a tU) \\
&\quad + \mathrm{tr}\, A_{fU}^2 - \mathrm{tr}\, A_U^2 - 2\mathrm{tr}\, A_U A_{fU} P + \sum_a g(A_U t v_a, A_U t v_a) \\
&\quad + \frac{1}{2}|[P, A_U]|^2.
\end{aligned}
$$

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PROOF. For any vector field X on a Riemannian manifold, we generally have the equation [11]

$$\operatorname{div}(\nabla_X X) - \operatorname{div}((\operatorname{div}X)X) = S(X, X) + \tfrac{1}{2}|L_X g|^2 - |\nabla X|^2 - (\operatorname{div}X)^2, \quad (3.1)$$

where S denotes the Ricci tensor and $(L_X g)\,(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y)$.

Suppose that U is a parallel section of the normal bundle of M. From the equation of Gauss,

$$S(tU, tU) = (n-1)g(tU, tU) + 3g(PtU, PtU) - \sum_a g(A_a tU, A_a tU).$$

On the other hand, since $(\nabla_X t)V = -PA_V X + A_{fV} X$ for any V normal to M, we have $\nabla_X(tU) = -PA_U X + A_{fU} X$. This implies that $\operatorname{div}(tU) = \operatorname{tr} A_{fU} = 0$. Also

$$|\nabla tU|^2 = \operatorname{tr} A_{fU}^2 + \operatorname{tr} A_U^2 - 2\operatorname{tr} A_U A_{fU} P - \sum_a g(A_U tv_a, A_U tv_a),$$
$$|L_{tU} g|^2 = |[P, A_U]|^2 + 4\operatorname{tr} A_{fU}^2 - 8\operatorname{tr} A_U A_{fU} P.$$

Substituting these equations into (3.1), we have our lemma. □

LEMMA 3.4. *Let M be an n-dimensional minimal submanifold in CP^m with flat normal connection. Then*

$$\begin{aligned}
&-g(\nabla^2 A, A) - 2\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) - 2\sum_i g(FPe_i, FPe_i)\\
&\quad + \frac{1}{2}\Big(\sum_a \operatorname{tr} A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \operatorname{tr} A_a A_{fa} P\Big)\\
&\quad + \sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2)\\
&= \sum_a \operatorname{tr} A_a^2 - \sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j)\\
&\quad + 8\sum_i g(FPe_i, FPe_i) - \frac{1}{2}\sum_a \operatorname{tr} A_{fa}^2 - 2\sum_a \operatorname{div}(\nabla_{tv_a} tv_a).
\end{aligned}$$

PROOF. By a straightforward computation, we obtain

$$\sum_a g(tfv_a, tfv_a) = \sum_a g(Ptv_a, Ptv_a) = \sum_i g(FPe_i, FPe_i), \quad (3.2)$$

$$\begin{aligned}
&\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2)\\
&\quad = (n-1)\sum_a g(tv_a, tv_a) - \sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a)\\
&\qquad + \sum_a g(Ptv_a, Ptv_a). \quad (3.3)
\end{aligned}$$

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Thus, using Lemmas 2.3 and 3.2,

$$\begin{aligned}-g(\nabla^2 A, A) = &-\sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) + 3\sum_a \operatorname{tr} A_a^2 - 3\sum_a \operatorname{tr} A_{fa}^2 \\ &+ 6\sum_a \operatorname{tr} A_a A_{fa} P - \frac{3}{2}\sum_a |[P, A_a]|^2 - 2(n-1)\sum_a g(tv_a, tv_a) \\ &+ 2\sum_{i,a} g(Pe_i, Pe_i) g(tv_a, tv_a) + 4\sum_a g(Ptv_a, Ptv_a) \\ &- \sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2).\end{aligned}$$

Since the normal connection of M is flat, we can choose an orthonormal basis $\{v_a\}$ of $T(M)^{\perp}$ such that $Dv_a = 0$ for all a. Thus, from Lemma 3.3,

$$\begin{aligned}\operatorname{div}(\nabla_{tv_a} tv_a) = &(n-1)g(tv_a, tv_a) + 3g(Ptv_a, Ptv_a) \\ &+ \operatorname{tr} A_{fa}^2 - \operatorname{tr} A_a^2 - 2\operatorname{tr} A_a A_{fa} P + \tfrac{1}{2}|[P, A_a]|^2.\end{aligned}$$

From these equations, we have our assertion. □

If M is compact, we have $\int_M |\nabla A|^2 = -\int_M g(\nabla^2 A, A)$ (see [9]). Therefore Lemma 3.4 implies the following.

THEOREM 3.5. *Let M be an n-dimensional compact minimal submanifold in a complex projective space CP^m with flat normal connection. Then*

$$\begin{aligned}\int_M \Big(&|\nabla A|^2 - 2\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) - 2\sum_i g(FPe_i, FPe_i) \\ &+ \frac{1}{2}\Big(\sum_a \operatorname{tr} A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \operatorname{tr} A_a A_{fa} P\Big) \\ &+ \sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2)\Big) \\ = \int_M \Big(&\sum_a \operatorname{tr} A_a^2 - \sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) \\ &+ 8\sum_i g(FPe_i, FPe_i) - \frac{1}{2}\sum_a \operatorname{tr} A_{fa}^2\Big).\end{aligned}$$

We next consider the properties of some terms of the equation in Theorem 3.5.

LEMMA 3.6. *Let M be an n-dimensional submanifold in CP^m. Then*

$$|\nabla A|^2 \geq 2\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) + 2\sum_i g(FPe_i, FPe_i).$$

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PROOF. We put

$$T_1(X, Y, Z) = (\nabla_X B)(Y, Z) + g(PX, Y)FZ + g(PX, Z)FY.$$

Then we obtain

$$|T_1|^2 = |\nabla B|^2 + 2\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) + 2\sum_i g(FPe_i, FPe_i) + 4\sum_{i,j} g((\nabla_{e_i} B)(Pe_i, e_j), Fe_j).$$

From the equation of Codazzi,

$$\sum_{i,j} g((\nabla_{e_i} B)(Pe_i, e_j), Fe_j) = \sum_{i,j} g((\nabla_{e_j} B)(e_i, Pe_i), Fe_j) - \sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) - \sum_i g(FPe_i, FPe_i).$$

Since B is symmetric and P is skew-symmetric, the first term on the right-hand side of the equation vanishes. So we have our assertion. □

LEMMA 3.7. *Let M be an n-dimensional submanifold in CP^m with parallel mean curvature vector field. If the equality*

$$|\nabla A|^2 = 2\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) + 2\sum_i g(FPe_i, FPe_i)$$

holds, then M is a CR submanifold.

PROOF. By the proof of Lemma 3.6, the equation holds if and only if $T_1 = 0$. Suppose that $T_1 = 0$. Then

$$\sum_i g((\nabla_{e_i} B)(e_i, X), v_a) = g(FPX, v_a)$$

for any X and v_a. On the other hand, since the mean curvature vector field is parallel, the equation of Codazzi implies that

$$\sum_i g((\nabla_{e_i} B)(e_i, X), v_a) = 3g(FPX, v_a).$$

From these equations, we have $FP = 0$. Then, from Theorem 2.2, M is a CR submanifold. □

LEMMA 3.8. *Let M be a n-dimensional submanifold of CP^m. Then*

$$\sum_a \operatorname{tr} A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \operatorname{tr} A_a A_{fa} P \geq 0.$$

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PROOF. We put

$$T_2(X, Y) = fB(X, Y) - B(X, PY) - B(PX, Y).$$

Then

$$\begin{aligned}|T_2|^2 &= \sum_{i,j} |fB(e_i, e_j) - B(e_i, Pe_j) - B(Pe_i, e_j)|^2 \\ &= \sum_a \operatorname{tr} A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \operatorname{tr} A_a A_{fa} P.\end{aligned}$$

Thus we have our inequality. □

REMARK. In Example 1, when the mean curvature vector field of M in CP^m is parallel, by Lemma 3.7, we see that the second fundamental form of a submanifold N in S^{2m+1} is parallel if and only if the second fundamental form M in CP^m satisfies $T_1 = 0$ and $T_2 = 0$. We see that the submanifold $\pi(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k))$ in CP^m satisfies $T_1 = 0$ and $T_2 = 0$. □

THEOREM 3.9. *Let M be an n-dimensional compact minimal submanifold in a complex projective space CP^m with flat normal connection. If the second fundamental form A satisfies $\sum_a \operatorname{tr} A_{fa}^2 \geq 16|FP|^2$, and if the sectional curvature K of M satisfies $K \geq 1/n$, then M is the geodesic minimal hypersphere $\pi(S^1(\sqrt{1/2m}) \times S^{2m-1}(\sqrt{(2m-1)/2m}))$ in CP^m.*

PROOF. From Lemmas 3.6 and 3.8, we see that the left-hand side of the equation in Theorem 3.5 is non-negative. Next we prove that the right-hand side of this equation is non-positive.

Choosing an orthonormal basis $\{e_i\}$ of $T_x(M)$ such that $A_a e_i = h_i^a e_i$, $i = 1, \ldots, n$,

$$\begin{aligned}\sum_{i,j} g((R(e_i, e_j)A)_a e_i, A_a e_j) &= \sum_{i,j} g(R(e_i, e_j)A_a e_i, A_a e_j) \\ &\quad - \sum_{i,j} g(A_a R(e_i, e_j)e_i, A_a e_j) \\ &= \frac{1}{2}\sum_{i,j}(h_i^a - h_j^a)^2 K_{ij},\end{aligned}$$

where K_{ij} denotes the sectional curvature of M with respect to the section spanned by e_i and e_j. Since $K_{ij} \geq 1/n$, we obtain

$$\sum_{i,j} g((R(e_i, e_j)A)_a e_i, A_a e_j) \geq \frac{1}{2n}\sum_{i,j}(h_i^a - h_j^a)^2 \geq \operatorname{tr} A_a^2.$$

The left-hand side of this inequality is independent of the choice of an orthonormal basis $\{e_i\}$. Hence

$$\sum_a \operatorname{tr} A_a^2 - \sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) \leq 0.$$

Consequently, Theorem 3.5 and Lemmas 3.6 and 3.8 imply that

$$|\nabla A|^2 - 2\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) - 2\sum_a g(Ptv_a, Ptv_a) = 0, \tag{3.4}$$

$$\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2) = 0, \tag{3.5}$$

$$8\sum_a g(FPe_i, FPe_i) - \frac{1}{2}\sum_a \operatorname{tr} A_{fa}^2 = 0. \tag{3.6}$$

By (3.4) and Lemma 3.7, M is a CR submanifold. Thus, from (3.6), we have $A_{fa} = 0$ for all v_a. On the other hand, (3.5) implies that $q = 1$ or $q = 0$.

Suppose that $q = 1$. Using Lemma 2.3, we obtain

$$\sum_{i,a} g([A_{fa}, A_a]e_i, Pe_i) = -2h(p-1) = 0.$$

When $p = 1$, from the theorem in [4], M is a geodesic minimal hypersphere. When $h = 0$, we have $n = q = 1$ and $K = 0$. This is a contradiction.

We next suppose that $q = 0$. Then M is a complex submanifold and $n = h$. On the other hand, again using Lemma 2.3, we have $hp = 0$, and hence $h = 0$. This is a contradiction. □

When M is a CR minimal submanifold, by Theorem 2.2, we have $FP = 0$. Hence the condition $\sum_a \operatorname{tr} A_{fa}^2 \geq 16|FP|^2$ in Theorem 3.9 is automatically satisfied. So we have the following theorem.

THEOREM 3.10. *Let M be an n-dimensional compact minimal CR submanifold in a complex projective space CP^m with flat normal connection. If the sectional curvature K of M satisfies $K \geq 1/n$, then M is the geodesic minimal hypersphere $\pi(S^1(\sqrt{1/2m}) \times S^{2m-1}(\sqrt{(2m-1)/2m}))$ in CP^m.*

4. Minimal submanifolds with semi-flat normal connection

In this section we give pinching theorems for minimal submanifolds in CP^m with semi-flat normal connection. First of all, using (3), (4) and Lemmas 2.4 and 3.2 we have the following lemma.

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LEMMA 4.1. *Let M be an n-dimensional compact minimal submanifold in CP^m with semi-flat normal connection. Then*

$$\begin{aligned}\int_M\Big(&|\nabla A|^2 - 2\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) - 2\sum_i g(FPe_i, FPe_i)\\ &+\frac{3}{2}\Big(\sum_a \operatorname{tr} A_{fa}^2 + \sum_a |[P, A_a]|^2 - \sum_a 4\operatorname{tr} A_a A_{fa} P\Big)\\ &+4\sum_i g(FPe_i, FPe_i)\Big)\\ =\int_M\Big(&-\sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) + 3\sum_a \operatorname{tr} A_a^2\\ &-\frac{3}{2}\sum_a \operatorname{tr} A_{fa}^2 - 2(n-1)\sum_a g(tv_a, tv_a)\\ &-\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2)\Big).\end{aligned}$$

From this, we have the following theorem.

THEOREM 4.2. *Let M be an n-dimensional compact minimal submanifold in a complex projective space CP^m with semi-flat normal connection. If the sectional curvature K of M satisfies $K \geq 3/n$, then M is the complex projective space $CP^{n/2}$ in CP^m.*

PROOF. From Lemmas 3.6 and 3.8, we see that the left-hand side of the equation in Lemma 4.1 is non-negative. Next we prove that the right-hand side of this equation is non-positive.

Since $K_{ij} \geq 3/n$, by a similar method in the proof of Theorem 3.9, we obtain

$$-\sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) + 3\sum_a \operatorname{tr} A_a^2 \leq 0.$$

Consequently,

$$\frac{3}{2}\sum_a \operatorname{tr} A_{fa}^2 + 2(n-1)\sum_a g(tv_a, tv_a) = 0.$$

Thus, we obtain $A_{fa} = 0$ for all v_a and $t = 0$. Therefore M is a complex submanifold in CP^m and $A_a = 0$ for all v_a. Thus M is a real n-dimensional totally geodesic complex submanifold in CP^m, that is, $CP^{n/2}$. □

Next we give a pinching theorem for a compact minimal CR submanifold in CP^m with semi-flat normal connection.

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THEOREM 4.3. *Let M be a compact n-dimensional minimal CR submanifold in a complex projective space CP^m with semi-flat normal connection. If the sectional curvature K of M satisfies $K \geq (1/n)$, then M is a totally geodesic complex projective space $CP^{n/2}$ or a geodesic minimal hypersphere $\pi(S^1(\sqrt{1/(n+1)}) \times S^n(\sqrt{n/(n+1)}))$ of some $CP^{(n+1)/2}$ in CP^m.*

PROOF. Since M is a CR submanifold in CP^m, we can take an orthonormal basis $\{v_a\}$ of $T_x(M)^\perp$ such that $\{v_1, \ldots, v_q\}$ form an orthonormal basis of $FT_x(M)$ and $\{v_{q+1}, \ldots, v_p\}$ form an orthonormal basis of $fT_x(M)^\perp$.

If $q = 0$, M is a complex submanifold in CP^m. Then the normal connection of M is semi-flat if and only if M is a totally geodesic complex projective space $CP^{n/2}$ by a theorem of Ishihara [3].

We next suppose that $q \geq 1$. Since the normal connection of M is semi-flat, we have $A_{fV}PX = 0$ and $A_{fV}tU = \beta tU$ for any vector X tangent to M and any vectors U, V normal to M (see Chen [1, Lemmas 5.3 and 5.6]). Thus, by the minimality of M, we see that $\beta = 0$ and $A_{fV} = 0$.

Let V be in $FT(M)$. Then

$$\begin{aligned} g(fD_XV, fU) &= -g((\nabla f)V, fU) \\ &= g(FA_VX, fU) + g(B(X, tV), fU) \\ &= g(A_{fU}X, tV) = 0. \end{aligned}$$

This means that $FT(M)$ is parallel, that is, D_XV is in $FT(M)$. Moreover, we have $R^\perp(X, Y)V = 0$ for any $V \in FT(M)$. So we can choose an orthonormal basis $\{v_\lambda\}$ in such a way that $D_Xv_\lambda = 0$, $\lambda = 1, \ldots, q$. We notice that $\nabla_X tv_\lambda = -PA_\lambda X$. Hence we have $\operatorname{div}(tv_\lambda) = -\operatorname{tr} PA_\lambda = 0$ since P is skew-symmetric and A_λ is symmetric.

From Lemmas 2.4 and 3.2, we obtain

$$\begin{aligned} g(\nabla^2 A, A) &= \sum_{i,j,\lambda} g((R(e_i, e_j)A)_\lambda e_i, A_\lambda e_j) \\ &\quad + 3\Big(-\sum_a \operatorname{tr} A_\lambda^2 + \frac{1}{2}\sum_a \|[P, A_\lambda]\|^2\Big) + 3q(q-1). \end{aligned}$$

On the other hand, Lemma 3.3 implies that

$$\sum_\lambda \operatorname{div}(\nabla_{tv_\lambda} tv_\lambda) = (n-1)q - \sum_\lambda \operatorname{tr} A_\lambda^2 + \frac{1}{2}\sum_\lambda \|[P, A_\lambda]\|^2.$$

Using these equations,

$$\begin{aligned} &-g(\nabla^2 A, A) - 2hq + \frac{1}{2}\sum_\lambda \|[P, A_\lambda]\|^2 + q(q-1) \\ &\quad = \sum_\lambda \operatorname{tr} A_\lambda^2 - \sum_{i,j,\lambda} g((R(e_i, e_j)A)_\lambda e_i, A_\lambda e_j) - 2\sum_\lambda \operatorname{div}(\nabla_{tv_\lambda} tv_\lambda). \end{aligned}$$

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Thus

$$\int_M \Big(|\nabla A|^2 - 2hq + \frac{1}{2} \sum_\lambda |[P, A_\lambda]|^2 + q(q-1) \Big) = \int_M \Big(\sum_\lambda \operatorname{tr} A_\lambda^2 - \sum_{i,j,\lambda} g((R(e_i, e_j)A)_\lambda e_i, A_\lambda e_j) \Big).$$

By Lemma 3.6, we see that the left-hand side of this equation is non-negative. Next we prove that the right-hand side of the equation above is non-positive. From the assumption on the sectional curvature of M, we have, by a similar method in the proof of Theorem 3.9,

$$\sum_\lambda \operatorname{tr} A_\lambda^2 - \sum_{i,j,\lambda} g((R(e_i, e_j)A)_\lambda e_i, A_\lambda e_j) \le 0.$$

Consequently, we obtain

$$|\nabla A|^2 = 2hq, \quad PA_\lambda = A_\lambda P, \quad q(q-1) = 0.$$

Hence we have $q = 1$ and M is a real hypersurface in some $CP^{(n+1)/2}$ in CP^m (see [10, p. 227]). Therefore, using Theorem 3.10, we have our result (see also [4]). □

If $n > p + 2$, we see that $\nabla f = 0$ and M is a CR submanifold in CP^m with $A_{fV} = 0$ for any vector V normal to M (see Okumura [7, 8]). Therefore, Theorem 4.3 implies the following result.

THEOREM 4.4. *Let M be a compact n-dimensional minimal submanifold in CP^m with semi-flat normal connection. If the sectional curvature K of M satisfies $K \ge 1/n$, and if $n > p + 2$, then M is a totally geodesic complex projective space $CP^{n/2}$ or a geodesic minimal hypersphere $\pi(S^1(\sqrt{1/(n+1)}) \times S^n(\sqrt{n/(n+1)}))$ of some $CP^{(n+1)/2}$ in CP^m.*

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