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Absolutely isolated singularities of a differential equation

F. SANCHO DE SALAS[1] and P. SANCHO DE SALAS[2]
[1]*Departamento de Matemática Pura y Aplicada, Universidad de Salamanca, Plaza de la Merced 1–4, E-37008 Salamanca, Spain*
[2]*Departamento de Matemáticas, Universidad de Extremadura, Avda. de Elvas s/n, Badajoz, Spain*

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Abstract. In this paper we solve the problem of desingularization of an absolutely isolated singularity of a differential equation, including the dicritical case. As an application, we prove the finiteness of the number of dicritical points in the blowing up tree of an absolutely isolated singularity.

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0. Introduction

Let X be a non singular variety of dimension n over a field k. A *differential equation* on X is an invertible subsheaf $\mathfrak{D}$ of the sheaf of derivations:

$$\mathfrak{D} \hookrightarrow T_X = \mathbf{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$$

such that the quotient $T_X/\mathfrak{D}$ is torsion free. A *singular point* of $\mathfrak{D}$ is a point of the critical locus of the injection $0 \to \mathfrak{D} \to T_X$, that is: a point p such that the morphism $\mathfrak{D}_p \otimes k(p) \to T_{X\,p} \otimes k(p)$ is not injective. If $x_1, \dots, x_n$ is a system of parameters at p and $f_1(\partial/\partial x_1) + \cdots + f_n(\partial/\partial x_n)$ is a local generator of $\mathfrak{D}$, then p is a singular point of $\mathfrak{D}$ if and only if $f_1(p) = \cdots = f_n(p) = 0$. The *multiplicity* of $\mathfrak{D}$ at a point $p \in X$, is

$$m_p(\mathfrak{D}) = \min_{a \in \mathcal{F}_p} \{v_p(a)\},$$

where $\mathcal{F}_p$ is the Fitting ideal $F_{n-1}((T_X/\mathfrak{D})_p)$ and v_p is the $\mathfrak{m}_p$-adic valuation corresponding to p. In terms of the local expression of $\mathfrak{D}$, one has

$$m_p(\mathfrak{D}) = \min\{v_p(f_1), \dots, v_p(f_n)\}.$$

A necessary and sufficient condition for a point p to be a singular point of $\mathfrak{D}$ is that $m_p(\mathfrak{D}) > 0$.

If p is a singular point of a differential equation $\mathfrak{D}$, and $\mathfrak{m}$ is the ideal of p then the differential equation induces an endomorphism of the cotangent space

$$\begin{aligned} D_T\colon \mathfrak{m}/\mathfrak{m}^2 &\to \mathfrak{m}/\mathfrak{m}^2 \\ f &\to Df \end{aligned}$$

being D a local generator of $\mathfrak{D}$ at p. We say that D_T is the *linear part* of $\mathfrak{D}$ at p. A singularity of a differential equation is said to be *irreducible* if the linear part, D_T, has at least one nonzero eigenvalue.

Let $\pi\colon X' \to X$ the blowing up of X with center at p, and let E be the exceptional fiber. If $\mathfrak{D}$ is a differental equation on X, there exists one and only one differential equation, $\mathfrak{D}'$, on X' such that its singular locus has codimension greater than 1 and $\mathfrak{D}'_{|X'-E} = \mathfrak{D}_{|X-p}$. $\mathfrak{D}'$ is called the *proper transform* of $\mathfrak{D}$ by π. If E is solution of $\mathfrak{D}'$ (that is, $\mathfrak{D}'\mathfrak{p} \subset \mathfrak{p}$, where $\mathfrak{p}$ is the ideal of E), we say that p (or $\pi\colon X' \to X$) is *non-dicritical*, and one has

$$\mathfrak{D}' = \pi^*\mathfrak{D} \otimes \mathcal{O}_{X'}(1-m) \quad (\mathfrak{p} = \mathcal{O}_{X'}(1)),$$

where m is the multiplicity of $\mathfrak{D}$ at p. On the contrary, if E is not a solution of $\mathfrak{D}'$, then we say that p is *dicritical*, and one has

$$\mathfrak{D}' = \pi^*\mathfrak{D} \otimes \mathcal{O}_{X'}(-m).$$

An isolated singularity p is said to be *absolutely isolated* if all the singularities of the blowing up tree of p are isolated. More precisely, if for any sequence of quadratic transformations

$$X_n \xrightarrow{\pi_n} \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X$$

such that: the center, p_i, of $X_{i+1} \to X_i$ is a closed point in the fiber of p and is a singular point of $\mathfrak{D}_i$, where $\mathfrak{D}_i$ is the proper transform of $\mathfrak{D}$ by $X_i \to X$, then all the singularities of $\mathfrak{D}_n$ over the exceptional fiber are isolated.

The definition of an absolutely isolated singularity given in [1] is adapted to the non-dicritical case. Our definition does not make restrictions about the dicriticalness, that is, it includes the dicritical case.

In this paper we prove the following.

THEOREM. *If p is an absolutely isolated singularity, then after a finite number of quadratic transformations all the singularities become irreducible.*

To prove this theorem one introduces a number associated with the singularity and to the exceptional divisor obtained after a sequence of blowing up's. This is the critical length relative to a divisor at a point p. If the divisor is empty, this number is the classical Milnor number of the differential equation; if the divisor has only one component, it is the ‘adapted Milnor number’ defined in [1]. These numbers are calculated in term of Chern classes (Theorem 3), and this fact permits us to compute the variation of the critical length under blowing up (Theorem 4). One proves that this number never increases, and it decreases strictly whenever one blows up a point of multiplicity greater than 1. The desingularization theorem follows easily. As an application of the desingularization theorem, one can prove the following

THEOREM. *The number of dicritical points in the blowing up tree of an absolutely isolated singularity is finite.*

In the non-dicritical case, the above desingularization theorem is due to C. Camacho, F. Cano and P. Sad [1]. When the dimension of X is two, the desingularization is due to Seidenberg [6]. In dimension three, the desingularization of a non-dicritical differential equation is due to Cano [2, 3].

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1. A fundamental theorem on Chern classes

The results of this section can be found in [5]. Let X be a non singular cuasi-projective variety over a field k. The intersection theory used here will be that of the graded group $\mathrm{GK}(X)$ of the Grothendieck group of coherents modules on X with respect to the filtration defined by the codimension of the support; the product, direct and inverse image, Chern classes, etc, are supposed to be the ones defined with respect to this graded ring.

Let E and F be two locally free $\mathcal{O}_X$-modules and let

$$r = \text{rank of } F$$

$$n = \text{rank of } E.$$

DEFINITION. Let $f\colon F \to E$ be a morphism of $\mathcal{O}_X$-modules. The natural morphism $\mathrm{id}\oplus f\colon F \to F \oplus E$ induces a closed immersion $\mathbb{P}(F) \to \mathbb{P}(F \oplus E)$. The closed subscheme defined by this immersion shall be called the *projective graph of f*.

The cohomology class of the projective graph of f in $\mathrm{GK}(\mathbb{P}(F \oplus E))$ shall be denoted by Γ_f.

PROPOSITION. *Let ξ be the obstruction class of $\mathcal{O}_{\mathbb{P}(F\oplus E)}(-1)$. For any morphism f, the cohomology class of the projective graph of f is*

$$\Gamma_f = \sum_{i=0}^{n} c_i(E) \cdot \xi^{n-i}.$$

THEOREM 1. *The Chern class $c_{n-r+1}(E-F)$ is equal to the projection on X of the self-intersection class of the projective graph of the null morphism. That is, if Γ_0 is the cohomology class of the null morphism, and $\pi\colon \mathbb{P}(F\oplus E)\to X$ is the natural projection, then*

$$c_{n-r+1}(E-F)=\pi_*(\Gamma_0\cdot\Gamma_0).$$

COROLLARY. *The Chern class $c_{n-r+1}(E)$ is equal to the projection on X of the self-intersection class of the projective graph of the null morphism in $\mathbb{P}(\mathcal{O}^r\oplus E)$. This gives a geometric construction of Chern classes.*

COROLLARY. *Let $f\colon F\to E$ be a morphism of $\mathcal{O}_X$ modules. One can see that*

$$c_{n-r+1}(E-F)=\pi_*(\Gamma_f\cdot\Gamma_0).$$

In the rest of this section f will be supposed to be an injective morphism of sheaves.

DEFINITION. Let $f\colon F\to E$ be an injective morphism of locally free modules. The *critical locus* of f is the locus of the points p such that the morphism $F\otimes k(p)\to E\otimes k(p)$ is not injective. This locus is a closed subset, defined by the Fitting ideal $F_{n-r}(E/F)$. We shall consider it to be a closed subscheme: $\mathrm{Spec}\,\mathcal{O}_X/F_{n-r}(E/F)$. The *critical cycle* of f, $\mathrm{Cr}(f)$, is the cycle (in $\mathrm{GK}(X)$) associated with the critical locus. By definition:

$$\mathrm{Cr}(f)=\sum_p n_p\cdot p,$$

where p runs over the generic points of the critical locus of f and n_p is the length at p of the module $(\mathcal{O}_X/(F_{n-r}(E/F)))_p$.

THEOREM 2. *Let $f\colon F\to E$ be an injective morphism of locally free modules, and let $r=$ rank of F, $n=$ rank of E. If the critical locus of f has codimension greater than or equal to $n-r+1$, then the Chern class $c_{n-r+1}(E-F)$ is the critical cycle of f. That is,*

$$c_{n-r+1}(E-F)=\mathrm{Cr}(f)$$

2. Critical length relative to a divisor: Variation under blowing up

Let $i\colon Y\hookrightarrow X$ be a closed subscheme of X defined by a sheaf of ideals $\mathfrak{I}$. Let us denote by T_X^Y the submodule of T_X formed by the derivations of X which are tangent to Y, that is,

$$T_X^Y=\{D\in T_X \text{ such that } D\,\mathrm{Rad}(\mathfrak{I})\subset\mathrm{Rad}(\mathfrak{I})\}\,,$$

where $\mathrm{Rad}(\mathfrak{I})$ is the radical of $\mathfrak{I}$. Let us suppose that Y is reduced. Let $N_{Y/X} = (\mathfrak{I}/\mathfrak{I}^2)^*$ be the module of sections of the the normal bundle of Y in X. We have the exact sequence:

$$\begin{aligned} &0 \to T_X^Y \to T_X \to i_* N_{Y/X}, \\ &D \mapsto D\colon \mathfrak{I}/\mathfrak{I}^2 \to \mathcal{O}_Y, \\ &f \mapsto Df, \end{aligned}$$

where the morphism $T_X \to i_* N_{Y/X}$ is obtained by adjuntion from the natural morphism $i^* T_X \to N_{Y/X}$. If Y is a smooth subscheme, the sequence $0 \to T_X^Y \to T_X \to i_* N_{Y/X} \to 0$ is exact.

EXAMPLE. Let Y be a smooth hypersurface. Let $\{x_1, \ldots, x_n\}$ be local parameters at a point x, such that Y is defined by $x_1 = 0$. Then $\{(\partial/\partial x_1), \ldots, (\partial/\partial x_n)\}$ is a basis of $T_{X,x}$ and $\{x_1(\partial/\partial x_1), (\partial/\partial x_2), \ldots, (\partial/\partial x_n)$ is a basis of $T_{X,x}^Y$.

The following proposition is immediate:

PROPOSITION.

(a) $T_X^Y = T_X^{Y_{\mathrm{red}}}$.

(b) *If $Y = \lambda_1 E_1 + \cdots + \lambda_m E_m$ is an effective divisor whose irreducible components are smooth and have normal crossings, then one has the exact sequence*

$$0 \to T_X^Y \to T_X \to N_{E_1} \oplus \cdots \oplus N_{E_m} \to 0,$$

where N_{E_j} is the module of sections of the normal bundle of E_j in X. Moreover T_X^Y is a locally free module of rank $n =$ dimension of X. Locally, if $\{x_1, \ldots, x_n\}$ is a system of parameters at x, such that Y is defined by $x_1^{\lambda_1} \cdots x_r^{\lambda_r} = 0$, then $T_{X,x}^Y$ is a free module with basis

$$\left\{ x_1 \frac{\partial}{\partial x_1}, \ldots, x_r \frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_{r+1}}, \ldots, \frac{\partial}{\partial x_n} \right\}.$$

Let $Y = \lambda_1 E_1 + \cdots + \lambda_m E_m$ be an effective smooth divisor (that is, E_j is smooth) with normal crossings. Let $\mathfrak{D}$ be a differential equation on X and let

$$Y(\mathfrak{D}) \subset Y$$

be the effective divisor formed by the components of Y that are a solution of $\mathfrak{D}$ (that is, $\mathfrak{D}$ is tangent to them).

DEFINITION. The *critical locus of $\mathfrak{D}$ relative to Y*, $\mathbf{Sin}(\mathfrak{D}, Y)$, is the critical locus of the injection $0 \to \mathfrak{D} \to T_X^{Y(\mathfrak{D})}$. This is the closed subscheme defined by the Fitting ideal $F_{n-1}(T_X^{Y(\mathfrak{D})}/\mathfrak{D})$; this ideal will be denoted by $\mathfrak{C}(\mathfrak{D}, Y)$.

The critical locus of $\mathfrak{D}$ relative to Y is contained in the critical locus of $\mathfrak{D}$, that is,

$$\mathbf{Sin}(\mathfrak{D}, Y) \subset \mathbf{Sin}(\mathfrak{D})$$

This follows immediately from the fact that the injection $0 \to \mathfrak{D} \to T_X$ factors through $T_X^{Y(\mathfrak{D})}$.

DEFINITION. Let p be a generic point of the critical locus of $\mathfrak{D}$ relative to Y. The *critical length of* $\mathfrak{D}$ *at the point* p *relative to* Y, $n_p(\mathfrak{D}, Y)$, is the length at p of $(\mathcal{O}_X/\mathfrak{C}(\mathfrak{D}, Y))_p$

$$n_p(\mathfrak{D}, Y) = \text{length}_{\mathcal{O}_p} \left(\frac{\mathcal{O}_X}{\mathfrak{C}(\mathfrak{D}, Y)} \right)_p.$$

The *critical cycle of* $\mathfrak{D}$ *relative to* Y, $\mathrm{Cr}(\mathfrak{D}, Y)$, is the cycle associated with the critical locus of $\mathfrak{D}$ relative to Y. By definition,

$$\mathrm{Cr}(\mathfrak{D}, Y) = \sum_p n_p(\mathfrak{D}, Y) \cdot p,$$

where p runs over the generic points of the critical locus of $\mathfrak{D}$ relative to Y.

THEOREM 3. *Let* Y *be an effective smooth divisor with normal crossings,* $\mathfrak{D}$ *a differential equation on* X *with isolated singularities, and let us denote by* $T_X^{Y(\mathfrak{D})}$ *the module of derivations tangent to* $Y(\mathfrak{D})$*. Then the Chern class* $c_n(T_X^{Y(\mathfrak{D})} - \mathfrak{D})$ *is the critical cycle relative to* Y*, that is*

$$c_n(T_X^{Y(\mathfrak{D})} - \mathfrak{D}) = \mathrm{Cr}(\mathfrak{D}, Y).$$

Proof. It is immediate from theorem 2 applied to the injection $0 \to \mathfrak{D} \to T_X^{Y(\mathfrak{D})}$.

THEOREM 4. *Let* Y *be an effective smooth divisor (it may be empty) with normal crossings, and* $\mathfrak{D}$ *a differential equation on* X *which is tangent to* Y*. Let* p *be an isolated singularity of* $\mathfrak{D}$ *of multiplicity* m, $\pi\colon X' \to X$ *the blowing up with center at* p *and* E *the exceptional fiber. Let* r *be the number of components of* Y *which contain* p*. If the proper transform* $\mathfrak{D}'$ *of* $\mathfrak{D}$ *has only isolated singularities, then*

(1) *If* p *is non-dicritical, (that is, the exceptional fiber is a solution of the differential equation* $\mathfrak{D}'$*), then*

$$n_p(\mathfrak{D}, Y) - \sum_{q \in E} n_q(\mathfrak{D}', \pi^* Y) \cdot d_q = (m-1)^r m^{n-r},$$

where $n_q(\mathfrak{D}', \pi^* Y)$ *are the corresponding critical lengths of* $\mathfrak{D}'$ *at* q *relative to* $\pi^* Y$ *and* d_q *is the degree of the field extension* $k(p) \to k(q)$.

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(2) *If p is dicritical (that is, the exceptional fiber is not a solution of the differential equation $\mathfrak{D}'$), then*

$$n_p(\mathfrak{D}, Y) - \sum_{q \in E} n_q(\mathfrak{D}', \pi^* Y) \cdot d_q$$
$$= \begin{cases} (m+1)^n - \dfrac{(m+1)^n - 1}{m}, & \text{if } r = 0, \\ (m^r - m^{r-1})(m+1)^{n-r}, & \text{if } r > 0. \end{cases}$$

In particular, $n_p(\mathfrak{D}, Y) - \sum_{q \in E} n_q(\mathfrak{D}', \pi^* Y) \cdot d_q$ is always greater or equal to 0, and it is 0 if and only if p is a singularity of multiplicity 1 and $r > 0$.

Proof. We only have to consider the components of the divisor that contain p, so we can suppose that $Y = \lambda_1 E_1 + \cdots + \lambda_r E_r$. By the proposition we may suppose that Y is reduced, $Y = E_1 + \cdots + E_r$, and $\pi^* Y = E_1' + \cdots + E_r' + E$, where E_i'= proper transform of E_i. Let us denote $n_p = n_p(\mathfrak{D}, Y)$, $n_q = n_q(\mathfrak{D}', \pi^* Y)$ and $Y' = \pi^* Y(\mathfrak{D}')$.

As the question is local, one can suppose that X is a projective variety. Moreover, it is not difficult to see that one can suppose that the only singularity of the differential equation is p (replacing T_X by a locally free module M of rank n such that: $\mathfrak{D} \hookrightarrow M$, $M_{|U} \simeq T_{X\,|U}$ for some open neighbourhood U of p, and $\mathfrak{D}$ is locally a direct summand of M outside of p). By Theorem 3, $c_n(T_X^Y - \mathfrak{D}) = n_p \cdot p$ and $c_n(T_{X'}^{Y'} - \mathfrak{D}') = \sum_{q \in E} n_q \cdot q$. Moreover

$$\left(n_p - \sum_{q \in E} n_q \cdot d_q \right) \cdot p = \pi_*(\pi^* c_n(T_X^Y - \mathfrak{D}) - c_n(T_{X'}^{Y'} - \mathfrak{D}'))$$

because $\pi_*(q) = d_q \cdot p$ and $\pi_* \pi^* = \mathrm{id}(\pi_* \pi^*(a) = a \cdot \pi_*(1) = a$, since $\pi_*(1) = 1$ because π is birational). So one has to compute $\pi^* c_n(T_X^Y - \mathfrak{D}) - c_n(T_{X'}^{Y'} - \mathfrak{D}')$. But π is an isomorphism outside of p, so $\pi^* c_n(T_X^Y - \mathfrak{D}) - c_n(T_{X'}^{Y'} - \mathfrak{D}')$ is concentrated in E. As E is a projective space, it must be

$$\pi^* c_n(T_X^Y - \mathfrak{D}) - c_n(T_{X'}^{Y'} - \mathfrak{D}') = \lambda E^n, \quad \lambda \in \mathbb{Z}.$$

Remark. With the object of computing λ, one can substitute X by any open neighbourhood of p, so one may suppose that the $\mathcal{O}_X$-modules that appear are free.

We have the exact sequences

$$0 \to T_{X'}^{E_1' + \cdots + E_r' + E} \to T_{X'}^{E_1' + \cdots + E_r'} \to N_E \to 0,$$
$$0 \to T_{X'}^{E_1' + \cdots + E_r'} \to T_{X'} \to N_{E_1'} \oplus \cdots \oplus N_{E_r'} \to 0,$$
$$0 \to T_{X'} \to \pi^* T_X \to T_E \otimes \mathcal{L}_E \to 0.$$

Now, since E is a projective space, we have the exact sequence

$$0 \to \Omega_E \to \mathcal{O}_E(-1)^n \to \mathcal{O}_E \to 0,$$

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therefore

$$0 \to \mathcal{O}_E \to \mathcal{O}_E(1)^n \to T_E \to 0$$

and taking the tensor product with $\mathcal{L}_E$ (notice that $\mathcal{L}_{-E} = \mathcal{O}_{X'}(1)$), one has

$$0 \to N_E \to \mathcal{O}_E^n \to T_E \otimes \mathcal{L}_E \to 0.$$

Then $c(T_E \otimes \mathcal{L}_E) = 1/(1-E)^n(1+E)$, since

$$c(N_E) = c(\mathcal{L}_E \otimes \mathcal{O}_E) = c(\mathcal{L}_E \otimes (1 - \mathcal{L}_{-E})) = c(\mathcal{L}_E - 1) = 1 + E, \quad (1)$$

$$c(\mathcal{O}_E) = c(1 - \mathcal{L}_{-E}) = \frac{1}{1-E}. \quad (2)$$

Putting it all together, one has (taking into account the Remark)

$$c(T_{X'}^{E_1'+\cdots+E_r'}) = \frac{1}{c(T_E \otimes \mathcal{L}_E)c(N_{E_1'})\ldots c(N_{E_r'})} = (1-E)^{n-r}(1+E)$$

because $c(N_{E_j'}) = 1 + E_j' = 1 - E$, as $E_j' + E = \pi^*(E_j) = 0$ (by the Remark), and

$$c(T_{X'}^{E_1'+\cdots+E_r'+E}) = c(T_{X'}^{E_1'+\cdots+E_r'} - N_E) = (1-E)^{n-r}.$$

(1) In this case $Y' = E_1' + \cdots + E_r' + E$ and $\mathfrak{D}' = \pi^*\mathfrak{D} \otimes \mathcal{L}_{(m-1)E}$, hence

$$c_n(T_{X'}^{E_1'+\cdots+E_r'+E} - \mathfrak{D}') = \left[\frac{(1-E)^{n-r}}{1+(m-1)E}\right]_n$$

and one concludes by an easy computation (note that $\pi_* E^n = (-1)^{n-1} \cdot p$).

(2) In this case $Y' = E_1' + \cdots + E_r'$ and $\mathfrak{D}' = \pi^*\mathfrak{D} \otimes \mathcal{L}_{mE}$, therefore

$$c_n(T_{X'}^{E_1'+\cdots+E_r'} - \mathfrak{D}') = \left[\frac{(1-E)^{n-r}(1+E)}{1+mE}\right]_n$$

and again an easy computation allows us to conclude.

When Y is empty and p is non-dicritical, the formula is given in [1, Thm 2].

3. Desingularization

Our aim now is to prove that the absolutely isolated singularities become irreducible after a finite number of quadratic tranformations. Before proving the theorem let us see some previous lemmas:

LEMMA 1. *Let H_1 and H_2 be two hypersurfaces, which are solutions of a differential equation $\mathfrak{D}$ and with normal crossings. If $q \in H_1 \cap H_2$ is an absolutely isolated non-dicritical singularity of $\mathfrak{D}$ with multiplicity 1, then q is irreducible.*

Proof. Either H_1 or H_2 gives a nonzero eigenvalue; in fact: on the contrary, we would have $v_q(H_1) > 1$, $v_q(H_2) > 1$, and then q could not be an absolutely isolated singularity (blowing up at q, the critical locus would be of codimension less or equal than $n-1$, as can be shown by an easy computation).

LEMMA 2. *Let*

$$\cdots \to X_m \to X_{m-1} \to \cdots \to X_1 \to X_0 = X$$

be a sequence of blowing up's such that: $\pi_{i+1}: X_{i+1} \to X_i$ is a blowing up centered at p_i= non-dicritical singular point of $\mathfrak{D}_i$, with multiplicity 1, in the fiber of p_{i-1}, placed out of the corners, and which is rational over p (that is, the degree of the extension $k(p) \to k(p_i)$ is 1).

Then the formal curve in X defined by this sequence of points (that is, the curve whose successive proper transforms pass through the points p_i) is solution of the differential equation.

Proof. Let $\mathcal{O}_i$ be the local ring of p_i, with maximal ideal $\mathfrak{m}_i$, and let D_i be a local generator of $\mathfrak{D}_i$ at p_i ($\mathcal{O}_0 = \mathcal{O}$ and $D_0 = D$). Notice that $D_{i|\mathcal{O}} = D$ for every i, because p_i is non-dicritical of multiplicity 1, and then $\mathfrak{D}_i = \pi_i^* \mathfrak{D}_{i-1}$. Since $\mathfrak{m}_i$ is a singular point, one has that $D_i \mathfrak{m}_i \subset \mathfrak{m}_i$, hence $D(\mathfrak{m}_i^{i+1} \cap \mathcal{O}) \subset \mathfrak{m}_i^{i+1} \cap \mathcal{O}$. Let $\widehat{\mathcal{O}}$ denote the formal completion of $\mathcal{O}$. The formal curve defined by the sequence of points p_i is given by the surjective map

$$\widehat{\mathcal{O}} \to \varprojlim \left(\mathcal{O}/\mathfrak{m}_i^{i+1} \cap \mathcal{O}\right) \simeq k(p)[[t]].$$

The kernel is an ideal I such that $DI \subset I$, because $D(\mathfrak{m}_i^{i+1} \cap \mathcal{O}) \subset \mathfrak{m}_i^{i+1} \cap \mathcal{O}$, so the curve is a solution of $\mathfrak{D}$.

THEOREM 5. *Let p be an absolutely isolated singularity of a differential equation. Then, after a finite number of quadratic transformations, all the singularities become irreducible.*

Proof. By induction on the critical length relative to the exceptional divisor. Notice that, by Theorem 4, this number never increases under blowing up. If it is 0, then some of the components of the exceptional divisor must give a nonzero eigenvalue. Let q be a singular point, with critical length n_q and multiplicity m_q. If m_q is greater than 1, then, blowing up at q, the critical length decreases (by Theorem 4) and one ends by induction. So we may suppose the multiplicity to be 1 and that it remains equal to 1 after blowing up. Moreover, we may suppose that the points we are blowing up at are rational over q, since, to the contrary, the critical length also decreases (Theorem 4). Now, the singularities in the corners are irreducible (Lemma 1), and so are the dicritical points (of multiplicity 1); we

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may suppose then that we are always blowing up at non-dicritical points, with multiplicity 1, out of the corners and rational over q. By Lemma 2, one has a formal curve passing through q, which is transversal to the exceptional fiber, and solution of the differential equation. It is clear then that q is irreducible.

4. Finiteness of dicritical points

In this section the base field is supposed to be of characteristic 0 and algebraically closed. Our aim is to see that the dicritical points in the blowing up tree of an absolutely isolated singularity are finite in number. We shall first discuss the case when the variety has dimension two, since the argument is easily generalized for dimension n.

THEOREM 6. *The number of dicritical points in the blowing up tree of an absolutely isolated singularity is finite (*$\dim X = 2$*).*

Proof. It follows from the next proposition.

PROPOSITION. *Let D_T be the linear part at p of a differential equation of multiplicity* 1. *Suppose that $D_T = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ with $\mu \neq 0$. Then.*

(1) *If $\lambda/\mu \notin \mathbb{Q}^+$, then there is no dicritical points in the blowing up tree of p.*
(2) *If $\lambda/\mu \in \mathbb{Q}^+$, then there is, at most, one dicritical point in the blowing up tree of p.*

Proof. (1) It follows from the fact that the condition is stable under blowing up: an easy computation shows that blowing up at p two singularities appear and their linear part have the following Jordan form $\left(\begin{smallmatrix} \lambda-\mu & 0 \\ 0 & \mu \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu-\lambda \end{smallmatrix}\right)$, and $(\lambda - \mu)/\mu, \lambda/(\mu - \lambda) \notin \mathbb{Q}^+$ if $\lambda/\mu \notin \mathbb{Q}^+$.

Since the matrix of a dicritical point is $\left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu \end{smallmatrix}\right)$, $\mu/\mu = 1 \in \mathbb{Q}^+$, one concludes.

(2) Let $\lambda/\mu = n/m$, $n, m \in \mathbb{N}$, relatively prime. Suppose that $n \leqslant m$ (if $n \geqslant m$ the same argument is valid, interchanging λ by μ). If $n = m$, the point is dicritical and blowing up there are no more singular points. If $n < m$, then $(\lambda - \mu)/\mu = (n/m) - 1 \notin \mathbb{Q}^+$, so one of the points that appear can not have dicritical points in its blowing up tree, by (1). Dicritical points can only appear at the other point. If $\mu - \lambda = \lambda$ ($\mu = 2\lambda$), then the matrix of this point is either $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ (and then it is dicritical) or $\left(\begin{smallmatrix} \lambda & \gamma \\ 0 & \lambda \end{smallmatrix}\right)$, with $\gamma \neq 0$, and blowing up there is only one singular point with matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \lambda \end{smallmatrix}\right)$, that has no dicritical points above it, by (1). If $\lambda - \mu \neq \lambda$, then the matrix of the point is $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu-\lambda \end{smallmatrix}\right)$ and $\lambda/(\mu - \lambda) = n/(m - n) \in \mathbb{Q}^+$. But $\max\{n, m\} > \max\{n, m - n\}$, so the process is finite.

Now the theorem follows. In fact, we can suppose, by Theorem 5, that there is a nonzero eigenvalue, so $D_T = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ or $D_T = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$. In the first case we conclude by the proposition above. In the second: blowing up, only one singular point appears and its matrix is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \lambda \end{smallmatrix}\right)$, so we conclude again by the proposition.

Now, the proof in dimension n is clarified.

THEOREM 7. *The number of dicritical points in the blowing up tree of an absolutely isolated singularity is finite.*

Proof. It follows from Theorem 5 and the next proposition.

PROPOSITION. *Let D_T be the linear part at p of a differential equation with a nonzero eigenvalue. Then*

(1) *If D_T has two eigenvalues $\mu \neq 0$ and λ, such that $\lambda/\mu \notin \mathbb{Q}^+$, then there are no dicritical points in the blowing up tree of p. If D_T does not diagonalize, then there is not any dicritical point in the blowing up tree.*

(2) *If*

$$D_T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad and \quad \frac{\lambda_i}{\lambda_j} \in \mathbb{Q}^+ \; for \; any \; i, j,$$

then there is, at most, one dicritical point in the blowing up tree of p.

Proof. (1) The existence of two eigenvalues $\mu \neq 0$ and λ, such that $\lambda/\mu \notin \mathbb{Q}^+$, is stable by blowing up, as an easy computation shows, so one concludes. If D_T does not diagonalize, then its Jordan matrix can be put in the form

$$\begin{pmatrix} \ddots & & 0 \\ & \lambda & 1 \\ 0 & & \lambda \end{pmatrix}.$$

If $\lambda = 0$, since there exists $\mu \neq 0$, we conclude by (1). If $\lambda \neq 0$, one blows up. Either a matrix of the same type comes out (which is non-dicritical), or the zero eigenvalue appears, so one concludes as above.

(2) Let $x_1, \ldots, x_n$ be parameters such that D_T is diagonal. One blows up. On the affine set $x_i \neq 0$ one has: either a singular point whose Jordan matrix has the form

$$\begin{pmatrix} \ddots & & 0 \\ & \lambda_i & 1 \\ 0 & & \lambda_i \end{pmatrix}$$

that has no dicritical points above it, or a singular point with diagonal matrix and eigenvalues $\lambda_1 - \lambda_i, \ldots, \lambda_{i-1} - \lambda_i, \lambda_i, \lambda_{i+1} - \lambda_i, \ldots, \lambda_n - \lambda_i$. But $(\lambda_j - \lambda_i)/\lambda_i \in \mathbb{Q}^+$ if and only if $\lambda_i/\lambda_j < 1$. So, dicritical points (if they exist) can only appear in the affine set $x_i \neq 0$ such that $\lambda_i/\lambda_j < 1$ for any $j \neq i$. But if

$\lambda_j/\lambda_i = m_j/n$, g.c.d.$\{n, m_j\} = 1$, then $(\lambda_j - \lambda_i/\lambda_i = (m_j - n)/n$ and max $\{n, m_j\} >$ max$\{n, m_j - n\}$, so the process is finite.

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