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NOTE ON A SUBALGEBRA OF $C(X)$

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$C(X)$ (resp. $C^*(X)$) will denote as usual the ring of all (resp. all bounded) continuous functions into the real line R. Define $C^{\#}(X)$ to consist of all $f \in C(X)$ whose image $M(f)$ in the residue class ring $C(X)/M$ is real for every maximal ideal M in $C(X)$. Then $C^{\#}$ shares with C^* the property of being an intrinsically determined subalgebra of C. The compactification corresponding to $C^{\#}$ (as uniformity determining subalgebra of C^*) is thus also an intrinsically determined one. We show that this compactification is well known and "natural" in the cases of several elementary spaces X. Some topological characterizations of $C^{\#}(X)$ are first obtained. For notation and background information we refer to [1]. All spaces are assumed completely regular.

The straightforward proof of the following proposition is omitted.

PROPOSITION 1. *For a function $f \in C(X)$ the following are equivalent.*

(1) $f \in C^{\#}(X)$.

(2) *Every z-ultrafilter on X has a member on which f is constant.*

(3) *For every z-ultrafilter $\mathfrak{A}$ on X the family $f^{\#}\mathfrak{A}$ of all closed sets in R whose preimages under f belong to $\mathfrak{A}$, is again a z-ultrafilter.*

It is not difficult to verify that $C^{\#}(X)$ is a subalgebra of $C^*(X)$ which is also a sublattice. It need not be uniformly closed but is closed in the m-topology. We now proceed to obtain another characterization of $C^{\#}(X)$ which is useful in special cases.

LEMMA. *Let $D=\{d_n : n \in N\}$ be a C-embedded copy of N in X. There exists a neighbourhood W_n of d_n for each n such that for every zero-set $Z_n \subset W_n$ and for every $M \subset N, \bigcup_{m\in M} Z_m$ is a zero-set. (Hence in a G_δ-space every C-embedded copy of N is a zero-set).*

Proof. There exists $u \in C(X)$ such that $u(d_n)=n$. Put $W_n=\{x : |u(x)-n| \leq \frac{1}{3}\}$ and let Z_n be any zero-set contained in W_n. Put $Y_n=\{x : |u(x)-n| \geq \frac{2}{3}\}$. Note that Z_n is disjoint from Y_n and $W_m \subset Y_n$ for all $n \in N$ and $m \neq n$.

We can choose a nonnegative $h_n \in C(X)$ which has the value 0 precisely on Z_n and the value 1 precisely on Y_n. Since each point x has a neighbourhood (e.g. $\{y : |u(y)-u(x)| < 1\}$) on which all but finitely many h_n have the same value, it follows that the function $\inf_{m\in M} h_m$ belongs to $C(X)$ and we have $Z(\inf_{m\in M} h_m) = \bigcup_{m\in M} Z_m$ as required.

PROPOSITION 2. *$f \in C^{\#}(X)$ if and only if f is bounded and $f[D]$ is closed for every C-embedded copy of N.*

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Proof. Suppose D is a C-embedded copy of N such that $\mathrm{cl} f[D]-f[D]$ contains a point r. Choose $y_n \in D(n \in N)$ such that $\lim_n f(y_n)=r$ and put $V_n=\{x:|f(x)-f(y_n)|\leq 1/n\}$. Choose W_n to be a *nbhd* of y_n as described in the above lemma. Then $Z_n=V_n \cap W_n$ is a zero-set *nbhd* of y_n such that $A_m=\bigcup_{n\geq m} Z_n$ is a zero-set for each m. The family $\{A_m : m \in N\}$ has the finite intersection property, so there exists a z-ultrafilter $Z[M]$ to which each A_m belongs. For any $\varepsilon>0$ we can take m so large that $0<|f(x)-r|\leq|f(x)-f(y_m)|+|f(y_m)-r|<\varepsilon$ holds for all $x \in A_m$. It follows that $M(f-r)=M(f)-M(r)$ is infinitely small [1, Ch. 5] so $M(f)$ cannot be real.

To prove the converse, take $f \in C^*(X)$ and suppose that $M^p(f)$ fails to be real for some maximal ideal M^p corresponding to $p \in \beta X$. Since M^p is hyper-real, there exists $g \in C(X)$ with $|M^p(g)|$ infinitely large. At the same time $|M^p(f)-f^\beta(p)|$ is infinitely small but positive. Hence for each $n \in N$ there is a neighbourhood U_n of p such that

$$0 < |f(x)-f^\beta(p)| < \frac{1}{n} < n < |g(x)|$$

for all $x \in U_n \cap X$. It follows that there exists a sequence (x_n) in X such that $g(x_n)$ is strictly increasing to ∞, $f(x_n)\to f^\beta(p)$ while $f(x_n)\neq f^\beta(p)$ for all n. We conclude that $D=\{x_n : n \in N\}$ is a C-embedded copy of N [1, 1.20] and that $f[D]$ is not closed. This completes the proof.

We now turn to some special cases. The C-embedded copies of N in any space X are formed by sequences (x_n) satisfying the condition $h(x_n)\to\infty$ for some $h \in C(X)$. This condition reduces in the case $X=R$ to the requirement that (x_n) tends to $\pm\infty$; in R^2 it is equivalent to saying that the distance from x_n to 0 tends to ∞; in Q it becomes (x_n) tends to $\pm\infty$ or to an irrational limit; in N it is automatically satisfied. Using Proposition 2 we conclude easily that $C^{\#}(R)$ consists of all f which have a constant value on $\{x:x\leq a\}$ and on $\{x:x\geq b\}$ for some $a, b \in R$. It is not difficult to verify that the compactification determined by $C^{\#}(R)$ is the extended real line. $C^{\#}(R^2)$ consists of all f which are constant on the complement of some compact set; the one point compactification is determined in this case. Both $C^{\#}(Q)$ and $C^{\#}(N)$ consist of functions which attain only finitely many values. Any two disjoint closed sets in Q (resp. N) have disjoint open-closed neighbourhoods and so they can be separated by a function in $C^{\#}$. The corresponding compactification can be verified to be βQ (resp. βN).

We note in conclusion that the compactification [0, 1] of the space of rational numbers in this interval appears to be a difficult one to describe intrinsically.

Reference

1. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, N.Y., 1960.

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